Contextuality Analysis

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Acknowledging collaboration with

Janne V. Kujala,

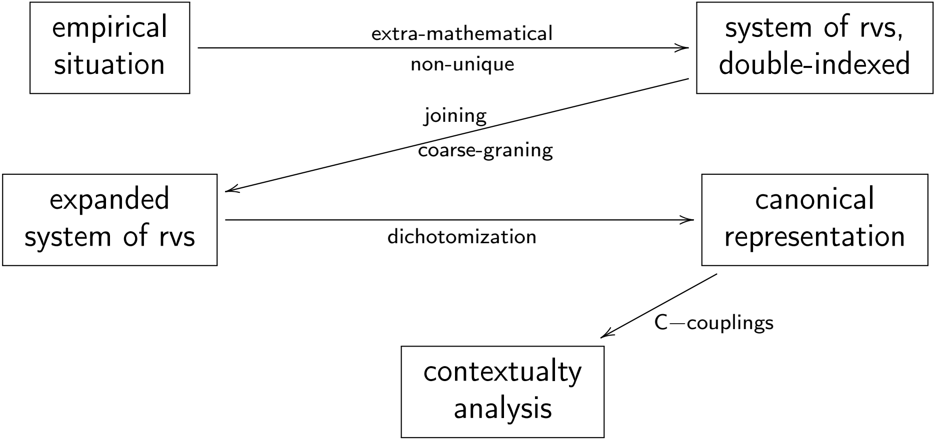
Victor Cervantes, Maria Kon

Benefited from recent discussions with

Matt Jones, Pawel Kurzynski, Andrei Khrennikov,

Federico Holic, Acacio de Barros

Preamble


empirical situation
extra-mathematical
non-unique
system of rvs, double-indexed
joining
coarse-graning
expanded system of rvs
dichotomization
canonical representation
C—couplings
contextualty analysis

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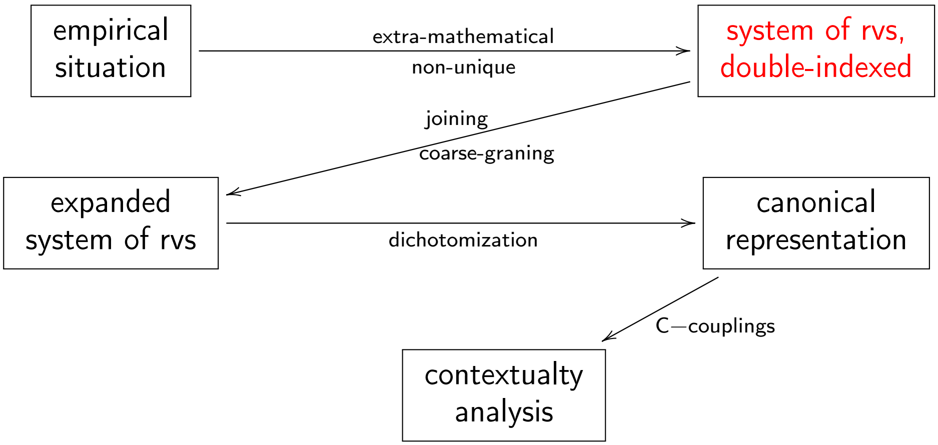
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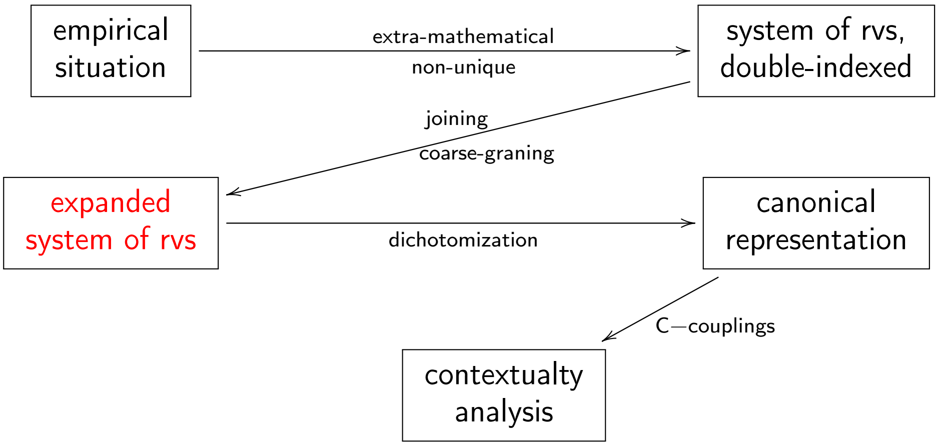
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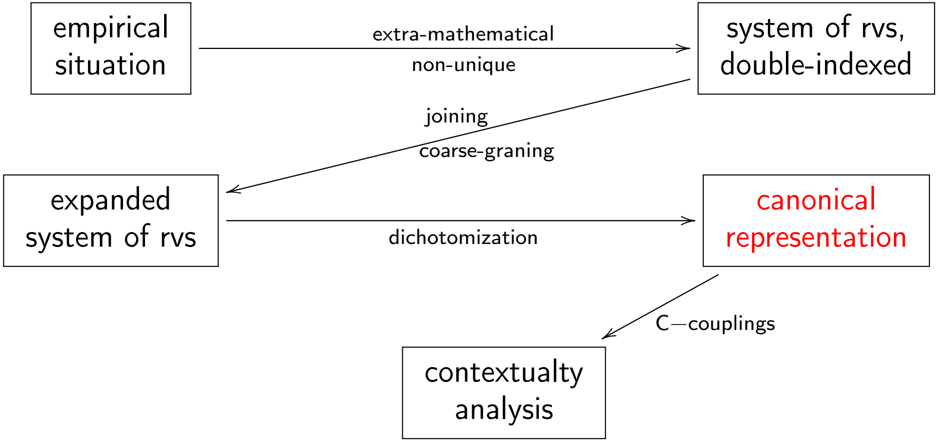
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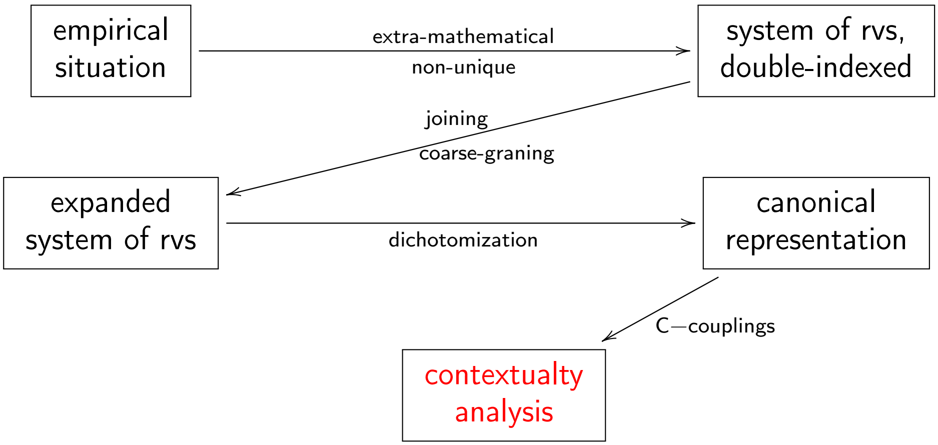

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Plan

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1. Preliminaries: Couplings, C-couplings.

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3. Double-Slit Paradigm.
4. More-than-binary random variables: Dichotomizations.

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1. Preliminaries: Couplings, C-couplings.
2. Basics: Systems of random variables; Definition of C-(non)contextuality; Measure of C-contextuality.
3. Double-Slit Paradigm.
4. More-than-binary random variables: Dichotomizations.
5. Directions of development, summary.

Preliminaries: Couplings

$$X_1, X_2, \ldots, X_n$$

$$(Y_1, Y_2, \ldots, Y_n)$$

Preliminaries: Couplings

	$+$	$-$
A	p	$1-p$
B	q	$1-q$

Preliminaries: Couplings

	$+$	$-$
A	p	$1-p$
B	q	$1-q$

$\Longrightarrow$

	$B' = +$	$B' = -$	
$A' = +$	?	?	p
$A' = -$	?	?	$1-p$
	q	$1-q$	

Preliminaries: Couplings

IND

	$B' = +$	$B' = -$	
$A' = +$	pq		p
$A' = -$			$1 - p$
	q	$1 - q$	

Preliminaries: Couplings

IND

	$B' = +$	$B' = -$	
$A' = +$	pq	$p(1-q)$	p
$A' = -$	$(1-p)q$	$(1-p)(1-q)$	$1-p$
	q	$1-q$	

Preliminaries: Couplings

$$MAX$$

	$B' = +$	$B' = -$	
$A' = +$	$\min(p, q)$		p
$A' = -$			$1 - p$
	q	$1 - q$	

Preliminaries: Couplings

$$MAX$$

	$B' = +$	$B' = -$	
$A' = +$	$\min(p, q)$	$p - \min(p, q)$	p
$A' = -$	$q - \min(p, q)$	$\min(1 - p, 1 - q)$	$1 - p$
	q	$1 - q$	

Preliminaries: Couplings

$$X_1, X_2, \ldots, X_n$$

$$(Y_1, Y_2, \ldots, Y_n)$$

Definition

A statement on (property of) a joint distribution of random variables is called a C-statement (C-property) if, for any n and any (allowable) $X_1, X_2, \ldots, X_n$, a coupling $(Y_1, Y_2, \ldots, Y_n)$ that satisfies C exists and is unique. This coupling is called the C-coupling of $X_1, X_2, \ldots, X_n$.

IND is a C-property

(for any class of random variables)

$MultiMax$ is a C-property

for dichotomic $X_1, X_2, \ldots, X_n$

$MultiMax$ is a C-property

for dichotomic $X_1, X_2, \dots, X_n$

$$(Y_1, Y_2, \dots, Y_n)$$

$$C \equiv \forall i, j : (Y_i, Y_j) \text{ is MAX coupling of } X_i, X_j$$

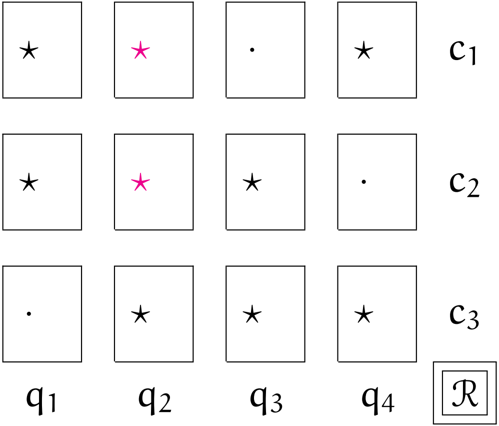
Contextuality-by-Default (CbD): Generalities

$$
\begin{array}{ccccc}
\boxed{R_1^1} & \boxed{R_2^1} & \boxed{\cdot} & \boxed{R_4^1} & c_1 \\
\boxed{R_1^2} & \boxed{R_2^2} & \boxed{R_3^2} & \boxed{\cdot} & c_2 \\
\boxed{\cdot} & \boxed{R_2^3} & \boxed{R_3^3} & \boxed{R_4^3} & c_3 \\
q_1 & q_2 & q_3 & q_4 & \boxed{\boxed{\mathcal{R}}}
\end{array}
$$

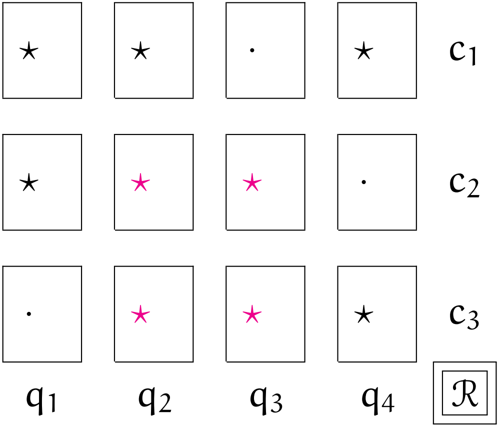
CbD: Generalities

$$
\begin{array}{ccccc}
\star & \star & \cdot & \star & c_1 \\
\star & \star & \star & \cdot & c_2 \\
\cdot & \star & \star & \star & c_3 \\
q_1 & q_2 & q_3 & q_4 & \boxed{\boxed{\mathcal{R}}}
\end{array}
$$

CbD: Generalities



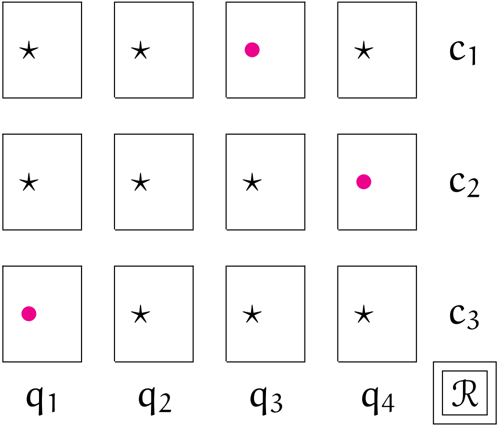
CbD: Generalities



CbD: Generalities

$\star$	$\star$	$\cdot$	$\star$	c_1
$\star$	$\star$	$\star$	$\cdot$	c_2
$\cdot$	$\star$	$\star$	$\star$	c_3
q_1	q_2	q_3	q_4	$\boxed{\boxed{\mathcal{R}}}$

CbD: Generalities



CbD: Generalities

$\star$	$\star$	$\cdot$	$\star$	c_1
$\star$	$\star$	$\star$	$\cdot$	c_2
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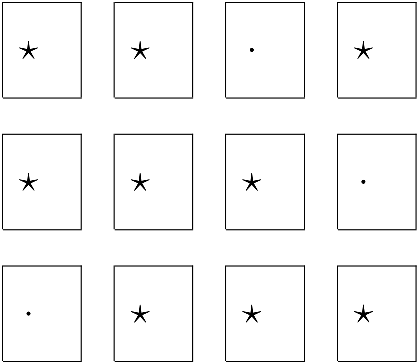
CbD: Generalities



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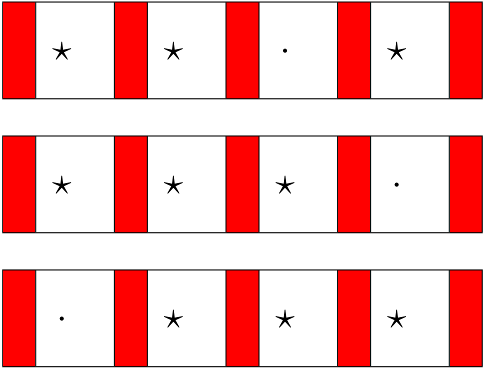
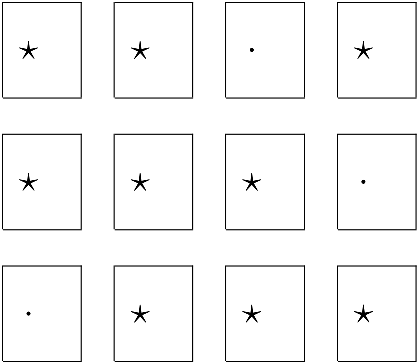


CbD: Generalities

a	b	$\cdot$	d	α_1
a'	b'	c'	$\cdot$	α_2
$\cdot$	b''	c''	d''	α_3

CbD: Generalities



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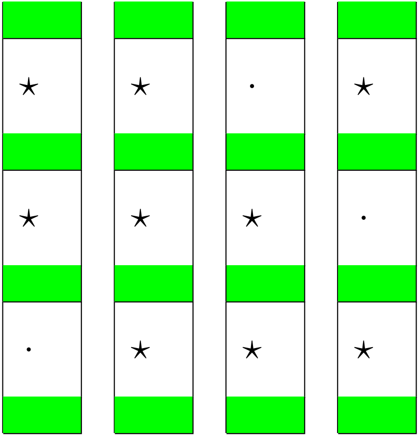

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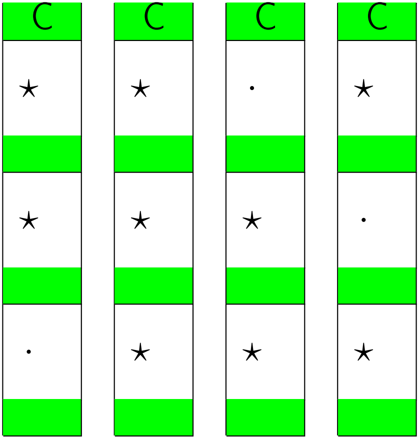
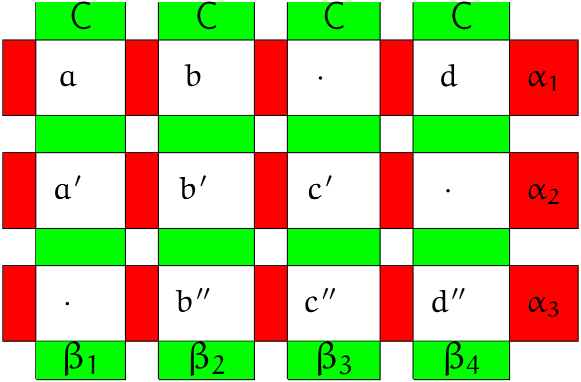


CbD: Generalities

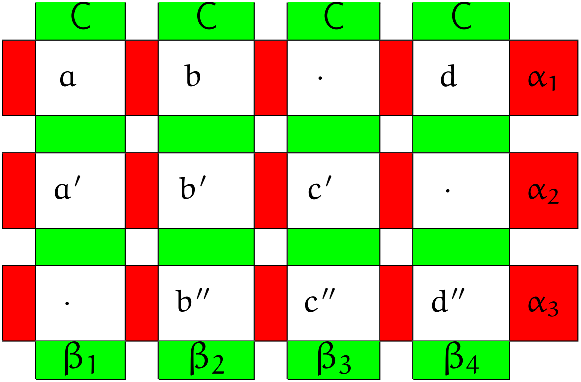
C	C	C	C
a	b	$\cdot$	d
a'	b'	c'	$\cdot$
$\cdot$	b''	c''	d''
β_1	β_2	β_3	β_4

C-(non)contextuality: Definition



$$\Pr\left[a, b, d, a', b', c', b'', c'', d''\right] \text{ exists?}$$

C-(non)contextuality: Definition



Pr $[a, b, d, a', b', c', b'', c'', d'']$ exists?

Yes $\rightarrow$ system is noncontextual (w.r.t.C)

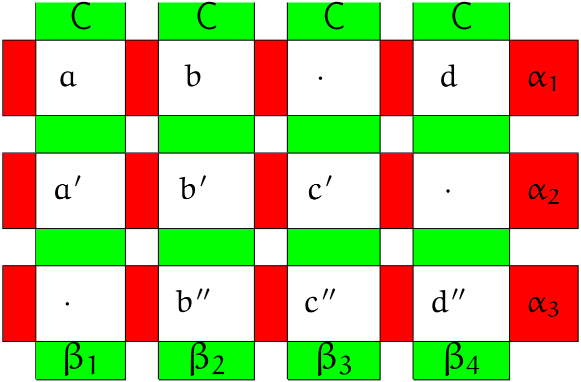
C-(non)contextuality: Definition

C	C	C	C	
a	b	$\cdot$	d	α_1
a'	b'	c'	$\cdot$	α_2
$\cdot$	b''	c''	d''	α_3
β_1	β_2	β_3	β_4	

$\Pr\left[a, b, d, a', b', c', b'', c'', d''\right]$ exists?

No $\rightarrow$ system is contextual (w.r.t.C)

C-(non)contextuality: Definition



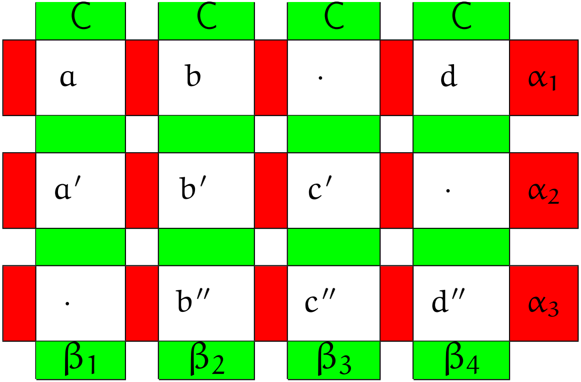
Any system is noncontextual w.r.t. $\mathsf{C} \equiv \mathrm{IND}$

C-(non)contextuality: Definition

C	C	C	C	
a	b	$\cdot$	d	α_1
a'	b'	c'	$\cdot$	α_2
$\cdot$	b''	c''	d''	α_3
β_1	β_2	β_3	β_4	

$\Pr\left[a, b, d, a', b', c', b'', c'', d''\right]$ for C $\equiv$MultiMax?

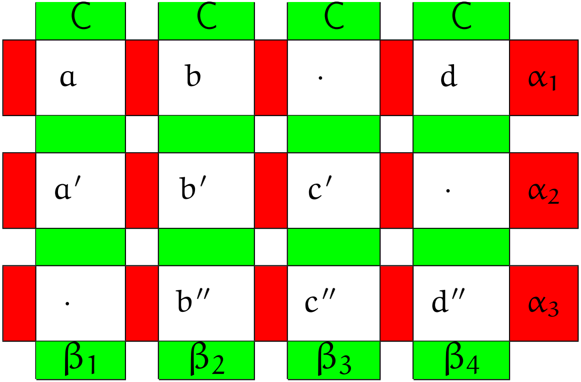
C-(non)contextuality: Definition



$\Pr[a, b, d, a', b', c', b'', c'', d'']$ for $C \equiv$MultiMax?

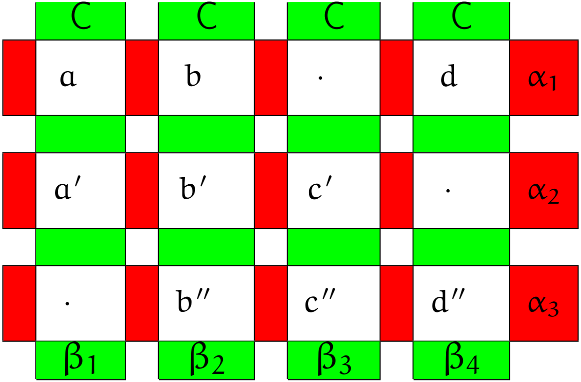
effectively determinable by LP

C-contextuality: Measure (through quasicouplings)



$$\mathrm{QPr}\left[a, b, d, a', b', c', b'', c'', d''\right]$$

C-contextuality: Measure (through quasicouplings)



$\mathrm{QPr}\left[a, b, d, a', b', c', b'', c'', d''\right]$ always exists

C-contextuality: Measure (through quasicouplings)

C	C	C	C	
a	b	$\cdot$	d	α_1
a'	b'	c'	$\cdot$	α_2
$\cdot$	b''	c''	d''	α_3
β_1	β_2	β_3	β_4	

$$\min \sum \left| \mathrm{QPr}\left[a, b, d, a', b', c', b'', c'', d''\right] \right| - 1$$

$= \min \mathrm{TV} - 1$, effectively determinable by LP

Example: Cyclic systems of binary random variables

R_1^1	R_2^1	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$	c_1
$\cdot$	R_2^2	R_3^2	$\cdot$	$\cdots$	$\cdot$	$\cdot$	c_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$	$\vdots$	$\vdots$	$\vdots$
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdots$	R_{n-1}^{n-1}	R_n^{n-1}	c_{n-1}
R_1^n	$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	R_n^n	c_n
q_1	q_2	q_3	q_4	$\cdots$	q_{n-1}	q_n	$\boxed{\mathcal{CYC}_n}$

Example: Cyclic systems of binary random variables

- $n > 5$ investigated in psychophysics (with Ru Zhang, Cervantes)
- $n = 5$— KCBS-type system
- $n = 4$— EPR/Bohm-Bell-type system ("Snow Queen" experiment, with Cervantes)
- $n = 3$— Suppes-Zanotti (origianl Bell) or Leggett-Garg-type system (with Basieva, Cervantes, Khrennikov)
- $n = 2$— question order (Moore-Wang-Busemeyer) type system

Example: Cyclic systems of binary random variables

Theorem

The degree of contextuality in a cyclic system of binary random variables is $1/2$ *of*

$$\max_{(\iota_1,\ldots,\iota_k)\in\{-1,1\}^n:\prod_{i=1}^{n}\iota_i=-1}\sum_{i=1}^{n}\iota_i\left\langle R_i^i R_{i\oplus 1}^i\right\rangle - n + 2 - \sum_{i=1}^{n}\left|\left\langle R_i^i\right\rangle - \left\langle R_i^{i\ominus 1}\right\rangle\right|.$$

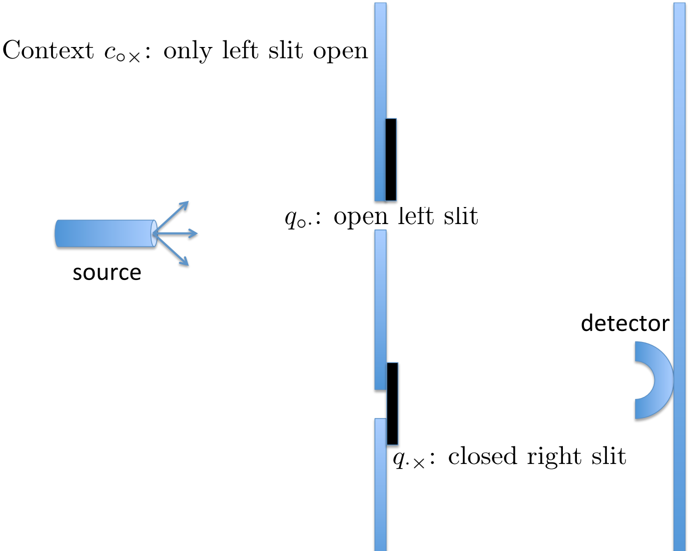
Example: Cyclic systems of binary random variables

Theorem

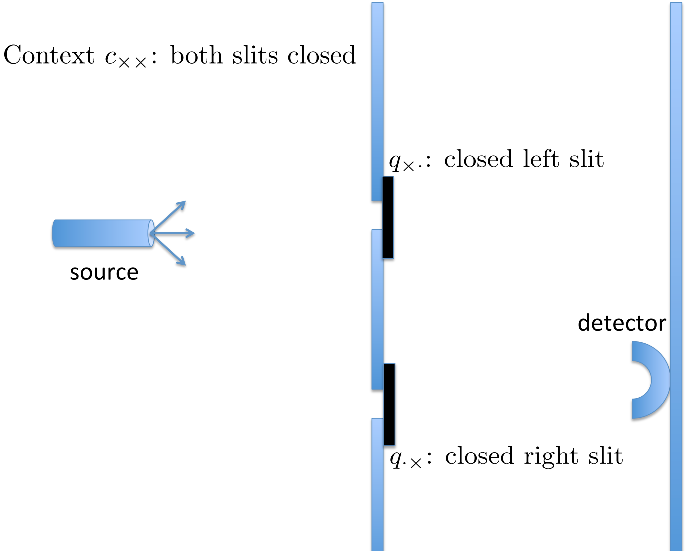
A cyclic system of binary random variables is contextual iff

$$\max_{(\iota_1,\ldots,\iota_k)\in\{-1,1\}^n:\prod_{i=1}^n \iota_i=-1} \sum_{i=1}^{n} \iota_i \left\langle R_i^i R_{i\oplus 1}^i \right\rangle > (n-2)+\sum_{i=1}^{n} \left|\left\langle R_i^i \right\rangle - \left\langle R_i^{i\ominus 1} \right\rangle\right|.$$

2-Slit, setup

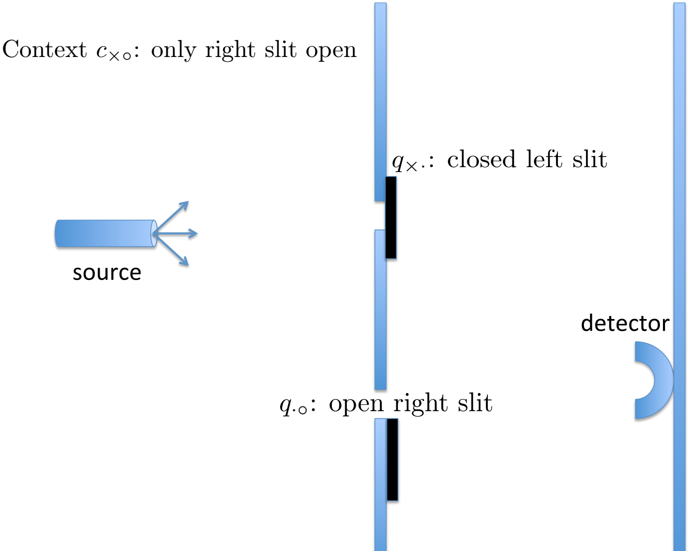

Context $c_{\circ\times}$: only left slit open
$q_{\circ\cdot}$: open left slit
source
detector
$q_{\cdot\times}$: closed right slit

2-Slit, setup



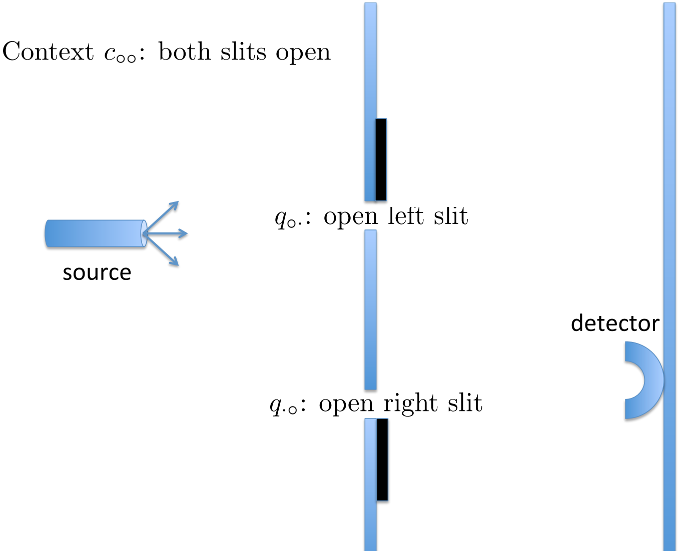
Context $c_{\times\times}$: both slits closed
$q_{\times\cdot}$: closed left slit
source
detector
$q_{\cdot\times}$: closed right slit

2-Slit, setup



Context $c_{\times\circ}$: only right slit open
$q_{\times\cdot}$: closed left slit
source
detector
$q_{\cdot\circ}$: open right slit

2-Slit, setup


Context $c_{\circ\circ}$: both slits open
$q_{\circ\cdot}$: open left slit
source
detector
$q_{\cdot\circ}$: open right slit

2-Slit, random variables chosen

$$R_{\circ.} = \begin{cases} +1 \text{ if } & \begin{matrix} \text{passes through } \mathbf{q}_{\circ}. \\ \text{and} \\ \text{hits detector} \end{matrix} \\ \\ -1 \text{ otherwise} & \end{cases}$$

2-Slit, random variables chosen

$$R^{\circ\circ}_{\circ\cdot} = \begin{cases} +1 \text{ if} & \begin{array}{c} \text{passes through } q_{\circ\cdot} \\ \text{and} \\ \text{hits detector} \end{array} \\ \\ -1 \text{ otherwise} & \end{cases}$$

2-Slit, random variables chosen

$$R^{\circ\times}_{\circ\cdot} = \begin{cases} +1 \text{ if } & \begin{array}{c} \text{passes through } q_{\circ}. \\ \text{and} \\ \text{hits detector} \end{array} \\ \\ -1 \text{ otherwise} & \end{cases}$$

2-Slit, random variables chosen

$$R^{\times\circ}_{\times\cdot} = \begin{cases} +1 \text{ if } & \begin{array}{c} \text{passes through } q_{\cdot\times} \\ \text{and} \\ \text{hits detector} \end{array} \\ \\ -1 \text{ otherwise} \end{cases}$$

2-Slit, random variables chosen

$$R^{\times\times}_{\times\cdot} = \begin{cases} +1 \text{ if } & \begin{array}{c}\text{passes through } q_{\cdot\times} \\ \text{and} \\ \text{hits detector}\end{array} \\ \\ -1 \text{ otherwise} & \end{cases}$$

2-Slit, observed probabilities and scenarios

context $c_{\circ\circ}$	$R^{\circ\circ}_{\cdot\circ} = +1$	$R^{\circ\circ}_{\cdot\circ} = -1$	
$R^{\circ\circ}_{\circ\cdot} = +1$	r	p	$r + p$
$R^{\circ\circ}_{\circ\cdot} = -1$	q	$\mathbf{1 - p - q - r}$	$1 - r - p$
	$r + q$	$1 - r - q$	

2-Slit, observed probabilities and scenarios

context $c_{\times\circ}$	$R^{\times\circ}_{\cdot\circ} = +1$	$R^{\times\circ}_{\cdot\circ} = -1$	
$R^{\times\circ}_{\times\cdot} = +1$	0	0	0
$R^{\times\circ}_{\times\cdot} = -1$	q'	$1 - q'$	1
	q'	$1 - q'$	

2-Slit, observed probabilities and scenarios

context $c_{\circ\times}$	$R^{\circ\times}_{\cdot\times} = +1$	$R^{\circ\times}_{\cdot\times} = -1$	
$R^{\circ\times}_{\circ\cdot} = +1$	0	p'	p'
$R^{\circ\times}_{\circ\cdot} = -1$	0	$\mathbf{1 - p'}$	$1 - p'$
	0	1	

2-Slit, observed probabilities and scenarios

context $c_{\times\times}$	$R^{\times\times}_{\cdot\times} = +1$	$R^{\times\times}_{\cdot\times} = -1$	
$R^{\times\times}_{\times\cdot} = +1$	0	0	0
$R^{\times\times}_{\times\cdot} = -1$	0	$\color{red}{1}$	1
	0	1	

2-Slit, content-context matrix

$\star$	$\star$			$c_{\circ\circ}$
	$\star$	$\star$		$c_{\times\circ}$
		$\star$	$\star$	$c_{\times\times}$
$\star$			$\star$	$c_{\circ\times}$
$q_{\circ\cdot}$	$q_{\cdot\circ}$	$q_{\times\cdot}$	$q_{\cdot\times}$	

2-Slit, content-context matrix

$\star$	$\star$			$c_{\circ\circ}$
	$\star$	$\star$		$c_{\times\circ}$
		$\star$	$\star$	$c_{\times\times}$
$\star$			$\star$	$c_{\circ\times}$
$q_{\circ\cdot}$	$q_{\cdot\circ}$	$q_{\times\cdot}$	$q_{\cdot\times}$	

2-Slit, content-context matrix

$\star$	$\star$			$c_{\circ\circ}$
	$\star$			$c_{\times\circ}$
				$c_{\times\times}$
$\star$				$c_{\circ\times}$
$q_{\circ\cdot}$	$q_{\cdot\circ}$	$q_{\times\cdot}$	$q_{\cdot\times}$	

2-Slit, content-context matrix

$\star$	$\star$	$\mathbf{c}_{\circ\circ}$
	$\star$	$\mathbf{c}_{\times\circ}$
		$\mathbf{c}_{\times\times}$
$\star$		$\mathbf{c}_{\circ\times}$
$\mathbf{q}_{\circ\cdot}$	$\mathbf{q}_{\cdot\circ}$	

2-Slit, content-context matrix

$\star$	$\star$	$c_{\circ\circ}$
	$\star$	$c_{\times\circ}$
		$c_{\times\times}$
$\star$		$c_{\circ\times}$
$q_{\circ\cdot}$	$q_{\cdot\circ}$	

2-Slit, content-context matrix

$\star$	$\star$	$\mathbf{c}_{\circ\circ}$
$\mathbf{q}_{\circ\cdot}$	$\mathbf{q}_{\cdot\circ}$	

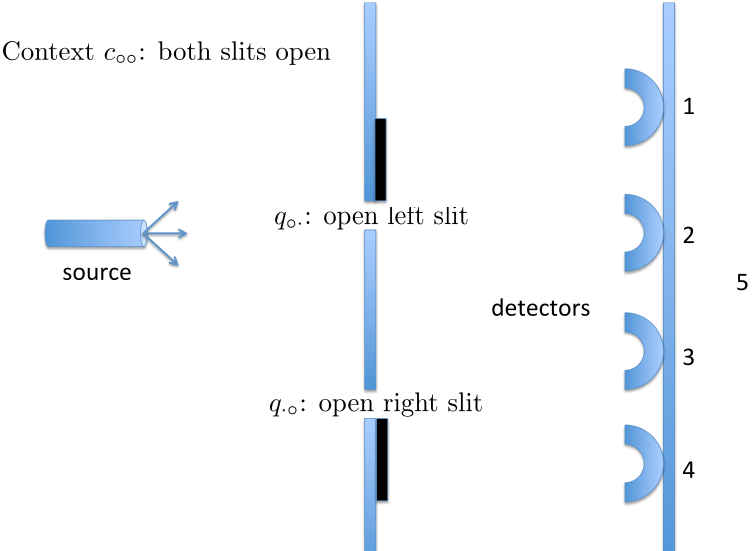
3-Slit, content-context matrix

			$\star$	$\star$	$\star$	$c_{\times\times\times}$
$\star$			$\star$	$\star$		$c_{\circ\times\times}$
	$\star$		$\star$		$\star$	$c_{\times\circ\times}$
		$\star$		$\star$	$\star$	$c_{\times\times\circ}$
$\star$		$\star$		$\star$		$c_{\circ\times\circ}$
$\star$	$\star$		$\star$			$c_{\circ\circ\times}$
	$\star$	$\star$			$\star$	$c_{\times\circ\circ}$
$\star$	$\star$	$\star$				$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	$q_{\cdot\cdot\times}$	$q_{\cdot\times\cdot}$	$q_{\times\cdot\cdot}$	

3-Slit, content-context matrix

$\star$		$\star$	$c_{\circ\times\circ}$
$\star$	$\star$		$c_{\circ\circ\times}$
	$\star$	$\star$	$c_{\times\circ\circ}$
$\star$	$\star$	$\star$	$c_{\circ\circ\circ}$
$q_{\circ\cdot\cdot}$	$q_{\cdot\circ\cdot}$	$q_{\cdot\cdot\circ}$	

2-Slit, many detectors



Canonical Representation

$$1\,|\,2\,|\,3\,|\,4\,|\,5 \longrightarrow \boxed{\begin{array}{c} 1\,|\,\text{not } 1 \\ 2\,|\,\text{not } 2 \\ 3\,|\,\text{not } 3 \\ 4\,|\,\text{not } 4 \\ 5\,|\,\text{not } 5 \end{array}}$$

Canonical Representation

$$1\,|\,2\,|\,3\,|\,4\,|\,5 \longrightarrow \boxed{\begin{array}{c} 12\,|\,\text{not } 12 \\ 13\,|\,\text{not } 13 \\ \cdots \\ 34\,|\,\text{not } 34 \\ 35\,|\,\text{not } 35 \\ 45\,|\,\text{not } 45 \end{array}}$$

Canonical Representation

R^1_1	$c = 1$
R^2_1	$c = 2$
$q = 1$	

$$\Longrightarrow$$

D^1_1	D^1_2		$D^1_{(2^{k-1}-1)}$
D^2_1	D^2_2	$\cdots$	$D^2_{(2^{k-1}-1)}$
$q' = 1$	2	$\cdots$	$\left(2^{k-1}-1\right)$

Canonical Representation

Theorem

The system (consisting of R_1^1 *and* R_1^2*) is noncontextual if and only if one of the* R_1^1 *and* R_1^2 *nominally dominates the other.*

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Definition

R_1^1 *nominally dominates* R_1^2 if $\Pr\left[R_1^1 = i\right] < \Pr\left[R_1^2 = i\right]$ for no more than one value of $i = 1, \ldots, k$.

Canonical Representation

Theorem

The system (consisting of R_1^1 and R_1^2) is noncontextual if and only if one of the R_1^1 and R_1^2 nominally dominates the other.

Example

x:	1	2	3	4	5
$\Pr\left[R_1^1 = x\right]$	.2	.3	.1	.3	.1
$\Pr\left[R_1^2 = x\right]$	.1	.2	.4	.3	0

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3. Contextually labeled observables in Hilbert spaces.

Summary

1. Contextuality analysis applies to systems of content-context-indexed random variables, after they have been transformed into a canonical (split representation) form, in which all random variables are dichotomous. Each of these dichotomous random variables answers the question: does the outcome of a given initial random variable fall within a given subset of its possible values?

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3. The minimal total variation (less one) across all possible quasi-couplings of the canonical representation of the system is a measure of contextuality of the system.

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1. any subsystem of a noncontextual system of random variables is noncontextual;
2. componentwise transformations of a noncontextual system of random variables are noncontextual;
3. a consistently connected ("non-signaling") system is (non)contextual iff it is (non)contextual in the traditional sense.