On Universality of Classical Probability with Contextually Labeled Random Variables

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Introduction

- It is often claimed that classical probability theory is unable to describe particular empirical phenomena.
- This general claim is exemplified by the three widespread statements below.

Methodology and Primary Aim

Apply the Contextuality-by-Default theory to show that these statements are mistakenly made due to misidentification of random variables.

Assumptions

Classical probability theory is:

- **Abstract** → unfalsifiable.
- **Universal** → can describe any probabilistic features.
- **Complementable** by special purpose computations.

The Contextuality-by-Default Theory: Key Features

Double labeling of random variables

$$R_q^c$$

Context: properties of the experiment, e.g., left slit open and right slit closed.

Content: what is being measured, e.g., the particle hit the detector after passing through right slit.

Empirically-defined joint distributions

Jointly distributed.

Shared probability space.

$R_q^c \quad R_{q'}^c$

$R_q^{c'} \quad R_{q'}^{c'}$

Stochastically unrelated.

Contextuality analysis

- A system of random variables is **noncontextual** if it has a coupling in which any two content-sharing random variables are equal with maximal probability. For example, for the system $R_q^c, R_{q'}^c, R_q^{c'}, R_{q'}^{c'}$, there is a joint distribution $(S_q^c, S_{q'}^c, S_q^{c'}, S_{q'}^{c'})$ such that

  $(S_q^i, S_{q'}^i) \sim (R_q^i, R_{q'}^i), i = c, c'$, and

  $S_j^c = S_j^{c'}$ with maximal probability, $j = q, q'$.
- A system is **contextual** otherwise.

The Three Statements and Analysis

Statement 1

"Classical probability requires that **specific (e.g., Bell-type) inequalities** hold for particular sets of random variables, but we know from quantum mechanics and from behavioral experiments that they may be violated."

Statement 2

"In classical probability the joint occurrence of two events is **commutative**, but we know from quantum mechanics and from behavioral experiments that the order of two events generally matters for their joint probability."

Statement 3

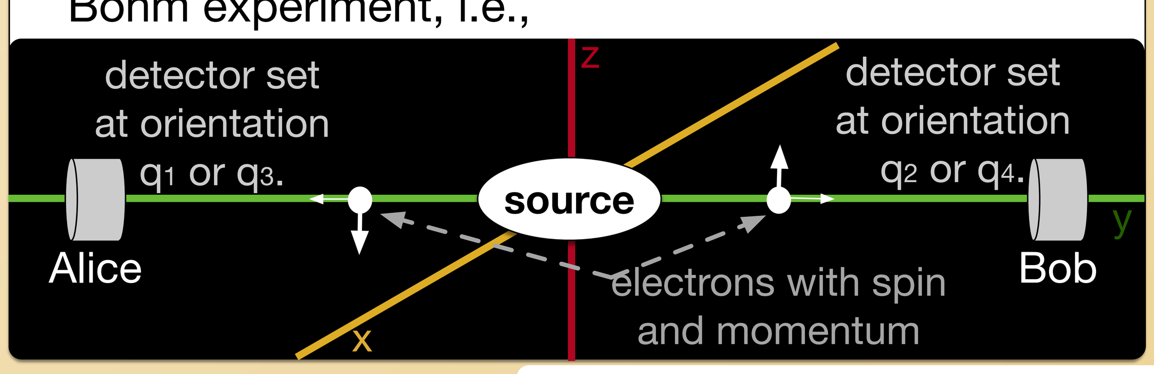
"Classical probability is **additive** (equivalently, obeys the law of total probability), but we know from quantum mechanics and from behavioral experiments that this additivity (the law of total probability) can be violated."

Example of Each Statement

- Classical probability theory requires that the CHSH/Fine inequality, i.e.,

$$\max_{j=1,\ldots,4}\left|\sum_{i=1}^{4}\langle R_iR_{i\oplus1}\rangle - 2\langle R_jR_{j\oplus1}\rangle\right| \le 2, \quad (1)$$

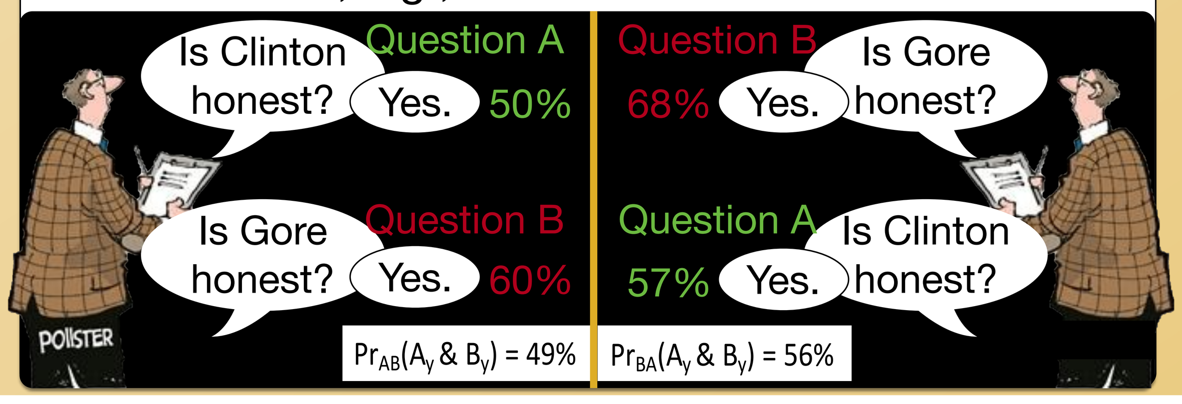
  holds for a set of four random variables with known distributions $(R_1,R_2),\ (R_2,R_3),\ (R_3,R_4)$, and (R_4,R_1).
- However, it is claimed that (1) is violated in the EPR-Bohm experiment, i.e.,



- Classical probability theory requires that commutativity, e.g.,

$$\Pr{}_{AB}(A_y \wedge B_y) = \Pr{}_{BA}(B_y \wedge A_y), \quad (2)$$

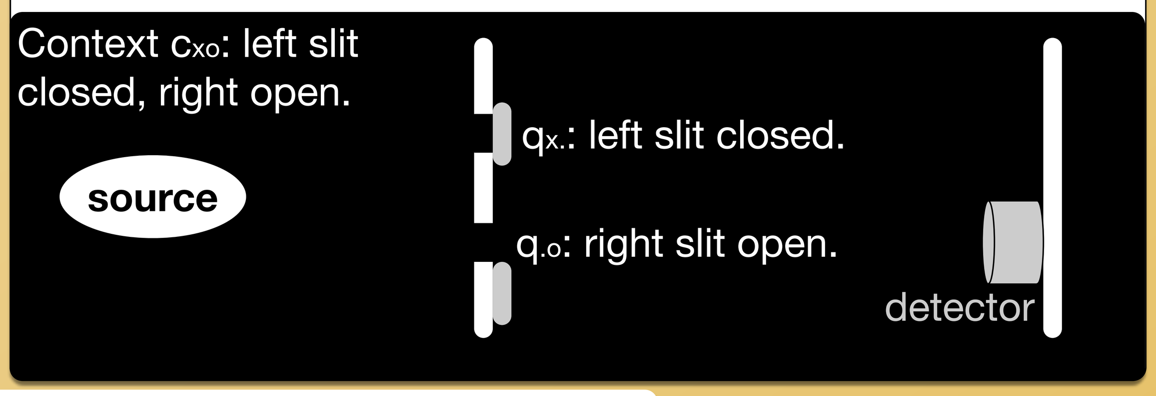
  holds for events A_y (question A is answered 'Yes') and B_y (question B is answered 'Yes').
- However, it is claimed that (2) is violated in question order effects, e.g.,



- Classical probability theory requires that additivity, i.e.,

$$\Pr[R \in A \cup B] = \Pr[R \in A] + \Pr[R \in B], \quad (3)$$

  holds for two disjoint events A and B in the codomain space of random variable R.
- However, it is claimed that (3) is violated in the double slit experiment, e.g.,



Results from Applying the Contextuality-by-Default Theory

The system of random variables is

R_1^1	R_2^1			c_1
	R_2^2	R_3^2		c_2
		R_3^3	R_4^3	c_3
R_1^4			R_4^4	c_4
q_1	q_2	q_3	q_4	

A possible pair of measurements that Alice and Bob can make simultaneously.

Axis along which Alice can make a measurement.

Axis along which Bob can make a measurement.

- There are eight random variables in play here, because of which (1) is not applicable without first imposing identity couplings column-wise. A "violation" of (1) means that these couplings are inconsistent with the row-wise distributions.

The system is noncontextual iff

$$\max_{j=1,\ldots,4}\left|\sum_{i=1}^{4}\left\langle R_i^iR_{i\oplus1}^i\right\rangle - 2\left\langle R_j^jR_{j\oplus1}^j\right\rangle\right| \le 2 + \sum_{i=1}^{4}\left|\left\langle R_{i\oplus1}^i\right\rangle - \left\langle R_{i\oplus1}^{i\oplus1}\right\rangle\right|, \quad (4)$$

which is a generalization of equation (1) [1].

The system of random variables is

R_A^{AB}	R_B^{AB}	c_{AB}
R_A^{BA}	R_B^{BA}	c_{BA}
q_A	q_B	

Condition where question A is asked before B.

Condition where question B is asked before A.

Response to A.

Response to B.

- There are four random variables in play here, because of which (2) is not applicable without first imposing identity couplings column-wise. A "violation" of (2) means that such couplings are inconsistent with the row-wise distributions.

The system is noncontextual iff

$$\left|\left\langle R_A^{AB}R_B^{AB}\right\rangle - \left\langle R_A^{BA}R_B^{BA}\right\rangle\right| \le \left|\left\langle R_A^{AB}\right\rangle - \left\langle R_A^{BA}\right\rangle\right| + \left|\left\langle R_B^{AB}\right\rangle - \left\langle R_B^{BA}\right\rangle\right|, \quad (5)$$

which is always satisfied if the Busemeyer-Wang QQ equality holds (the left-hand side is zero) [2].

- For the Clinton/Gore case, equation (5) holds.

The system of random variables is

$R_{\circ\cdot}^{\circ\circ}$	$R_{\cdot\circ}^{\circ\circ}$			$c_{\circ\circ}$
	$R_{\cdot\circ}^{\times\circ}$	$R_{\times\cdot}^{\times\circ}$		$c_{\times\circ}$
		$R_{\times\cdot}^{\times\times}$	$R_{\cdot\times}^{\times\times}$	$c_{\times\times}$
$R_{\circ\cdot}^{\circ\times}$			$R_{\cdot\times}^{\circ\times}$	$c_{\circ\times}$
$q_{\circ\cdot}$	$q_{\cdot\circ}$	$q_{\times\cdot}$	$q_{\cdot\times}$	

Condition with both slits open.

Condition with left slit open and right slit closed.

Left slit open.

Left slit closed.

- There are three different random variables in three probabilities of (3), because of which (3) is inapplicable without additional (wrong) assumptions.
- This system has probabilities:

$$\begin{array}{c|c|c} c_{\circ\times} & R_{\cdot\times}^{\circ\times}=\text{Yes} & \\ \hline R_{\circ\cdot}^{\circ\times}=\text{Yes} & 0 & p \\ \hline & 0 & \end{array},\quad \begin{array}{c|c|c} c_{\times\times} & R_{\cdot\times}^{\times\times}=\text{Yes} & \\ \hline R_{\times\cdot}^{\times\times}=\text{Yes} & 0 & 0 \\ \hline & 0 & \end{array},\quad \begin{array}{c|c|c} c_{\times\circ} & R_{\cdot\circ}^{\times\circ}=\text{Yes} & \\ \hline R_{\times\cdot}^{\times\circ}=\text{Yes} & 0 & 0 \\ \hline & q & \end{array},\quad \begin{array}{c|c|c} c_{\circ\circ} & R_{\cdot\circ}^{\circ\circ}=\text{Yes} & \\ \hline R_{\circ\cdot}^{\circ\circ}=\text{Yes} & r' & r'+p' \\ \hline & r'+q' & \end{array},$$

Assuming the detector is sufficiently small, the system is noncontextual iff [3,4]

$$\begin{aligned}&((1-2p)+(1)+(1-2q)+(1-2p'-2q'))\\ &-2\min\left((1-2p),(1),(1-2q),(1-2p'-2q')\right)\\ &\le 2+2|p-p'-r'|+|-1+1|+2|q-q'-r'|+|-1+1|.\end{aligned} \quad (6)$$

Conclusions

- The three statements above are based on misidentification of the random variables involved.
- Contextual labeling is a principled way to "automatically" ensure correct applicability of classical probability theory to an empirical situation.
- The use of contextual labeling enables a novel contextually analysis of random variables.

References

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Acknowledgements

This research was supported by:

- AFSOR grant FA9550-14-1-0318 (E.D.),
- Purdue University Lynn Fellowship (M.K.).