A simple explanation of Born's rule

Adán Cabello
University of Seville

QCQMB'18 Prague
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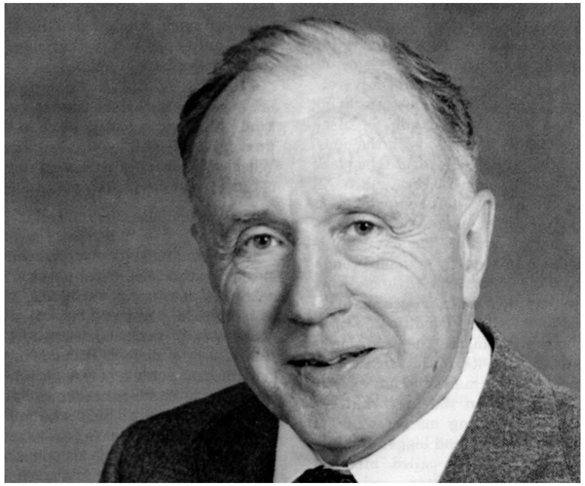
The problem

Where does it come from?

Quantum theory does not trouble me *at all*. It is just the way the world works. What eats me, gets me, drives me, pushes me, is to understand *how* it got that way. What is the deeper foundation underneath it? Where does it come from?

J. A. Wheeler



Introduction. Arguably, the "right view" in science is the one from which one learns things that remain hidden in other ways to look at nature.

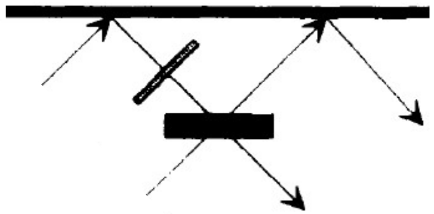
There are many ways to look at quantum theory (QT). Each of them is organized around a particular fundamental concept, e.g., "quantum system", e.g., "measurement scenario". The problem is that none of these concepts is given *a priori*, but all of them are defined through operations the experimenter performs.

Quantum systems and scenarios

- The world is not “made of” quantum systems.
- Quantum systems are “created” by they way the experimenter interrogates the world.

Quantum systems

One qubit



Quantum systems

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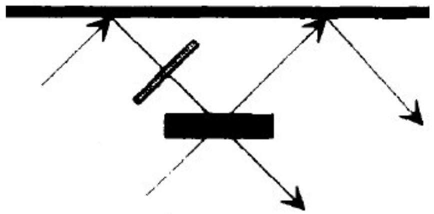
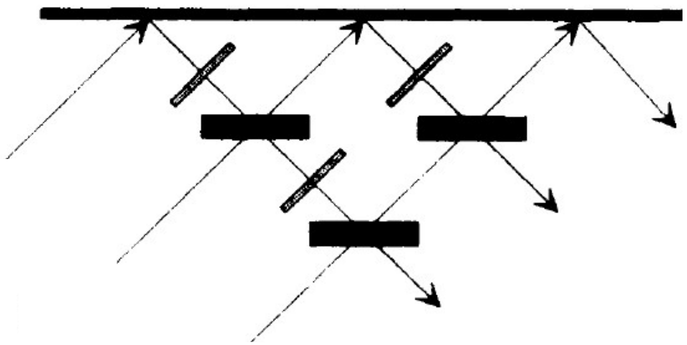


One qutrit



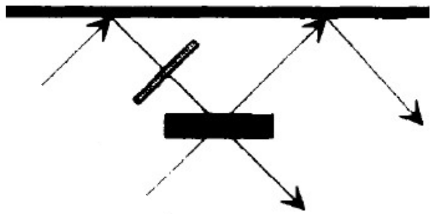
M. Reck, A. Zeilinger, H. J. Bernstein, and P. Bertani, Phys. Rev. Lett. **73**, 58 (1994).

Quantum systems and scenarios

- The world is not “made of” scenarios.
- Scenarios are “created” by they way and extent to which the experimenter interrogates the world.

Quantum systems Measurement scenarios

One qubit



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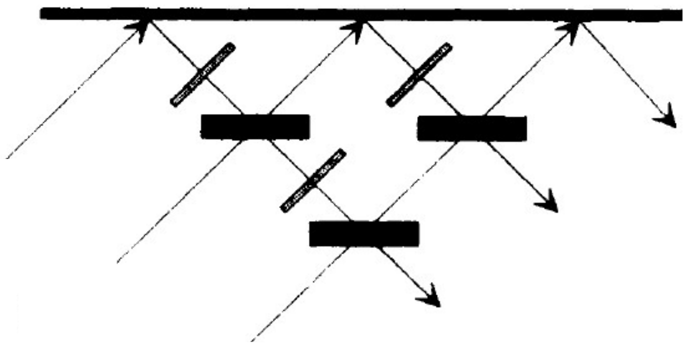

Two incompatible measurements

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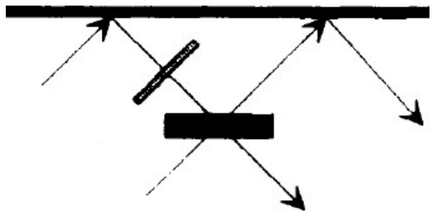
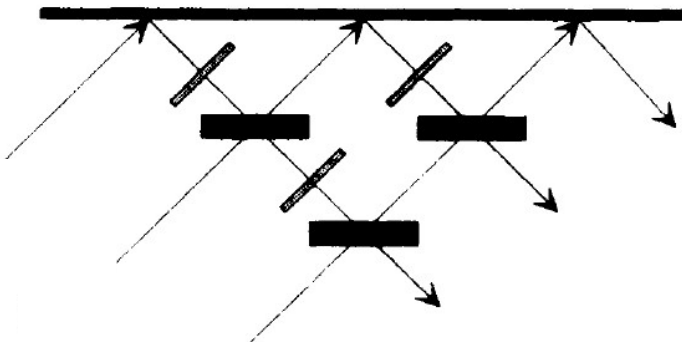
Quantum systems

One qubit



One qutrit



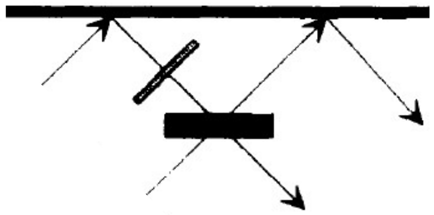
Measurement scenarios

Two incompatible measurements

Two compatible measurements

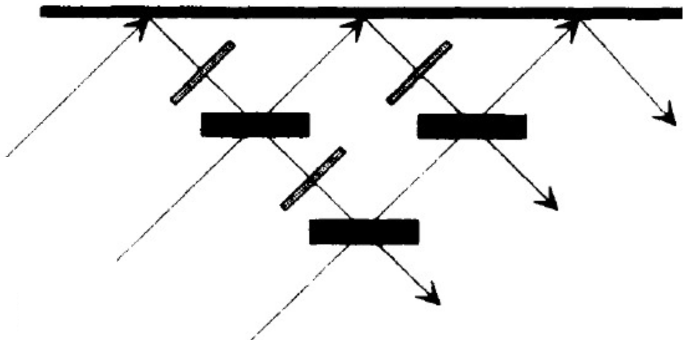
Quantum systems — Measurement scenarios

One qubit



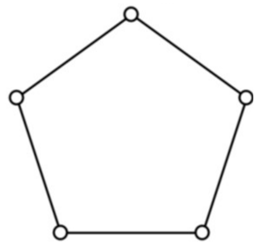
Two incompatible measurements

One qutrit



Two compatible measurements

5-cycle



The experimenter can always enlarge at will both the quantum system and the scenario. Consequently, any general probabilistic theory (GPT) that provides probabilities to the measurement outcomes should consistently connect the probabilities of all these extensions. Then, a reasonable question is whether there is an alternative way to address the problem of what probability assignments a theory allows, without separating the study into quantum systems or measurement scenarios, but adopting a perspective that naturally contains them all.

Here, we will introduce an approach, the graph-theoretic approach to GPTs, that does precisely that. This nonstandard way to look at QT will turn out to be particularly illuminating about what is the property of nature that enforces QT.

Measurements

Definition. A *measurement* $\mathcal{M}$ is an interaction that produces an outcome and transforms the state of the system (which encodes the probabilities of the outcomes of future measurements) into a new state. $\mathcal{M}$ can be characterized by a set $\{\mathcal{M}_x\}_{x \in X}$ of transformations, one for each outcome $x \in X$.

Measurements, events

Definition. A *measurement* $\mathcal{M}$ is an interaction that produces an outcome and transforms the state of the system (which encodes the probabilities of the outcomes of future measurements) into a new state. $\mathcal{M}$ can be characterized by a set $\{\mathcal{M}_x\}_{x \in X}$ of transformations, one for each outcome $x \in X$.

Definition. If the input state is ρ, then $\mathcal{M}_x\rho$ is the state after measuring $\mathcal{M}$ and obtaining outcome x. An *event* $(\mathcal{M}_x\rho|\rho)$ is the transformation $\mathcal{M}_x$ of the input state ρ when $\mathcal{M}$ gives outcome x. $P(\mathcal{M}_x\rho|\rho)$ denotes the probability of $(\mathcal{M}_x\rho|\rho)$.

Born's rule

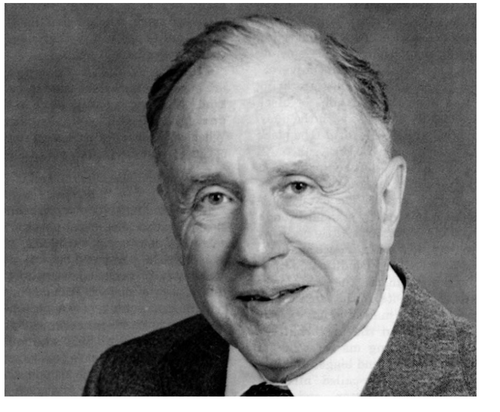
Born's rule.
(I) States are represented by unit vectors $|\rho\rangle$ and $|\mathcal{M}_x\rho\rangle$.
(II) $P(\mathcal{M}_x\rho|\rho) = |\langle\mathcal{M}_x\rho|\rho\rangle|^2$.

M. Born, Z. Physik **37**, 863 (1926).

Problem

Born's rule.
(I) States are represented by unit vectors $|\rho\rangle$ and $|\mathcal{M}_x\rho\rangle$.
(II) $P(\mathcal{M}_x\rho|\rho) = |\langle\mathcal{M}_x\rho|\rho\rangle|^2$.

Problem.
What property of the world enforces (I) and (II)?



Born’s rule is almost QT

(a) Connects the mathematical formalism (the Hilbert space) with the experiments.

(b) Provides the empirical content of the concepts of “state”, “system”, and “measurement”.

(c) Tells us how a state transforms after a measurement.

G. Lüders, Ann. Phys. (Leipzig) **8**, 322 (1951).

(d) Teaches us how to compute the probabilities for each measurement outcome.

(e) Is responsible for nearly all predictions of QT.

Ideal measurements

Definition. A measurement is *ideal* if:

(i) It gives the same outcome when performed consecutive times,

(ii) only disturbs measurements that are incompatible,

(iii) each of its coarse-grainings has a realization that satisfies (i) and (ii).

Axiom. Two ideal measurements, each on a different system, constitute an ideal measurement for the joint system.

G. Chiribella and X. Yuan, arXiv:1404.3348.

Disturbs, incompatible, coarse-graining

Definition. A measurement $\{\mathcal{M}_x\}_{x\in X}$ *disturbs* a measurement $\{\mathcal{N}_y\}_{y\in Y}$ if, from the outcome statistics of $\{\mathcal{N}_y\}_{y\in Y}$, we can detect whether $\{\mathcal{M}_x\}_{x\in X}$ was performed before $\{\mathcal{N}_y\}_{y\in Y}$.

Definition. Two measurements, $\{\mathcal{M}_x\}_{x\in X}$ and $\{\mathcal{N}_y\}_{y\in Y}$, are *incompatible* or not jointly measurable if there is no measurement $\{\mathcal{L}_{x,y}\}_{x\in X, y\in Y}$ such that, for all ρ and all $x \in X$, $P(\mathcal{M}_x\rho|\rho) = \sum_{y\in Y} P(\mathcal{L}_{x,y}\rho|\rho)$ and, for all ρ and all $y \in Y$, $P(\mathcal{N}_y\rho|\rho) = \sum_{x\in X} P(\mathcal{L}_{x,y}\rho|\rho)$.

Definition. A measurement $\{\mathcal{N}_y\}_{y\in Y}$ is a *coarse-graining* of a measurement $\{\mathcal{M}_x\}_{x\in X}$ if, for all x, there is y such that, for all ρ, $P(\mathcal{M}_x\rho|\rho) \leq P(\mathcal{N}_y\rho|\rho)$.

QT is only about ideal measurements

Newmark's theorem. Generalized measurements are represented in QT by POVMs, but the Hilbert space in which each of these POVMs is defined can be extended to a larger Hilbert space such that the POVM corresponds to a PVM, i.e., to an ideal measurement.

M. A. Neumark, Izv. Akad. Nauk S.S.S.R. [Bull. Acad. Sci. U.S.S.R.] Sér. Mat. **4**, 53 (1940); Izv. Akad. Nauk S.S.S.R. [Bull. Acad. Sci. U.S.S.R.] Sér. Mat. **4** 277 (1940); C.R. (Dokl.) Acad. Sci. U.R.S.S. (N.S.) **41** 359 (1943).

Physics and idealization

Observation. We do not assume that ideal measurements can be implemented with infinite precision or in a perfectly noiseless way. We only assume that ideal measurements exist as idealizations and that physical theories must (at least) give the probabilities of the outcomes of ideal measurements.

Assumption

Assumption. Physical theories are possible. There are *similar experiments* for which the theory assigns the same probabilities and *independent experiments*, one with events $\{(\mathcal{M}_x\rho|\rho)\}$ and the other with events $\{(\mathcal{N}_y\rho'|\rho')\}$, such that the joint events $\{(\mathcal{M}_x\rho, \mathcal{N}_y\rho'|\rho, \rho')\}$ satisfy $P(\mathcal{M}_x\rho, \mathcal{N}_y\rho'|\rho, \rho') = P(\mathcal{M}_x\rho|\rho)P(\mathcal{N}_y\rho'|\rho')$.

The graph-theoretic approach to GPTs

A. Cabello, S. Severini, and A. Winter, Phys. Rev. Lett. **112**, 040401 (2014).

Equivalence

Definition. Two events $(\mathcal{M}_x\rho|\rho)$ and $(\mathcal{N}_y\rho|\rho)$ are *equivalent*, denoted $(\mathcal{M}_x\rho|\rho) \sim (\mathcal{N}_y\rho|\rho)$, if they correspond to indistinguishable transformations of the same state ρ. Two transformations $\mathcal{M}_x$ and $\mathcal{N}_y$ are *equivalent*, denoted $\mathcal{M}_x \sim \mathcal{N}_y$, if $(\mathcal{M}_x\rho|\rho) \sim (\mathcal{N}_y\rho|\rho)$ for all ρ.

Equivalence, exclusivity

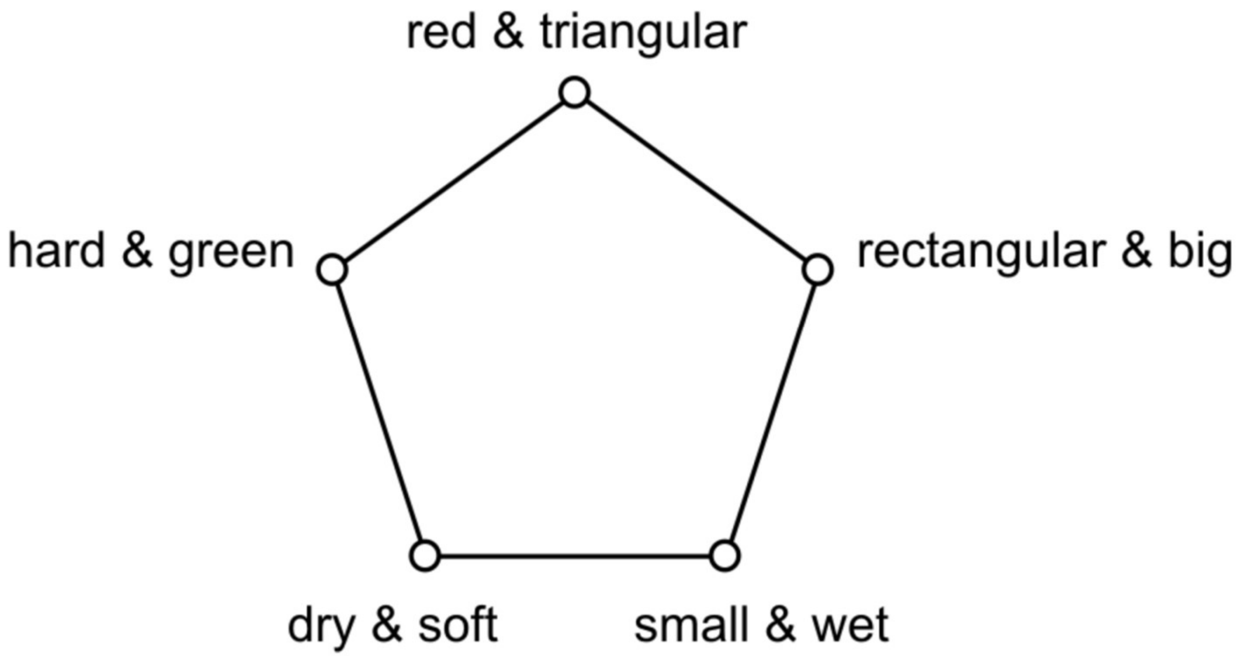
Definition. Two events $(\mathcal{M}_x\rho|\rho)$ and $(\mathcal{N}_y\rho|\rho)$ are *equivalent*, denoted $(\mathcal{M}_x\rho|\rho) \sim (\mathcal{N}_y\rho|\rho)$, if they correspond to indistinguishable transformations of the same state ρ. Two transformations $\mathcal{M}_x$ and $\mathcal{N}_y$ are *equivalent*, denoted $\mathcal{M}_x \sim \mathcal{N}_y$, if $(\mathcal{M}_x\rho|\rho) \sim (\mathcal{N}_y\rho|\rho)$ for all ρ.

Definition. Two events $(\mathcal{M}_x\rho|\rho)$ and $(\mathcal{N}_y\rho|\rho)$ are *exclusive*, denoted $(\mathcal{M}_x\rho|\rho) \perp (\mathcal{N}_y\rho|\rho)$, if there is an ideal measurement $\{\mathcal{L}_z\}_{z\in Z}$ with two different outcomes z and z' such that $(\mathcal{M}_x\rho|\rho) \sim (\mathcal{L}_z\rho|\rho)$ and $(\mathcal{N}_y\rho|\rho) \sim (\mathcal{L}_{z'}\rho|\rho)$. If $(\mathcal{M}_x\rho|\rho) \sim (\mathcal{N}_y\rho|\rho)$ and $(\mathcal{M}_x\rho|\rho) \perp (\mathcal{L}_z\rho|\rho)$, then $(\mathcal{N}_y\rho|\rho) \perp (\mathcal{L}_z\rho|\rho)$. Two transformations $\mathcal{M}_x$ and $\mathcal{N}_y$ are *exclusive*, denoted $\mathcal{M}_x \perp \mathcal{N}_y$, if $(\mathcal{M}_x\rho|\rho) \perp (\mathcal{N}_y\rho|\rho)$ for all ρ.

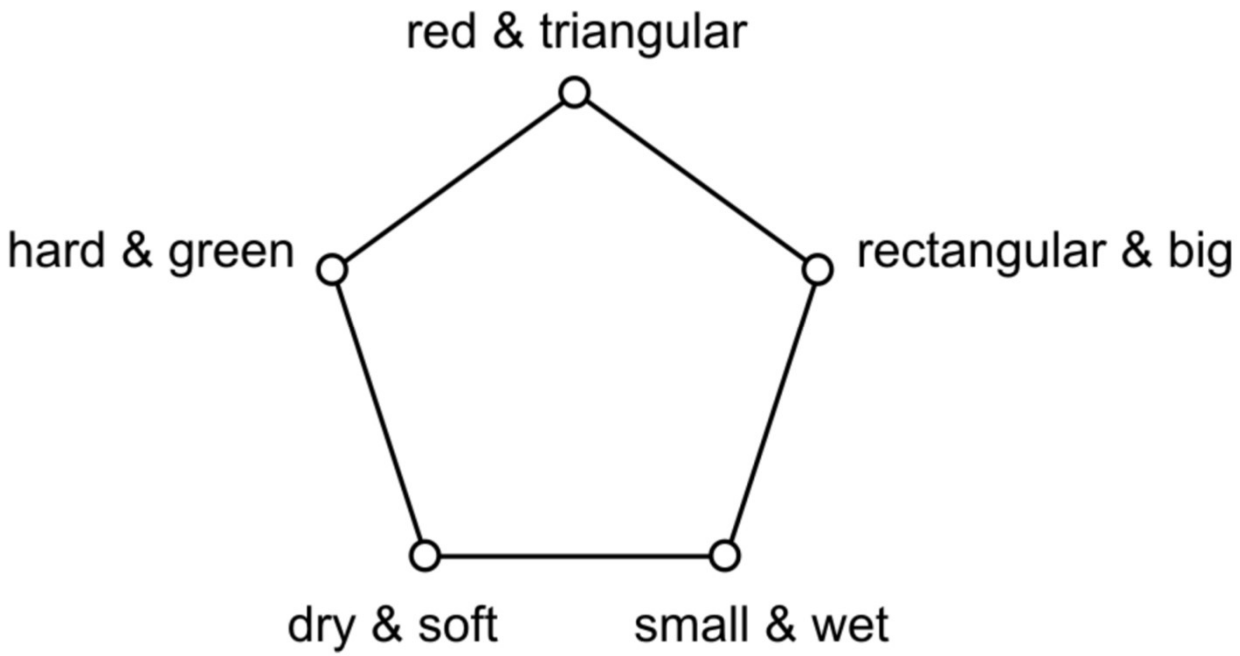
Graph of exclusivity

Definition. For a given theory, the *graph of exclusivity* G represents the set of all sets of equivalence classes of transformations produced by ideal measurements that have G as graph of exclusivity.

Graph of exclusivity for a set of transformations



Graph of exclusivity for a set of transformations



The reason why the transformations “red & triangular” and “rectangular & big” are exclusive is because there is an ideal measurement $\mathcal{M}$ (of the color, shape, and size) such that, for any initial state ρ, the tranformations “red & triangular” and “rectangular & big” correspond to different outcomes of $\mathcal{M}$.

Graph of exclusivity

Definition. For a given theory, the *graph of exclusivity* G represents the set of all sets of equivalence classes of transformations produced by ideal measurements that have G as graph of exclusivity.

Observation. All exclusivity graphs are realizable in classical probability theory.

Graph of exclusivity, probability assignment

Definition. For a given theory, the *graph of exclusivity* G represents the set of all sets of equivalence classes of transformations produced by ideal measurements that have G as graph of exclusivity.

Definition. A *probability assignment for the graph of exclusivity* G is a vector of probabilities

$$p : V(G) \longrightarrow [0,1]^{|V(G)|}$$

such that $p_i \in [0,1]$ for every $i \in V(G)$ and $p_i + p_j \le 1$ whenever $(i,j) \in E(G)$. $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G.

The E principle

Theorem. The events produced by ideal measurements satisfy the *exclusivity principle*, i.e., if every two events in a set are exclusive (i.e., if they are pairwise exclusive), then all the events in the set are mutually exclusive.

E.g., if every two events in the set $\{i, j, k\}$ are pairwise exclusive, then the valid probability assignments $\{p_i, p_j, p_k\}$ must satisfy $p_i + p_j + p_k \leq 1$ (and not only $p_i + p_j \leq 1$, $p_i + p_k \leq 1$, and $p_j + p_k \leq 1$).

G. Chiribella and X. Yuan, arXiv:1404.3348.

Investigating GPTs by their sets of prob for all E graphs

Problem. Given a theory, what is the set $S(G)$ of valid probability assignments for any graph of exclusivity G.

Classical probability theory

Problem. Given a theory, what is the set $S(G)$ of valid probability assignments for any graph of exclusivity G.

Result.
For classical probability theory,

$$S(G) = STAB(G) = \text{convex hull}\{p \in \{0,1\}^{|V(G)|} : p_i p_j = 0 \text{ if } (i,j) \in E(G)\},$$

i.e., the *stable set polytope* of G.

QT

Problem. Given a theory, what is the set $S(G)$ of valid probability assignments for any graph of exclusivity G.

Result.
For QT,

$$S(G) = TH(G) = \{p \in [0,1]^{|V(G)|} : p_i = |\langle \mathcal{M}_{x_i}^{(i)} \rho | \rho \rangle|^2,$$
$$|\langle \rho | \rho \rangle| = 1,\ |\langle \mathcal{M}_{x_i}^{(i)} \rho | \mathcal{M}_{x_i}^{(i)} \rho \rangle| = 1,$$
$$\langle \mathcal{M}_{x_i}^{(i)} \rho | \mathcal{M}_{x_j}^{(j)} \rho \rangle = 0,\ \forall (i,j) \in E(G)\},$$

i.e., the *theta body* of G.

More general probabilistic theories

Problem. Given a theory, what is the set $S(G)$ of valid probability assignments for any graph of exclusivity G.

Result.

For more general probabilistic theories,

$$S(G) = QSTAB(G) = \{p \in [0,1]^{|V(G)|} : \sum_{i \in c} p_i \leq 1,\ \forall c \in C(G)\},$$

i.e., the *clique-constrained stable set polytope* of G. $C(G)$ is the set of cliques of G. A *clique* is a subset of $V(G)$ such that every two vertices are adjacent.

A. Cabello, S. Severini, and A. Winter, Phys. Rev. Lett. **112**, 040401 (2014).

Born's rule

Problem. Given a theory, what is the set $S(G)$ of valid probability assignments for any graph of exclusivity G.

Result.
For QT,

$$S(G) = TH(G) = \{p \in [0,1]^{|V(G)|} : p_i = |\langle \mathcal{M}^{(i)}_{x_i}\rho|\rho\rangle|^2, \\ |\langle\rho|\rho\rangle| = 1,\ |\langle \mathcal{M}^{(i)}_{x_i}\rho|\mathcal{M}^{(i)}_{x_i}\rho\rangle| = 1, \\ \langle \mathcal{M}^{(i)}_{x_i}\rho|\mathcal{M}^{(j)}_{x_j}\rho\rangle = 0,\ \forall (i,j) \in E(G)\},$$

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Born's rule

Problem. Given a theory, what is the set $S(G)$ of valid probability assignments for any graph of exclusivity G.

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Born's rule

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$$S(G) = TH(G) = \{p \in [0,1]^{|V(G)|} : p_i = |\langle \mathcal{M}^{(i)}_{x_i} \rho | \rho \rangle|^2,$$
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$$\langle \mathcal{M}^{(i)}_{x_i} \rho | \mathcal{M}^{(j)}_{x_j} \rho \rangle = 0,\ \forall (i,j) \in E(G)\},$$

i.e., the *theta body* of G.

Problems

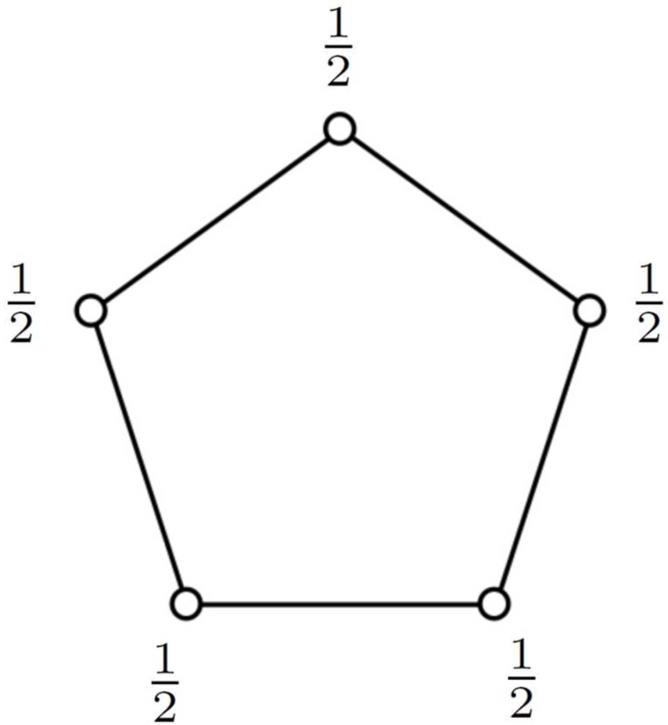
Problem 1. What is the physical principle that singles out $TH(G)$?

Problems

Problem 1. What is the physical principle that singles out $TH(G)$?

Problem 2. What is the *largest* set of valid probability assignments $S(G)$ that is not logically inconsistent (and, in particular, satisfies the exclusivity principle in every situation)?

Is it a valid probability assignment?



R. Wright, in *Mathematical Foundations of Quantum Mechanics*, edited by A. R. Marlow (Academic, San Diego, 1978), p. 255.

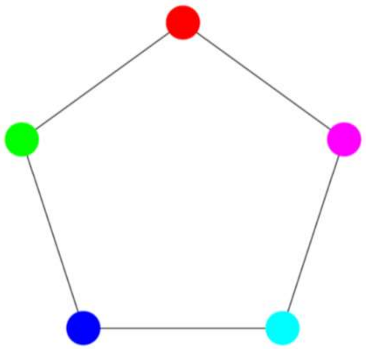
S. Popescu and D. Rohrlich, Found. Phys. **24**, 379 (1994).

The old results

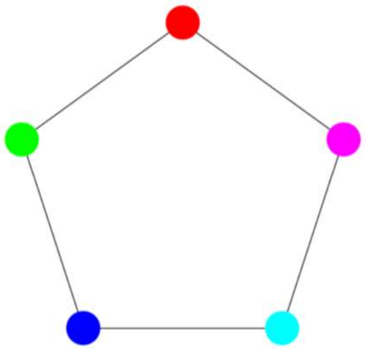
A. Cabello, Phys. Rev. Lett. **110**, 060402 (2013).

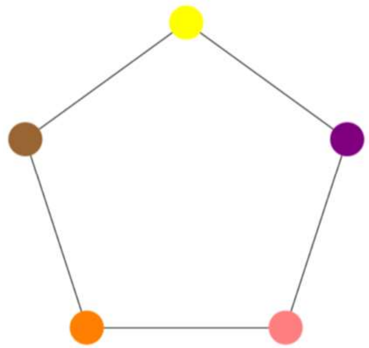
B. Amaral, M. Terra Cunha, and A. Cabello, Phys. Rev. A **89**, 030101(R) (2014).

One experiment (its E graph)

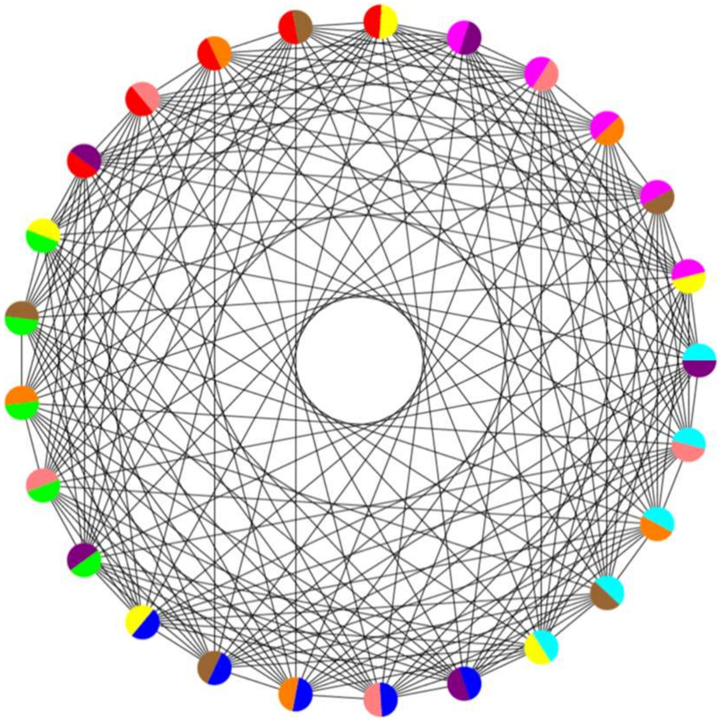


Another similar experiment

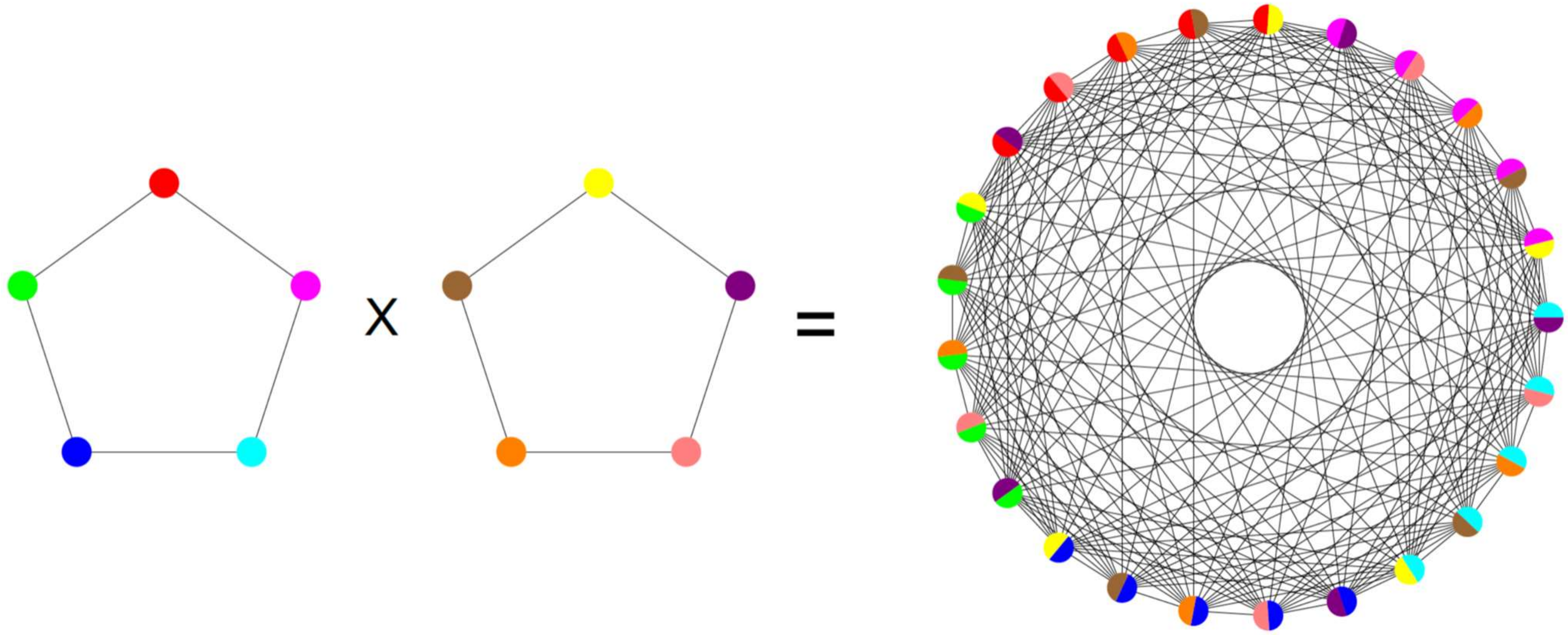




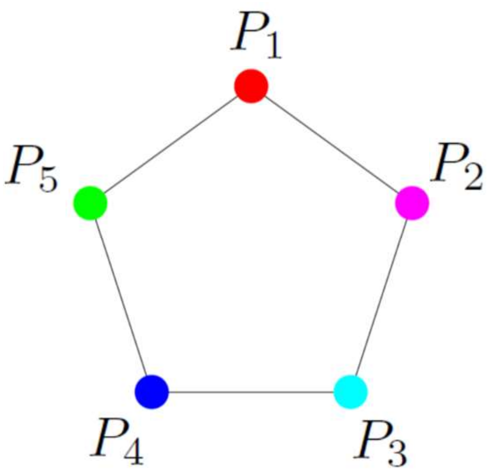
The two experiments can be seen as a single one



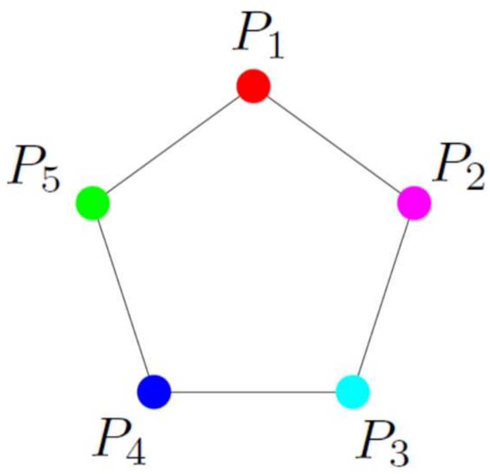
OR product

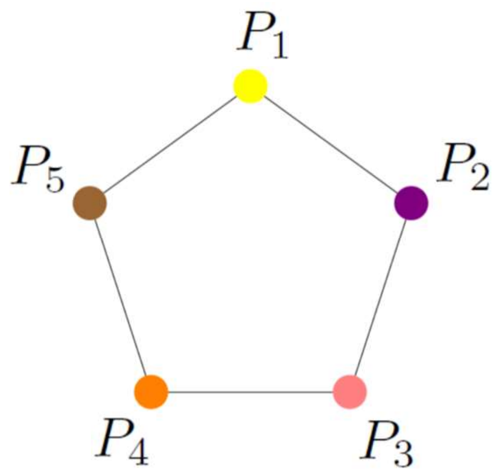


What are the valid assignments?

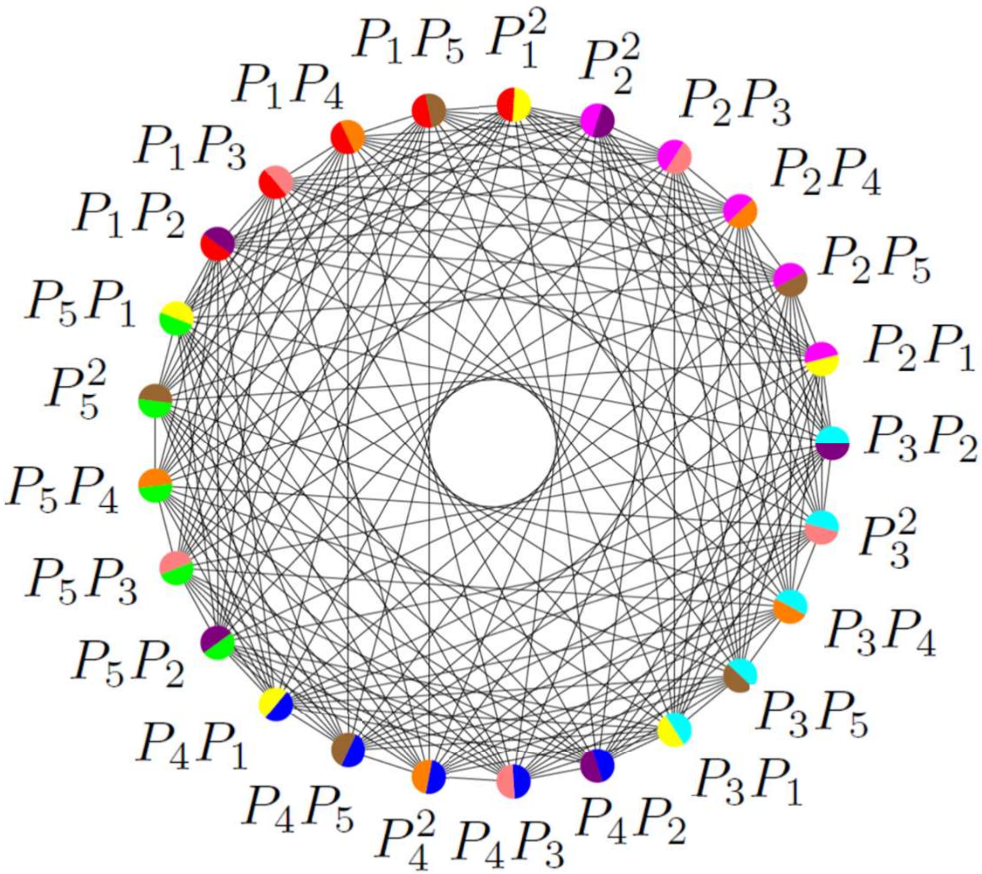


The second experiment is similar

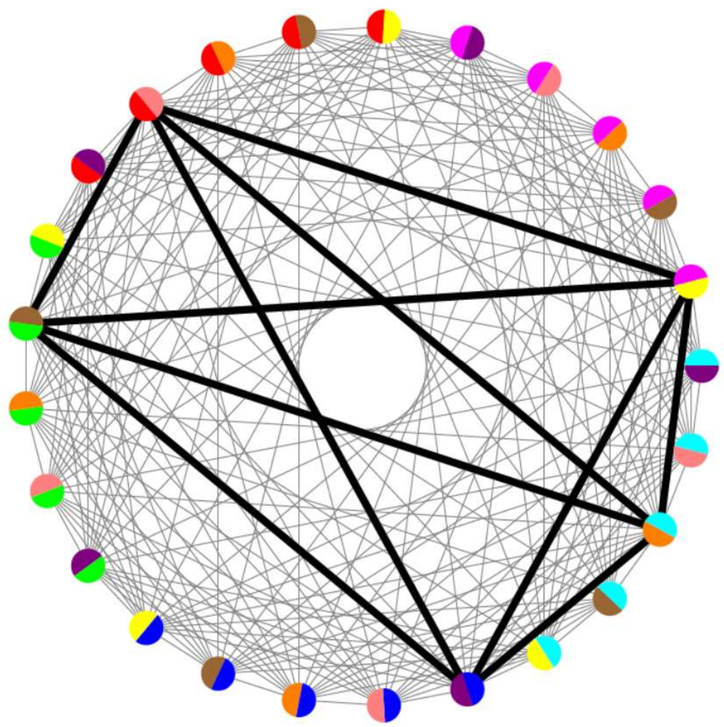




The two experiments are <u>independent</u>

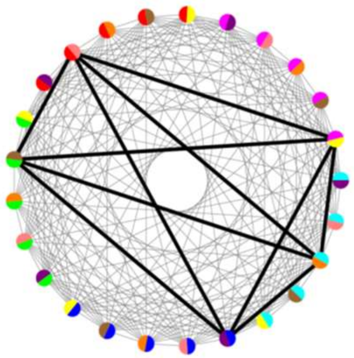


The E principle implies



$$P_1 P_3 + P_2 P_1 + P_3 P_4 + P_4 P_2 + P_5^2 \leq 1$$

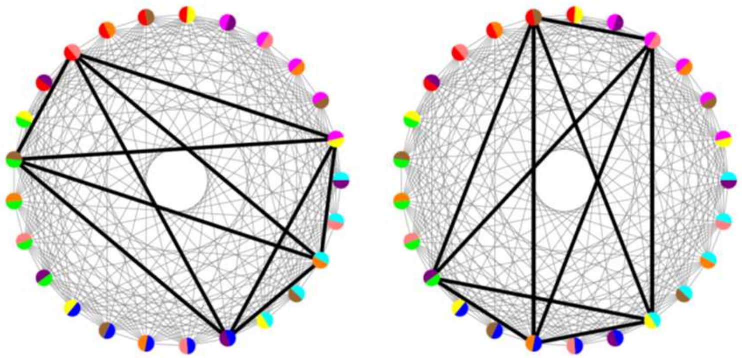
The E principle implies



$$P_1P_3 + P_2P_1 + P_3P_4 + P_4P_2 + P_5^2 \leq 1$$

The E principle implies

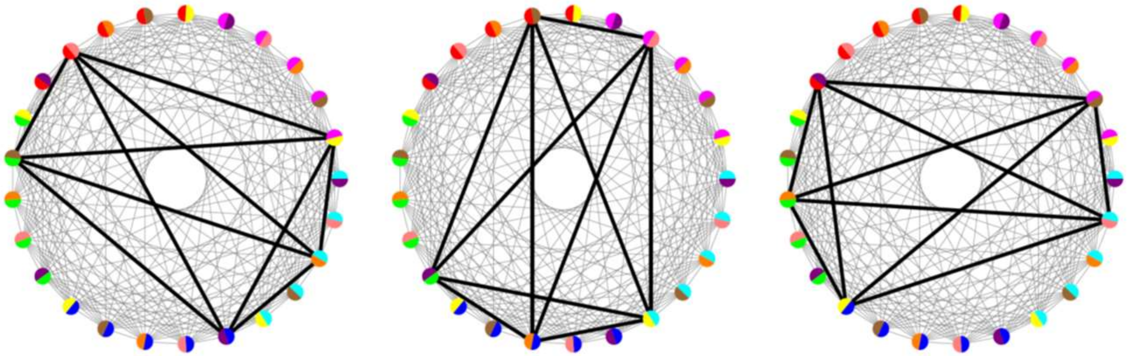
$$P_1P_5 + P_2P_3 + P_3P_1 + P_4^2 + P_5P_2 \leq 1$$



$$P_1P_3 + P_2P_1 + P_3P_4 + P_4P_2 + P_5^2 \leq 1$$

The E principle implies

$$P_1P_5 + P_2P_3 + P_3P_1 + P_4^2 + P_5P_2 \leq 1$$



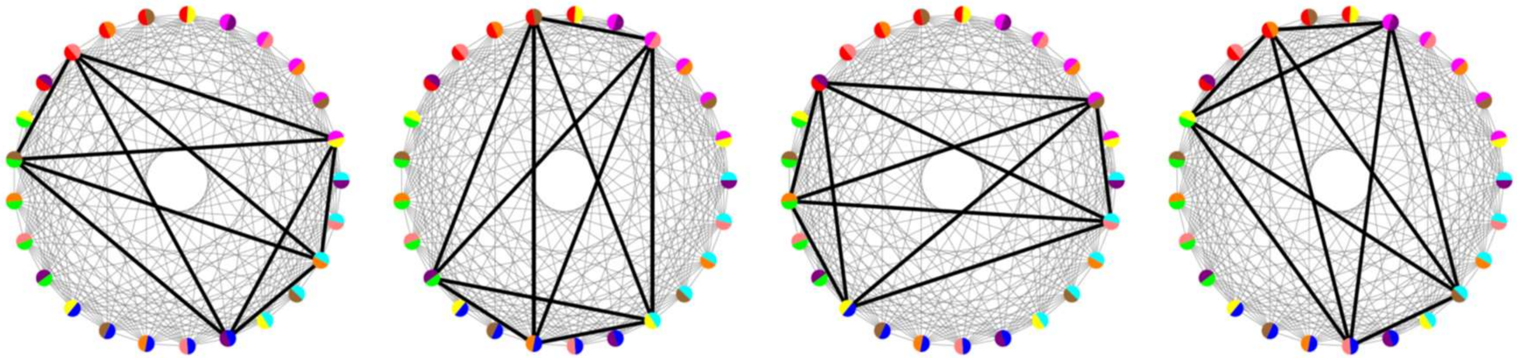
$$P_1P_3 + P_2P_1 + P_3P_4 + P_4P_2 + P_5^2 \leq 1$$

$$P_1P_2 + P_2P_6 + P_3^2 + P_4P_1 + P_5P_4 \leq 1$$

The E principle implies

$$P_1P_5 + P_2P_3 + P_3P_1 + P_4^2 + P_5P_2 \leq 1$$

$$P_1P_4 + P_2^2 + P_3P_5 + P_4P_3 + P_5P_1 \leq 1$$



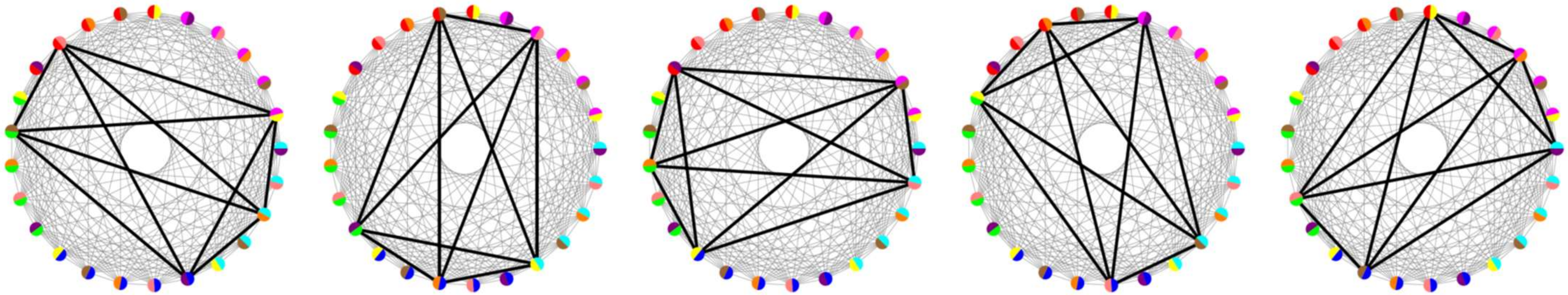
$$P_1P_3 + P_2P_1 + P_3P_4 + P_4P_2 + P_5^2 \leq 1$$

$$P_1P_2 + P_2P_6 + P_3^2 + P_4P_1 + P_5P_4 \leq 1$$

The E principle implies

$$P_1P_5 + P_2P_3 + P_3P_1 + P_4^2 + P_5P_2 \leq 1$$

$$P_1P_4 + P_2^2 + P_3P_5 + P_4P_3 + P_5P_1 \leq 1$$



$$P_1P_3 + P_2P_1 + P_3P_4 + P_4P_2 + P_5^2 \leq 1$$

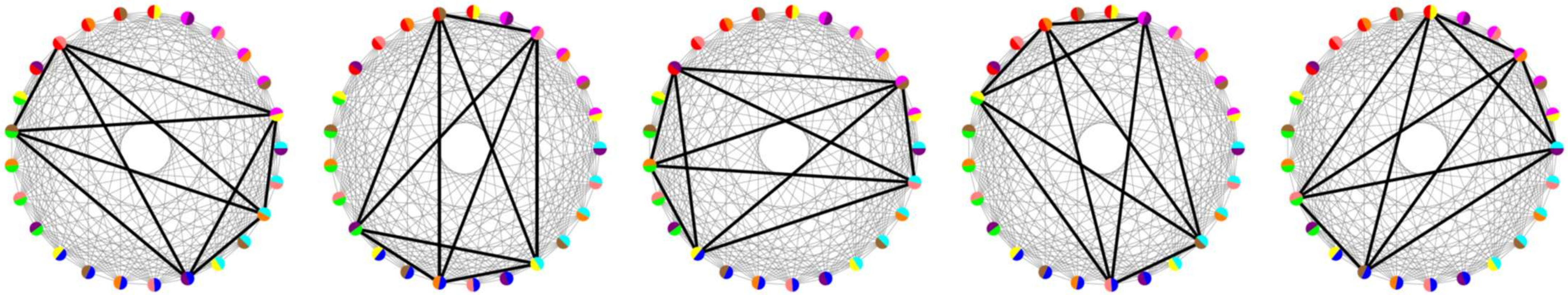
$$P_1P_2 + P_2P_6 + P_3^2 + P_4P_1 + P_5P_4 \leq 1$$

$$P_1^2 + P_2P_4 + P_3P_2 + P_4P_5 + P_5P_3 \leq 1$$

The E principle implies

$P_1P_5 + P_2P_3 + P_3P_1 + P_4^2 + P_5P_2 \leq 1$

$P_1P_4 + P_2^2 + P_3P_5 + P_4P_3 + P_5P_1 \leq 1$



$P_1P_3 + P_2P_1 + P_3P_4 + P_4P_2 + P_5^2 \leq 1$

$P_1P_2 + P_2P_6 + P_3^2 + P_4P_1 + P_5P_4 \leq 1$

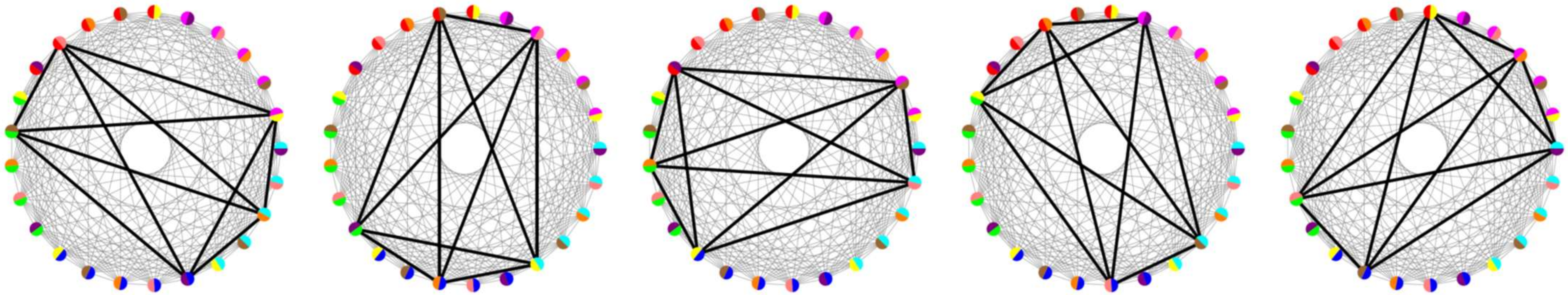
$P_1^2 + P_2P_4 + P_3P_2 + P_4P_5 + P_5P_3 \leq 1$

$$(P_1 + P_2 + P_3 + P_4 + P_5)^2 \leq 5$$

The E principle implies

$P_1P_5 + P_2P_3 + P_3P_1 + P_4^2 + P_5P_2 \leq 1$

$P_1P_4 + P_2^2 + P_3P_5 + P_4P_3 + P_5P_1 \leq 1$



$P_1P_3 + P_2P_1 + P_3P_4 + P_4P_2 + P_5^2 \leq 1$

$P_1P_2 + P_2P_6 + P_3^2 + P_4P_1 + P_5P_4 \leq 1$

$P_1^2 + P_2P_4 + P_3P_2 + P_4P_5 + P_5P_3 \leq 1$

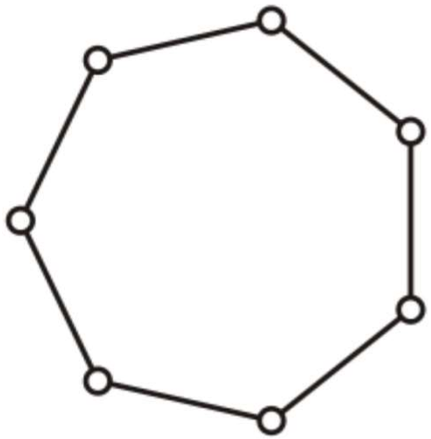
$$(P_1 + P_2 + P_3 + P_4 + P_5)^2 \leq 5$$

$$P_1 + P_2 + P_3 + P_4 + P_5 \leq \sqrt{5}$$

A. Cabello, Phys. Rev. Lett. **110**, 060402 (2013).

Complement

Definition. The *complement* of G, denoted $\overline{G}$, is the graph with the same vertices as G and such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.



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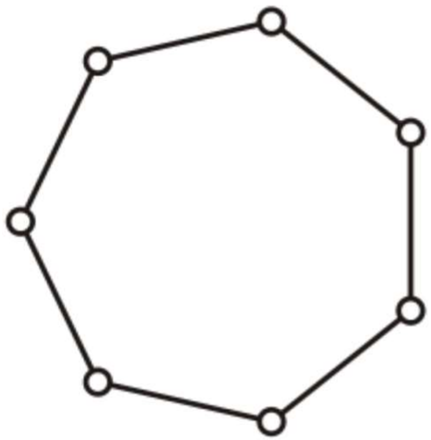
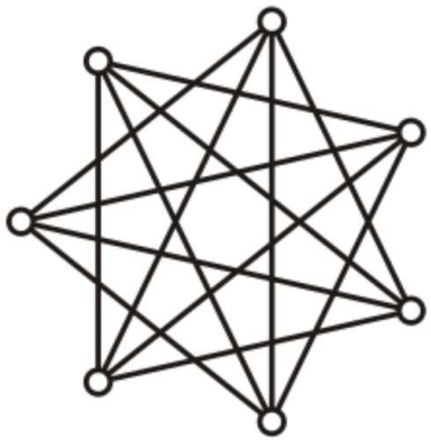
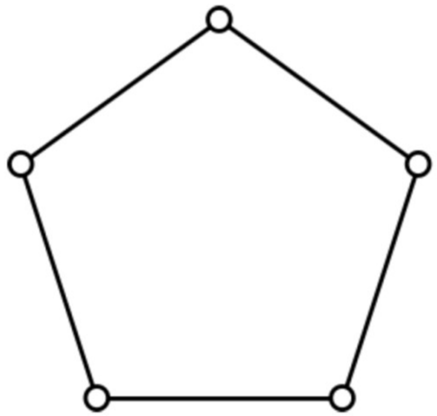
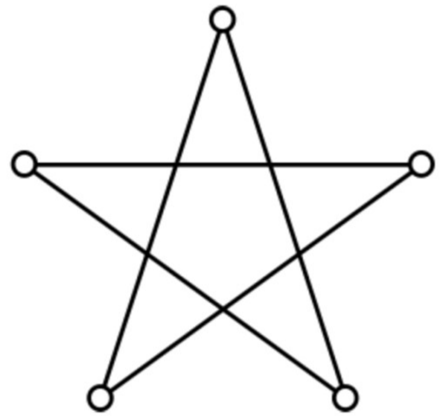




The EP is prowerful for self-complementary graphs





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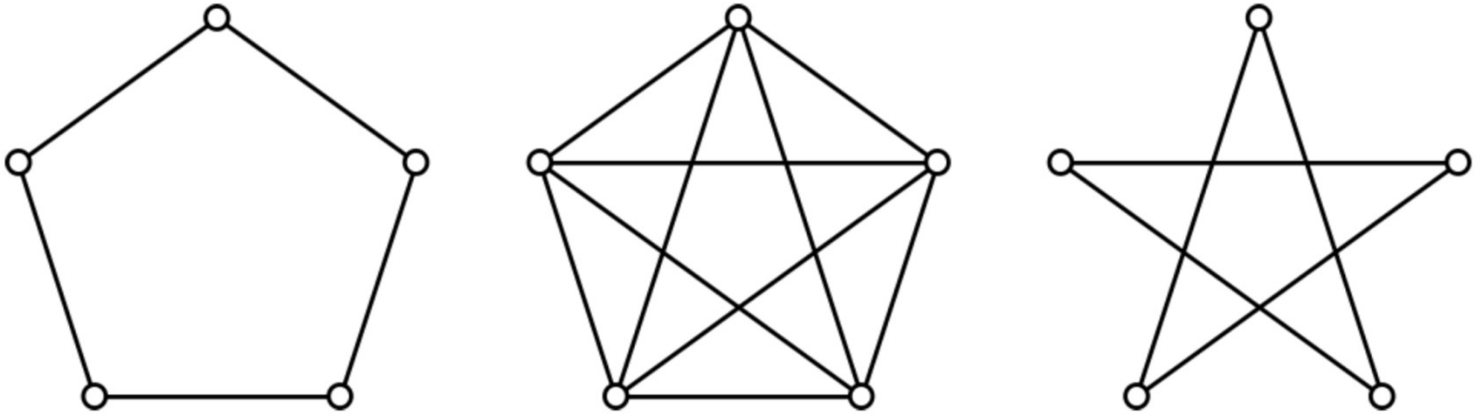

For self-complementary graphs

Result. If G is a self-complementary graph, the exclusivity principle excludes any set of probability assignments strictly larger than $TH(G)$.

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Proof. Let P be an assignment which is not in $TH(G)$. Then, there is an assignment $\overline{P}$ in $TH(\overline{G})$ such that $\sum_{i\in V(G)} P_i\overline{P}_i \geq 1$. Since G and $\overline{G}$ are isomorphic, after permuting the entries given by the isomorphism, $\overline{P}$ becomes an element of $TH(G)$ and the previous inequality implies that P is not allowed by the exclusivity principle. ■

B. Amaral, M. Terra Cunha, and A. Cabello, Phys. Rev. A **89**, 030101(R) (2014).

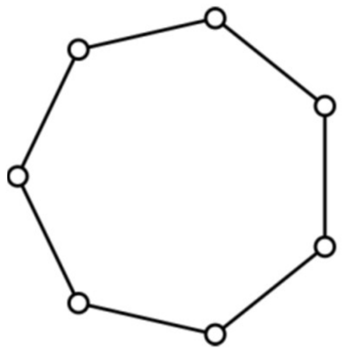
The new results

A. Cabello, arXiv:1801.06347.

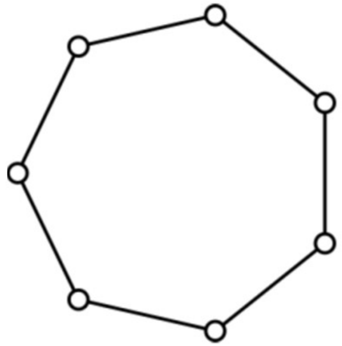
Result 1

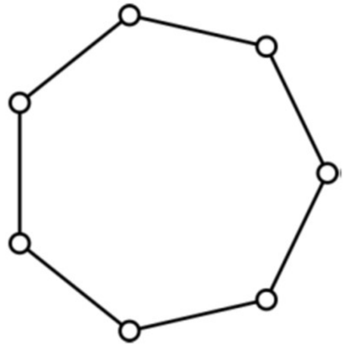
Result 1. For every graph of exclusivity G, the exclusivity principle, together with the assumptions that similar and independent experiments exist, single out $TH(G)$.

The proof is based on the following construction

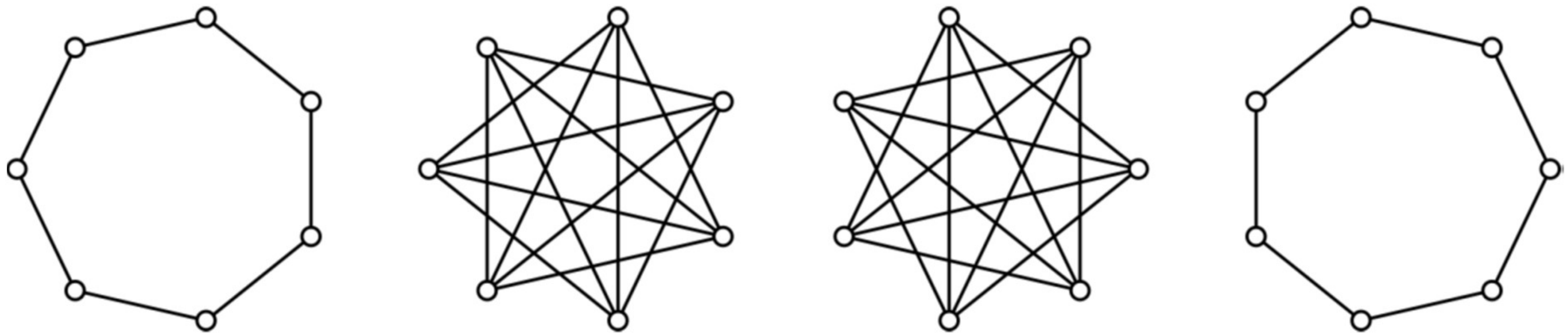


Step 1

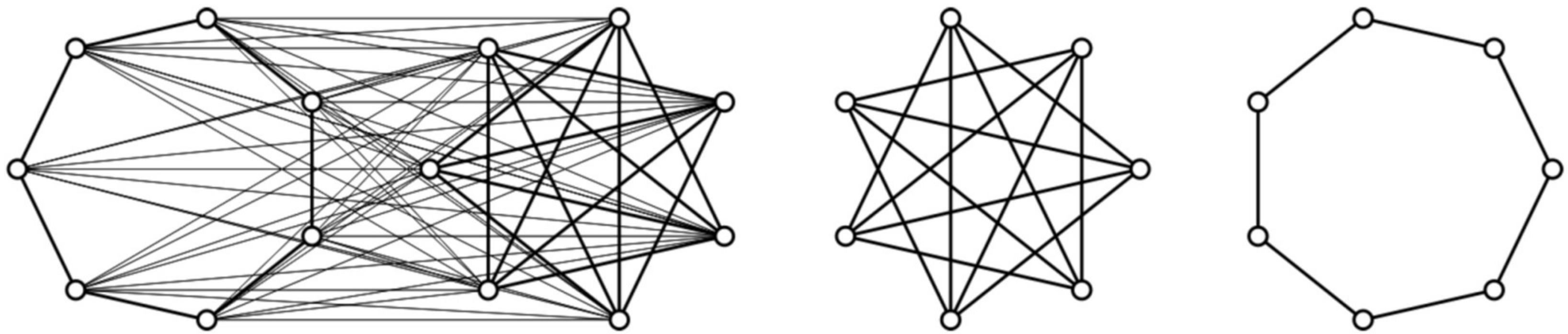




Step 2



Step 3



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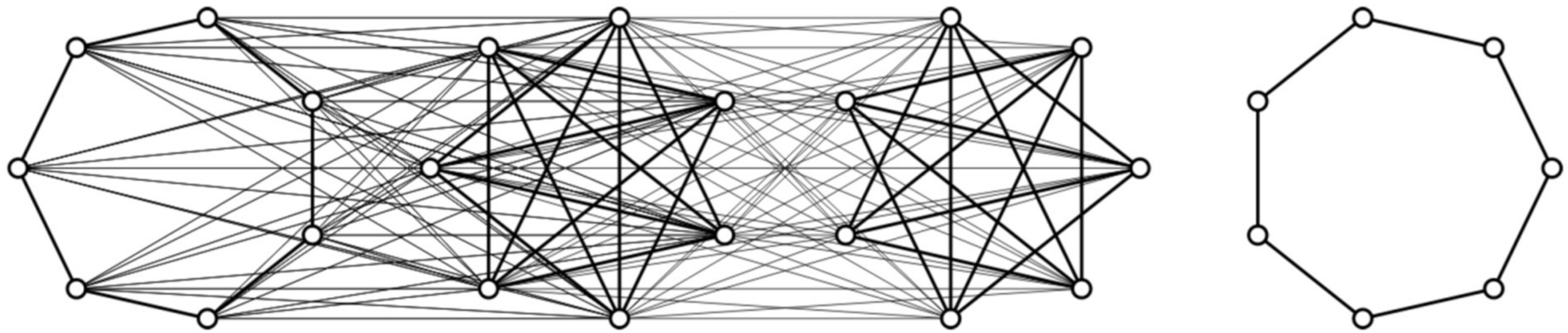

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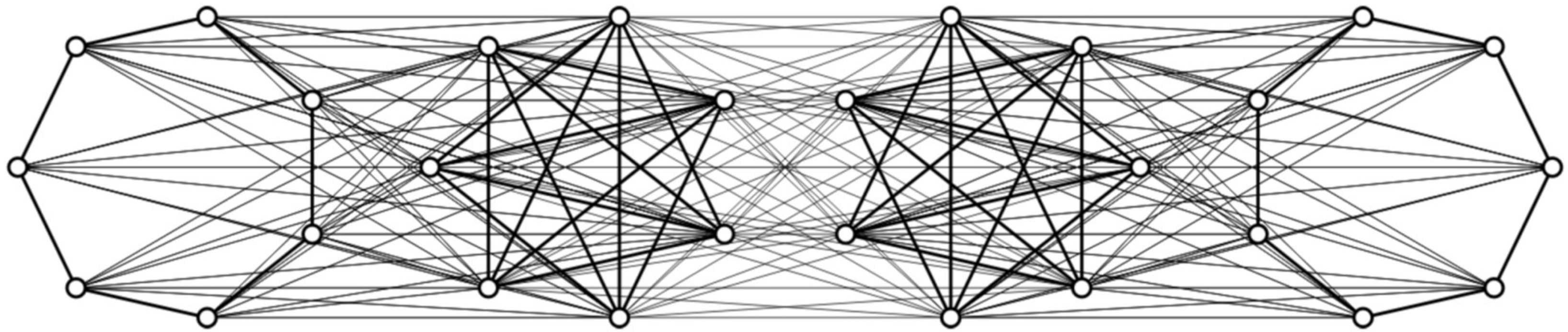
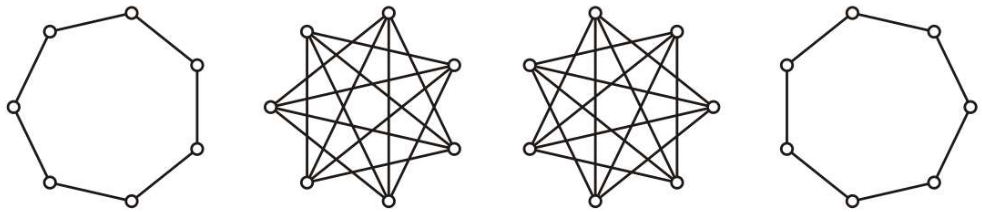


Generalized composition

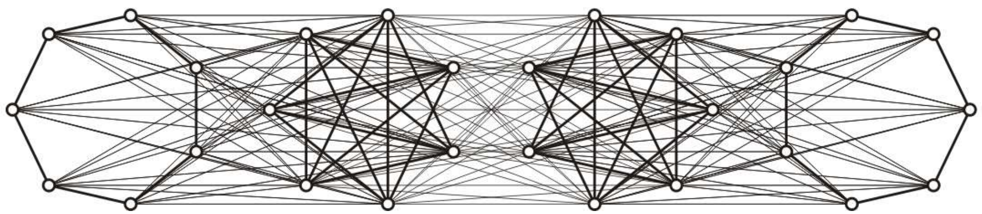
Definition (*generalized composition*). If $\mathcal{G}$ is a graph with n vertices, then the graph $\mathcal{G}[G_1, \ldots, G_n]$ is constructed by taking the disjoint graphs $G_1, \ldots, G_n$ and joining every vertex of G_i with every vertex of G_j whenever v_i and v_j are adjacent vertices in $\mathcal{G}$.

$\mathcal{G}$

$C_7, \overline{C_7}, \overline{C_7}, C_7$



$H(C_7) = \mathcal{G}[C_7, \overline{C_7}, \overline{C_7}, C_7]$



Proof

Step 1. For any G, $H(G)$ is isomorphic to $\overline{H(G)}$.

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$$
\begin{tikzcd}
\overline{G} \arrow[r, dash] \arrow[d, dash] & G \\
\overline{G} \arrow[r, dash] & G
\end{tikzcd}
$$

Proof

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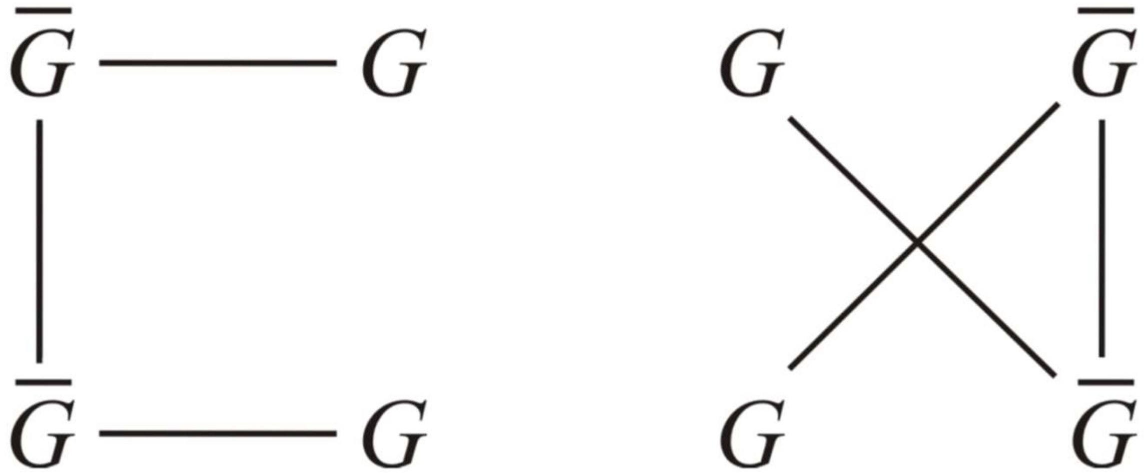

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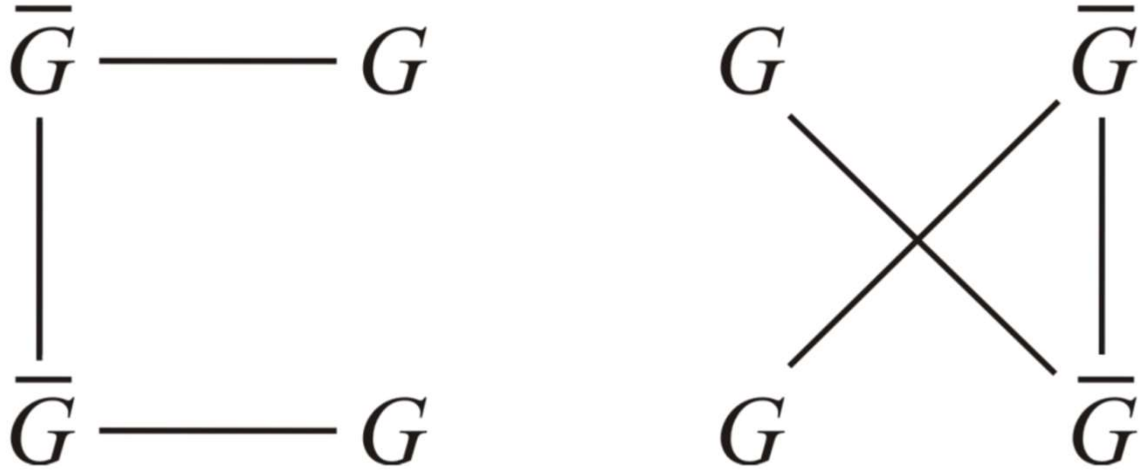

Therefore, $S[H(G)] \subseteq TH[H(G)]$.

B. Amaral, M. Terra Cunha, and A. Cabello, Phys. Rev. A **89**, 030101(R) (2014).

Proof

Step 2. For any G, the only induced subgraphs of $H(G)$ that can have odd cycles C_n, with $n \geq 5$, or their complements $\overline{C_n}$, are the ones in each of $G, \overline{G}, \overline{G}$, and G.

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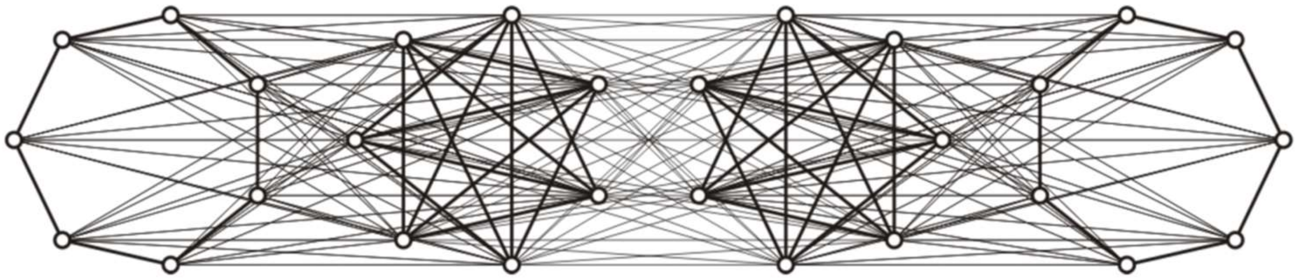

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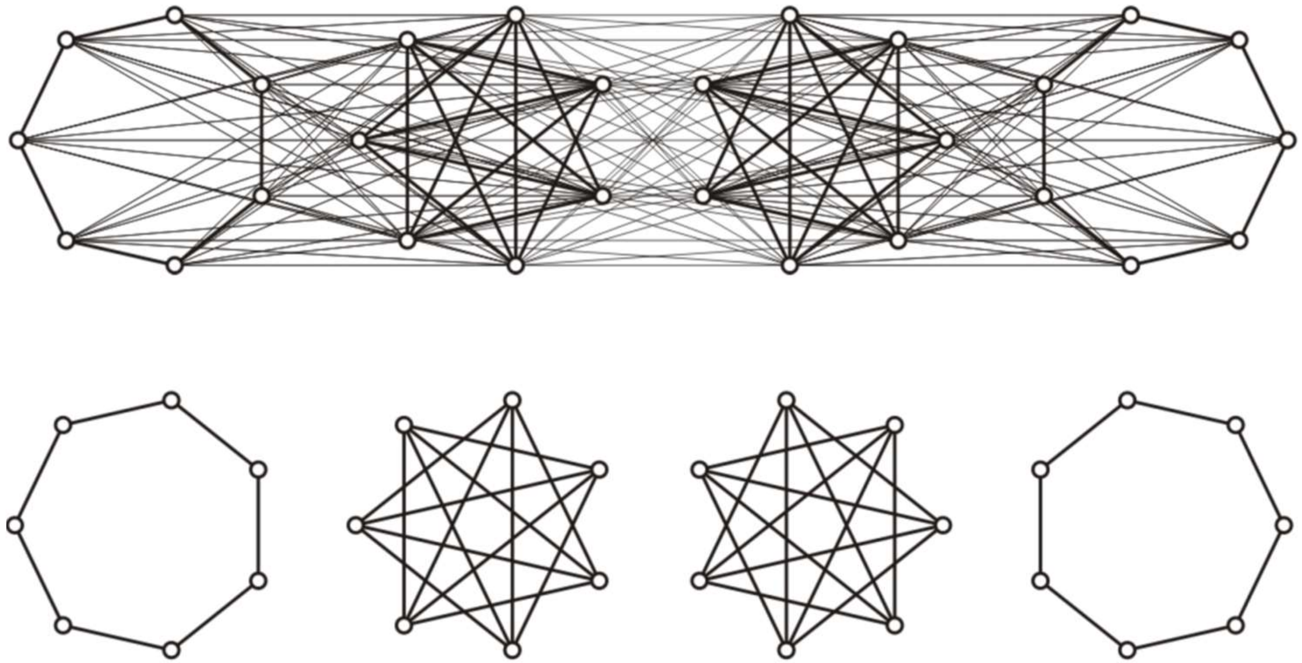

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For any graph not having as induced graphs odd cycles C_n, with $n \geq 5$, or their complements $\overline{C_n}$, the set of valid probability assignments for any theory satisfying the exclusivity principle is identical to the one of classical probability theory.

A. Cabello, S. Severini, and A. Winter, Phys. Rev. Lett. **112**, 040401 (2014).

Proof

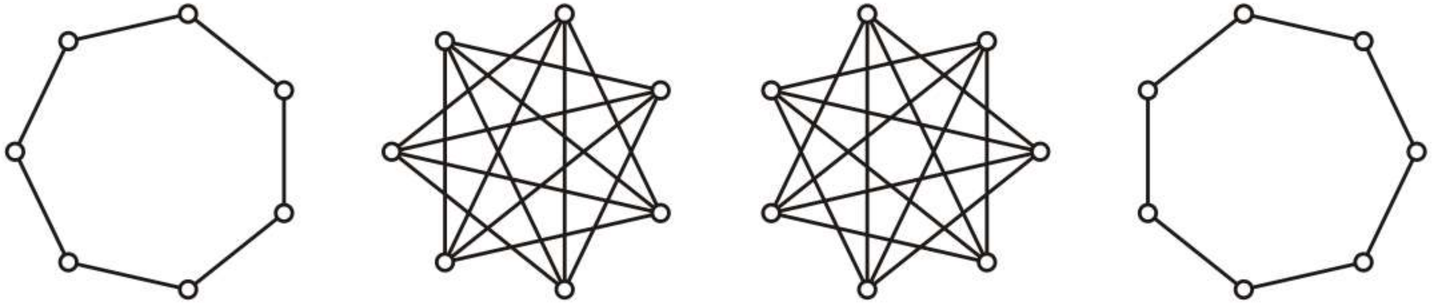
Step 2. For any G, the only induced subgraphs of $H(G)$ that can have odd cycles C_n, with $n \geq 5$, or their complements $\overline{C_n}$, are the ones in each of $G, \overline{G}, \overline{G}$, and G.

Therefore, the only nonclassical probability assignments in $H(G)$ are the ones that $S(G), S(\overline{G}), S(\overline{G})$, and $S(G)$ may have. This implies

$$S[H(G)] = \text{convex hull}\{p = (p_1, p_2, p_3, p_4)$$
$$\in \{(S(G), 0^{|V(G)|}, 0^{|V(G)|}, S(G)),$$
$$(S(G), 0^{|V(G)|}, S(\overline{G}), 0^{|V(G)|}),$$
$$(0^{|V(G)|}, S(\overline{G}), 0^{|V(G)|}, S(G))\}\}.$$

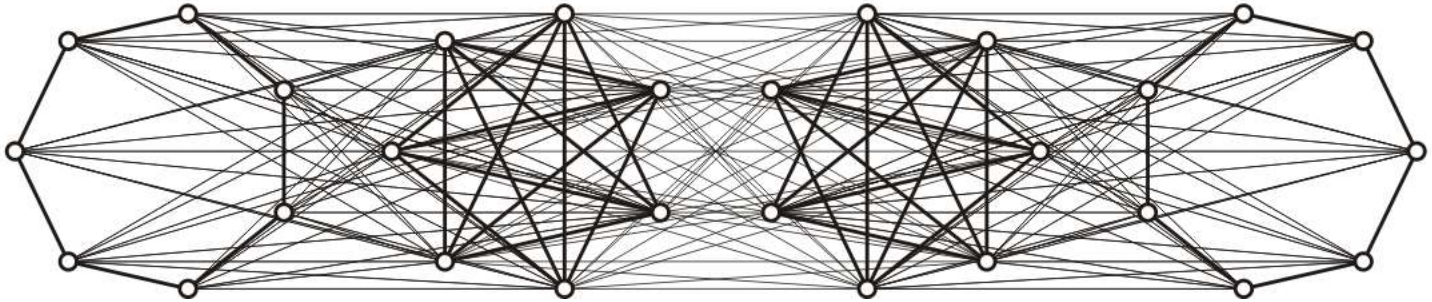
Therefore $S[H(G)]$ determines $S(G)$. Since we have already proven that $S[H(G)] \subseteq TH[H(G)]$, then $S(G) \subseteq TH(G)$. Since, for QT, $S(G) = TH(G)$, this finishes the proof. ■

Physical interpretation. Step 1



$\{e_k^{(1)}\}_{k=1}^n$ $\{e_k^{(2)}\}_{k=1}^n$ $\{e_k^{(3)}\}_{k=1}^n$ $\{e_k^{(4)}\}_{k=1}^n$

Physical interpretation. Step 2



$$\{(a_0, e_k^{(1)}), (a_1, b_0, e_k^{(2)}), (b_1, c_0, e_k^{(3)}), (c_1, e_k^{(4)})\}_{k=1}^{n}$$

Problem 1

Problem 1. What is the physical principle that singles out $TH(G)$?

Result 1

Result 1. For every graph of exclusivity G, the exclusivity principle, together with the assumptions that similar and independent experiments exist, single out $TH(G)$.

Problem 2

Problem 2. What is the *largest* set of valid probability assignments $S(G)$ that is not logically inconsistent (and, in particular, satisfies the exclusivity principle in every situation)?

Result 2

Result 2. For every graph of exclusivity G, no physical theory that assigns probabilities to the outcomes of ideal measurements can produce probability assignments outside $TH(G)$.

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Proof. For the outcomes of ideal measurements, Result 1 implies that the probability assignments outside $TH(G)$ are logically inconsistent. ■

A. Cabello, arXiv:1801.06347.

For years, we thought that we would understand QT when we identify the physical principle that singles out QT from the set of GPTs. Behind this way of thinking is, however, the assumption that there are logically consistent GPTs that can produce probabilities *beyond* those allowed by QT. The particular way to look at QT we have described here shows that this is not the case.

Conclusions

- There is a crucial message in the *specific* way nature is contextual: The sets of probability assignments to the outcomes of ideal measurements for every graph of exclusivity are indistinguishable from the ones of a universe with no laws.

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- The *ontological* message of QT is that nature has no laws governing the outcomes of certain "slicings of the world."

- Classical probability theory emerges when, in addition to (i)–(iii) in the definition of ideal measurements, we ask ideal measurements to produce outcomes admitting a joint probability distribution.