Towards a complete cohomology invariant for contextuality

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Quantum Contextuality in Quantum Mechanics and Beyond
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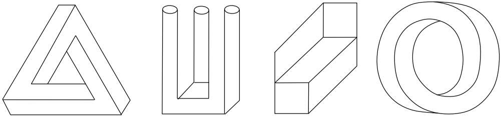
UNIVERSITY OF OXFORD
DEPARTMENT OF COMPUTER SCIENCE

QUANTUM GROUP

Outline

1. **Contextuality** and "**impossible figures**": a **topological viewpoint**.



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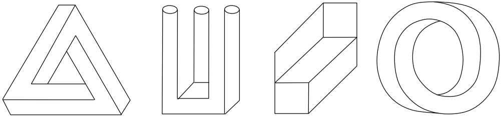

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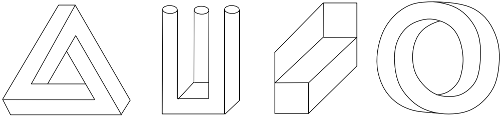

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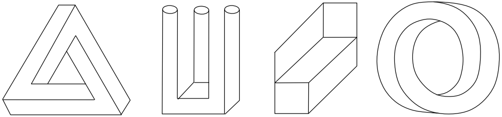

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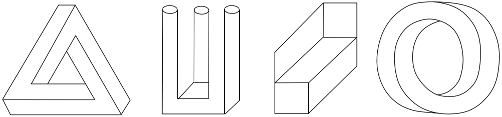

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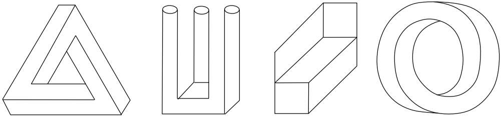

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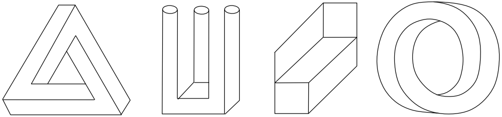

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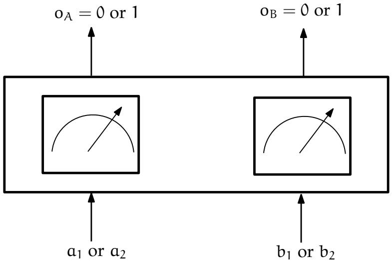

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- Such a **possibility table** is called a **possibilistic empirical model**.

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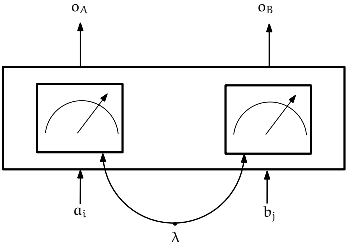

- In other words, the state of the whole system is determined by a **hidden variable** λ, whose value is independent of Alice and Bob's choices.

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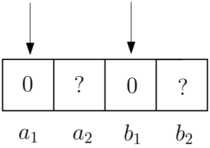

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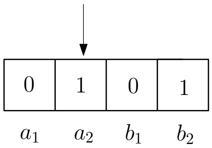

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No predetermined global assignment!

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What does topology have to do with this?

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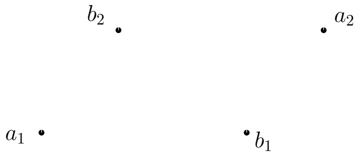

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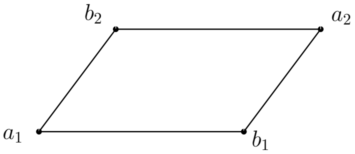

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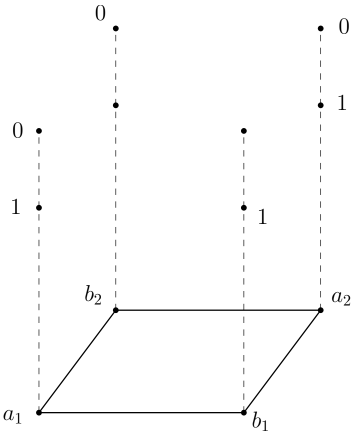

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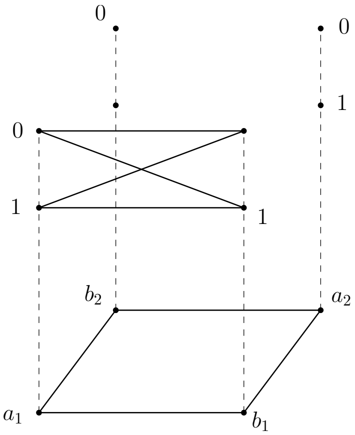

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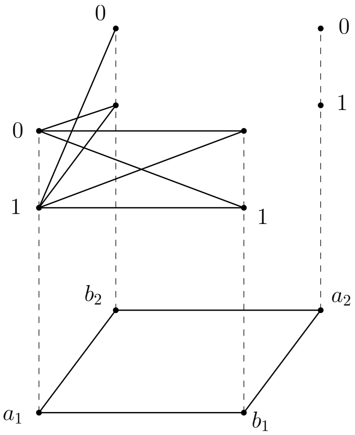

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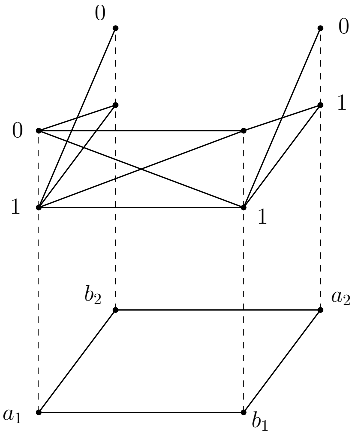

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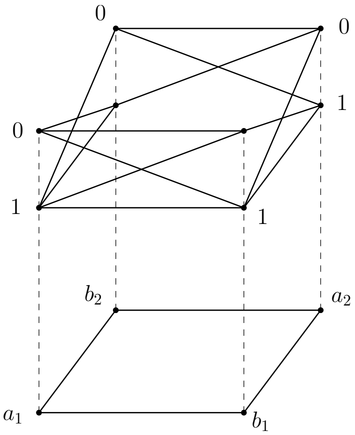

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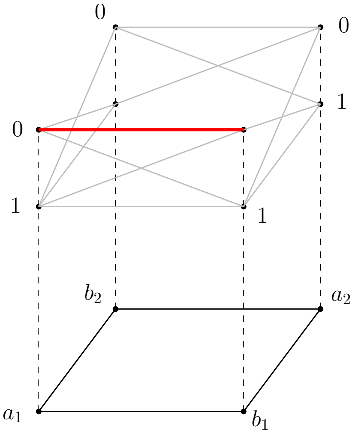

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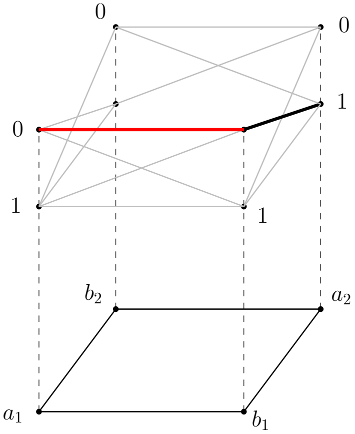

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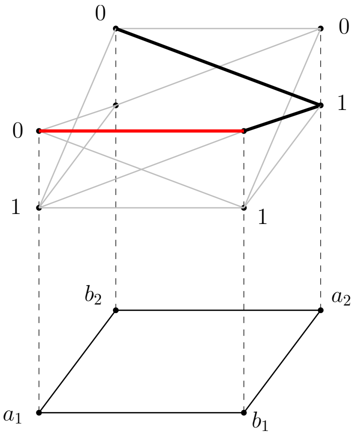

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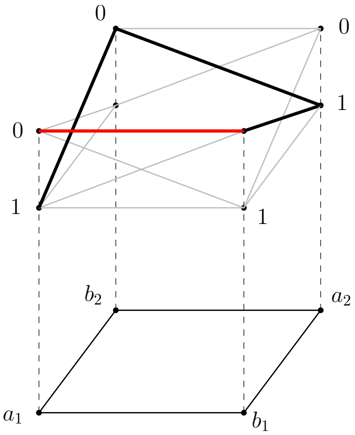

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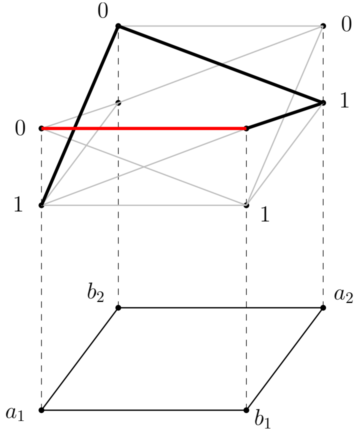

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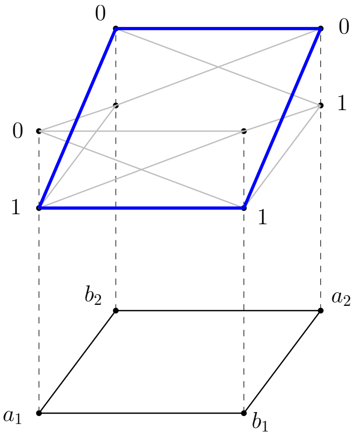

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However, a portion of the model can be explained deterministically

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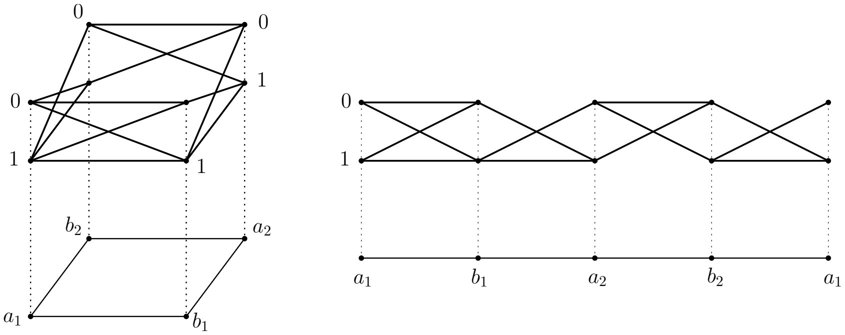

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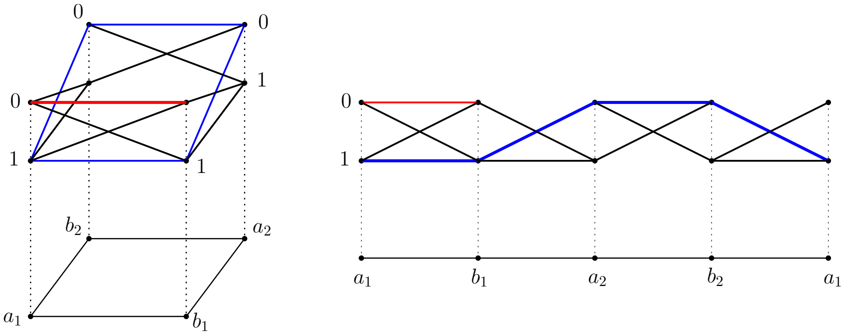

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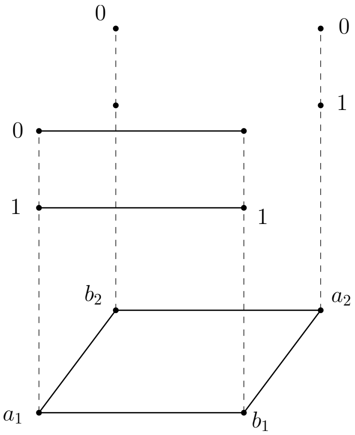

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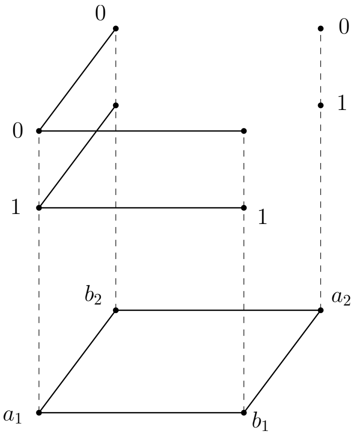

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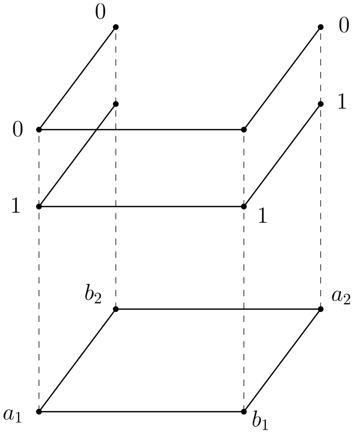

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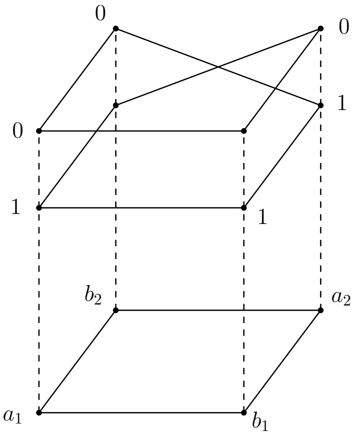

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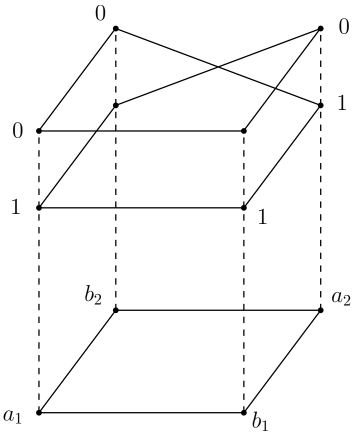

This model is contextual at **every section**. We say that it is **strongly contextual**

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- No-signalling corresponds to the fact that **each local section can always be extended to its adjacent contexts**.

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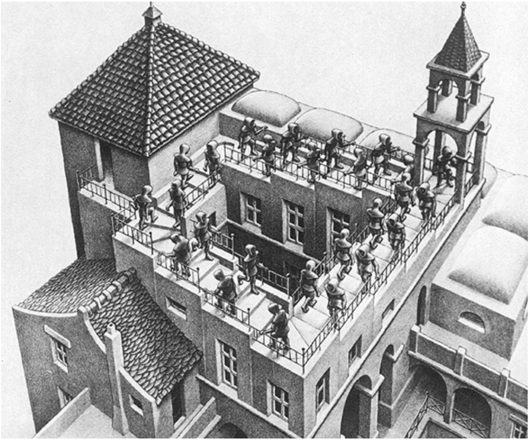
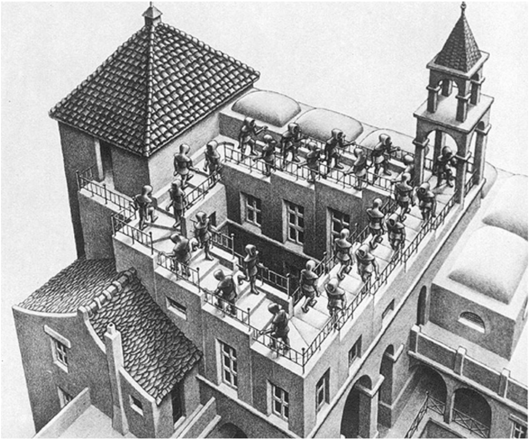
- Therefore, contextuality of no-signalling empirical models can be loosely interpreted as a **discrepancy between local consistency and global inconsistency**, which is ultimately a **topological property**:





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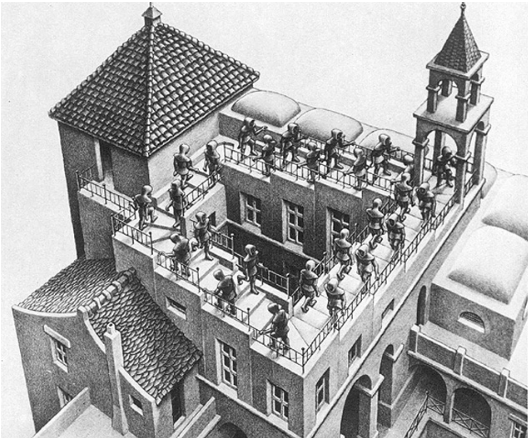

- This discrepancy is typical of **impossible figures**
- It has been studied in detail using **sheaf theory** and **sheaf cohomology**

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- **This is exactly what we are looking for!**

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$$\mathscr{S}(C) := \{(a_1, b_2) \mapsto (0,1), (1,0), (1,1)\}$$

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- The main ingredient? **Sheaf cohomology!**

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- However, this 'abelianisation' gives rise to a significant amount of **false positives**.

False positives

Hardy model

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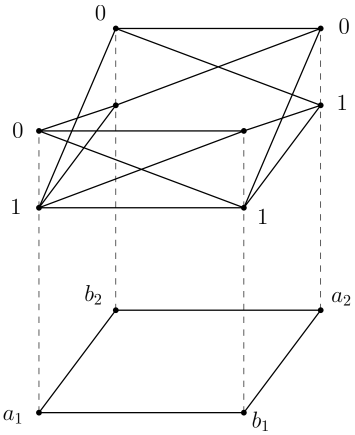

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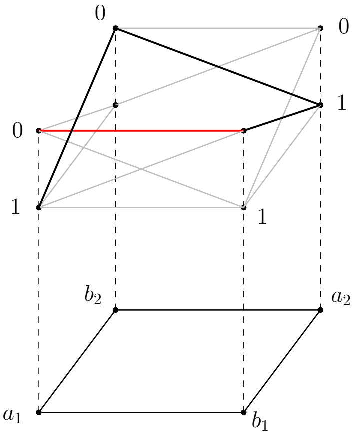

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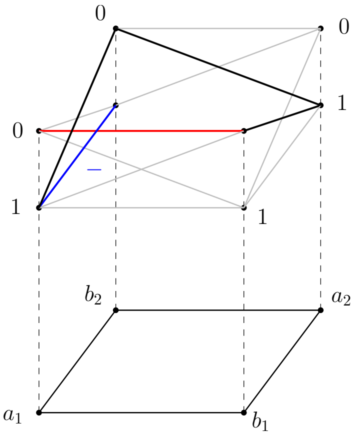

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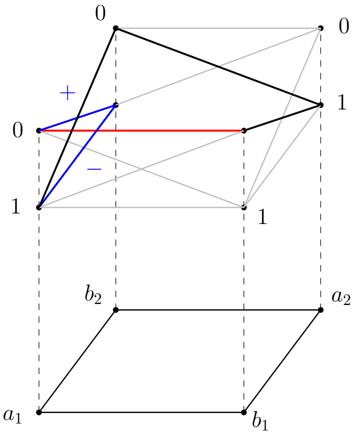

The possibility of **linearly adding sections** allows us to find a global section (for $\mathcal{F}$) containing the red section. Thus cohomology does not detect contextuality in this case!

False positives

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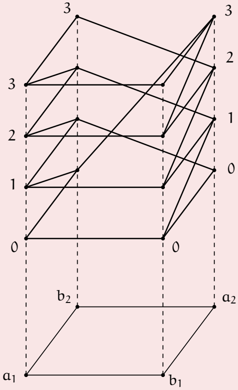
A strongly contextual model which is cohomologically non-contextual



This model presents a false positive **for every single local section**.

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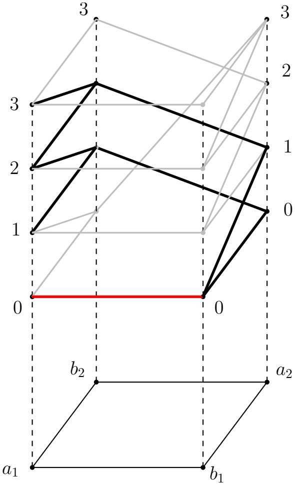

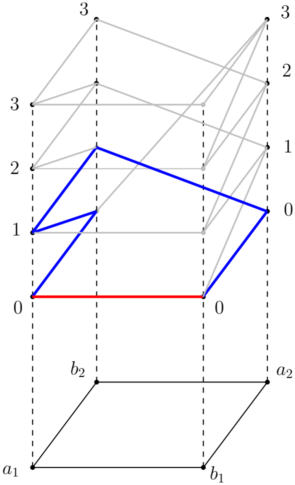

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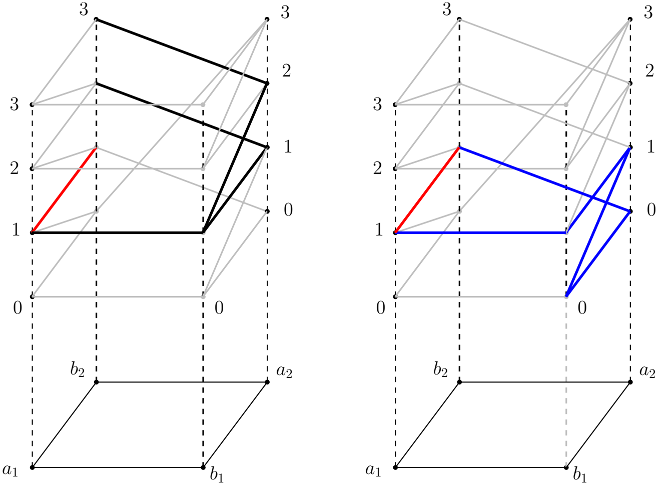

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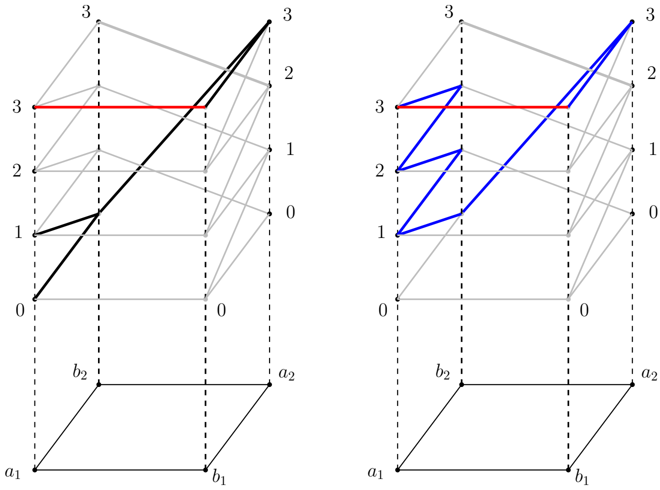

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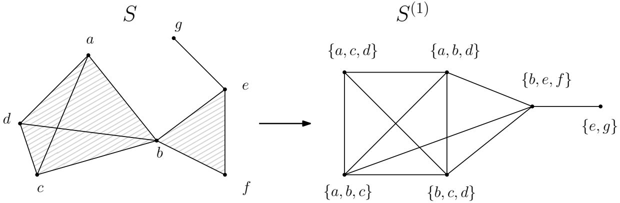

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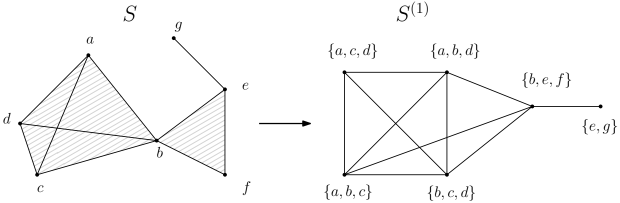

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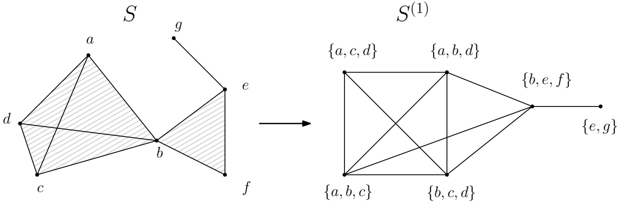

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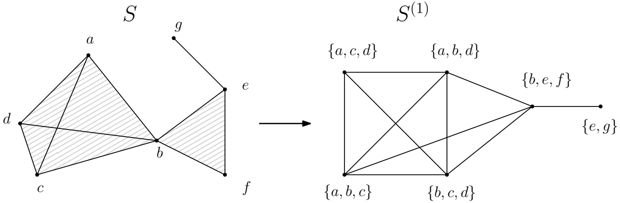

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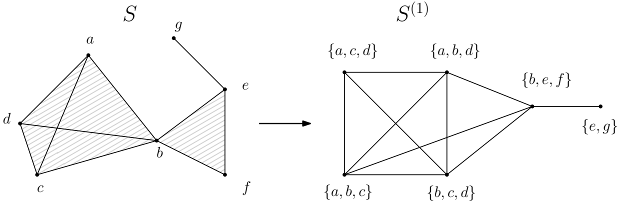

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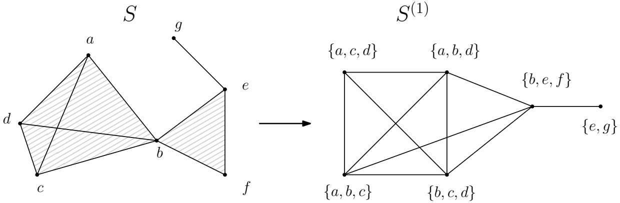

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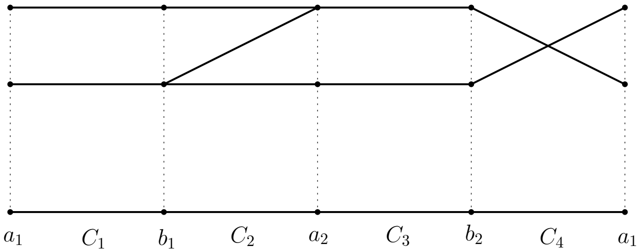
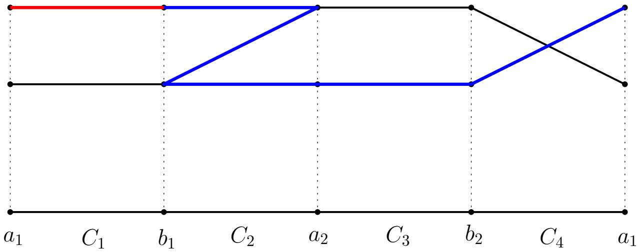


Example



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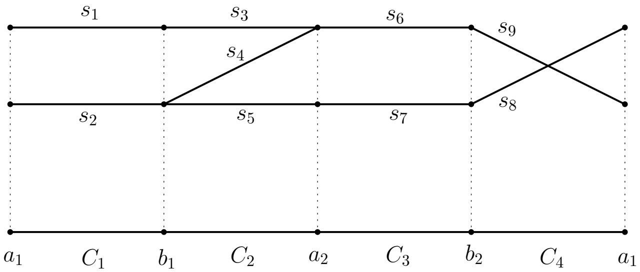

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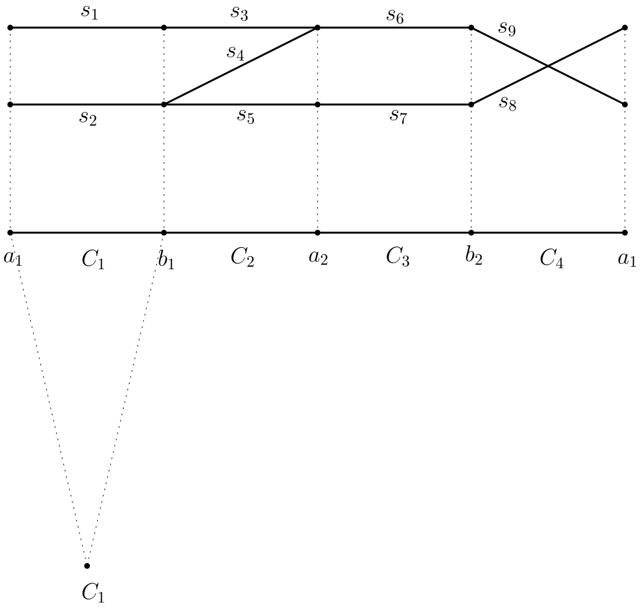

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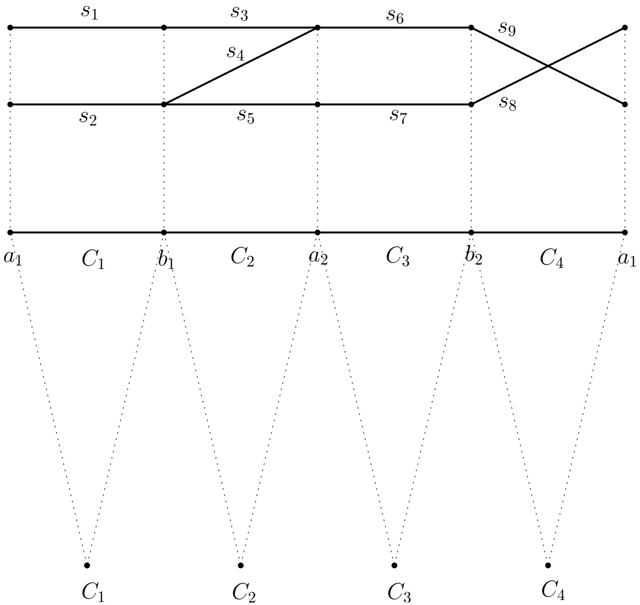

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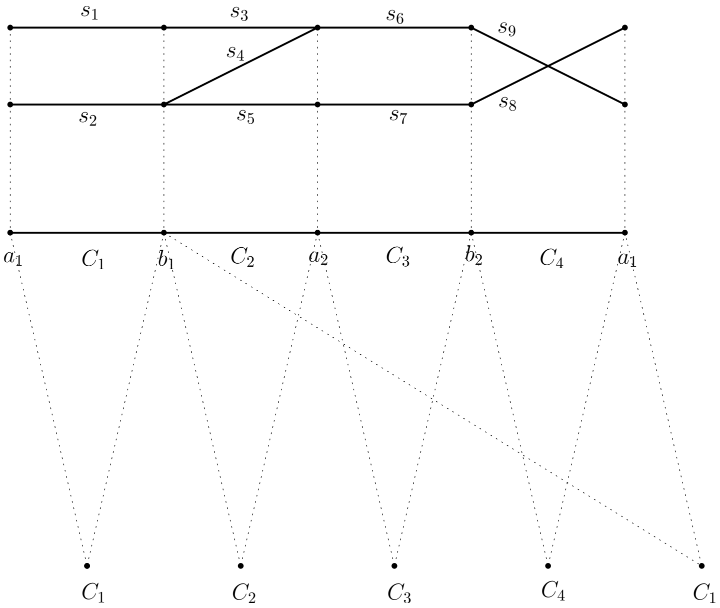

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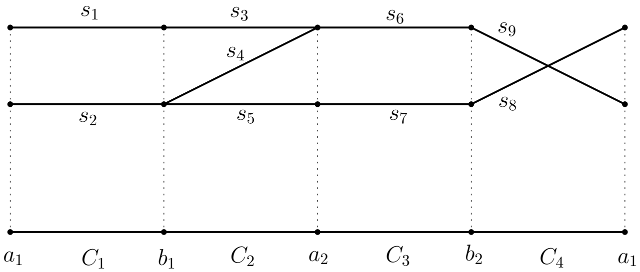

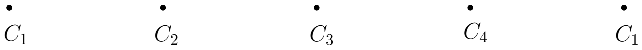

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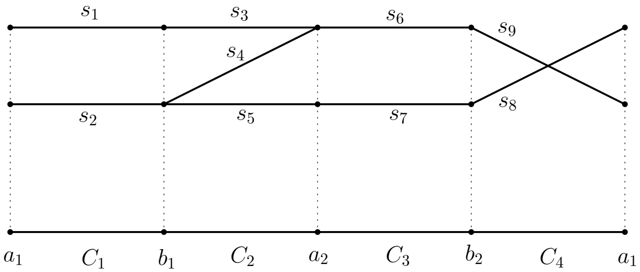

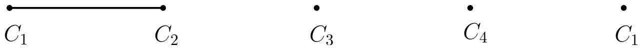

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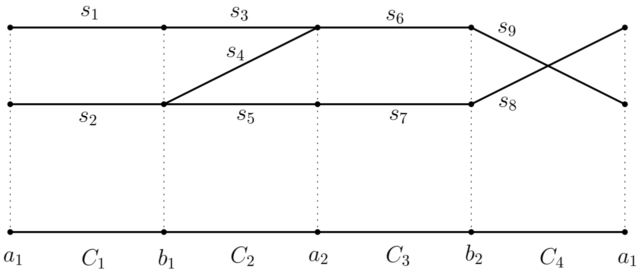

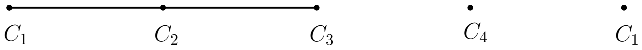

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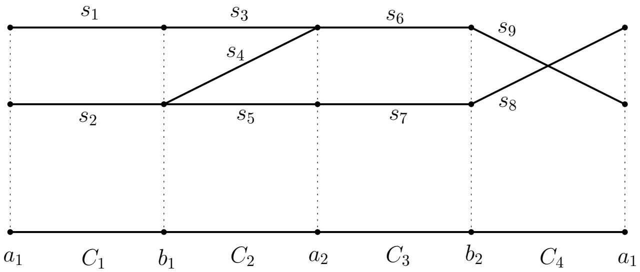

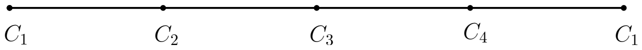

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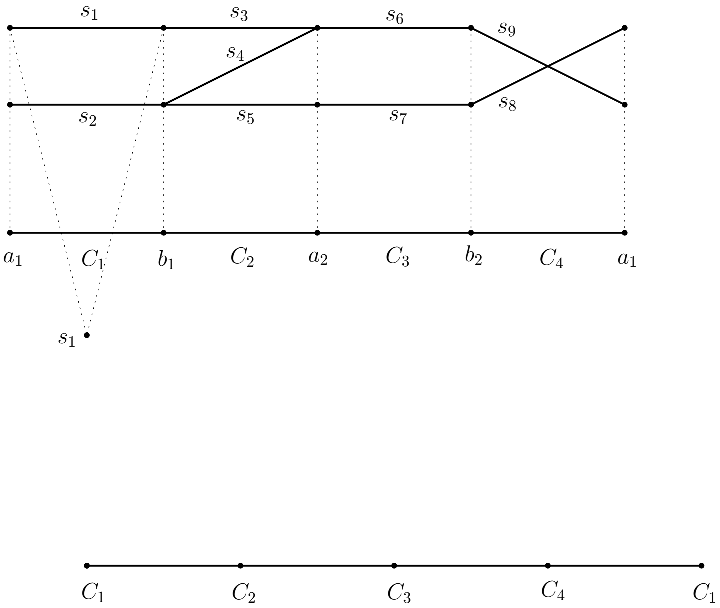

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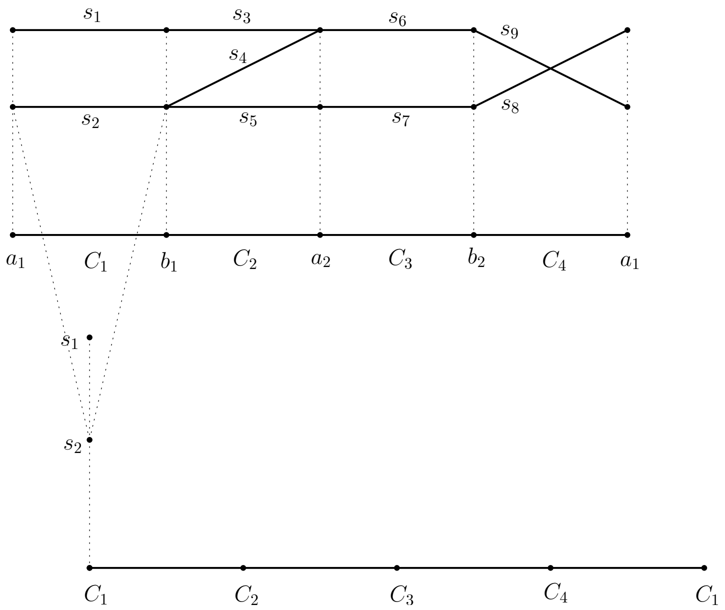

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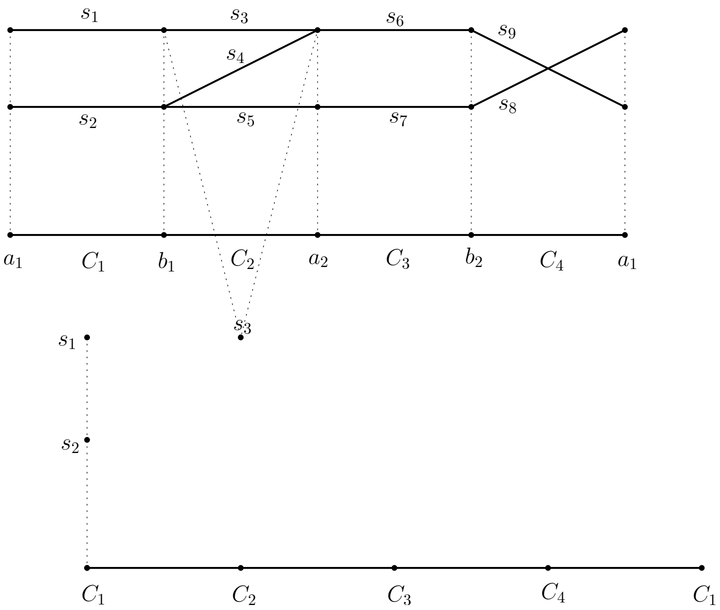

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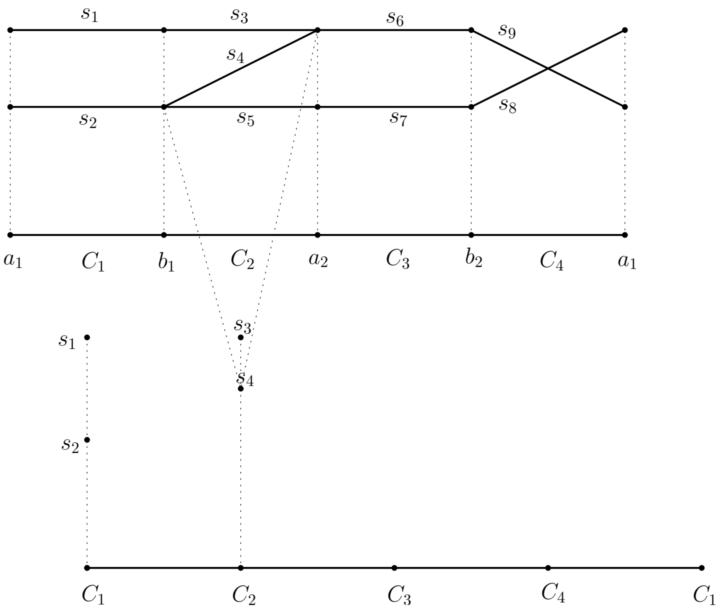
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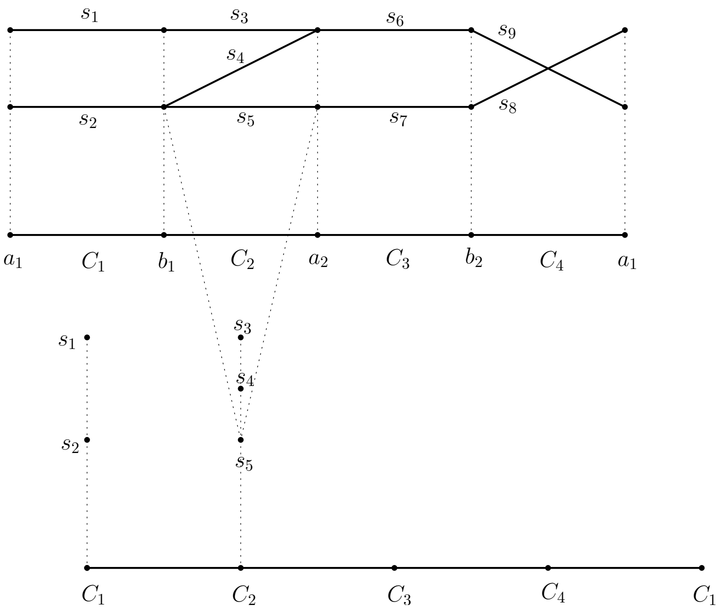

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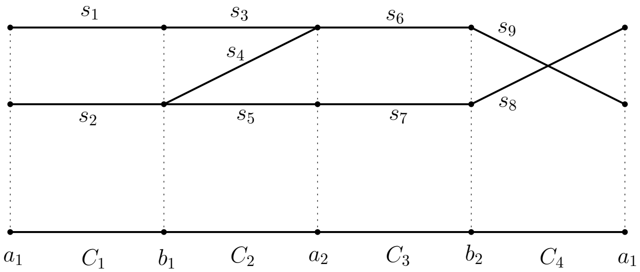
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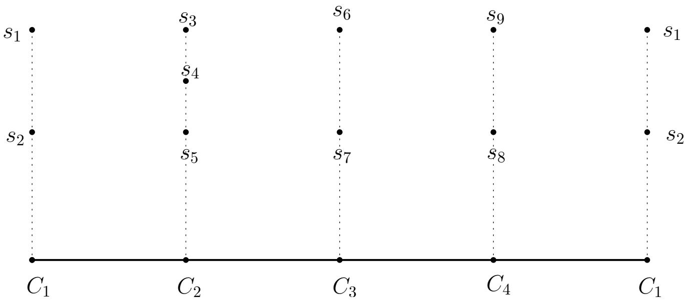
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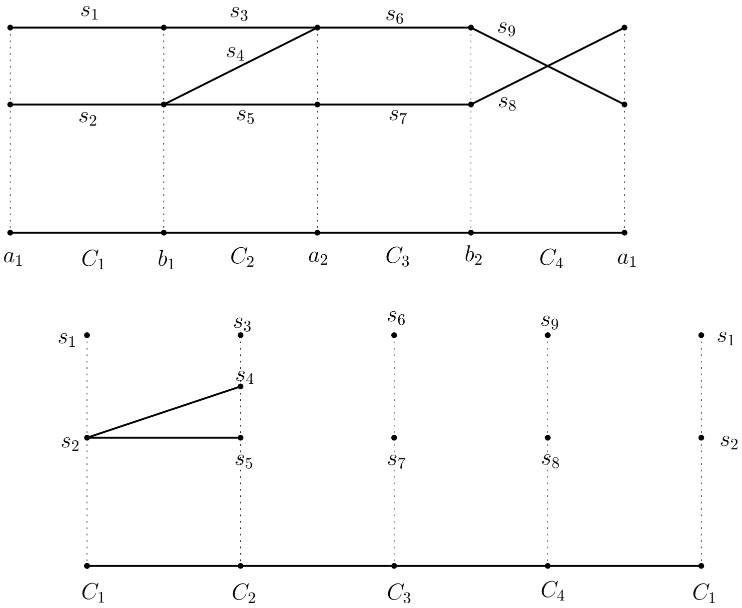

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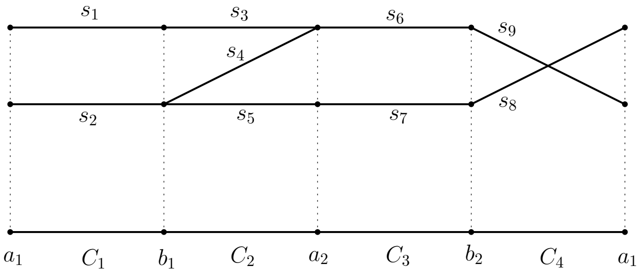

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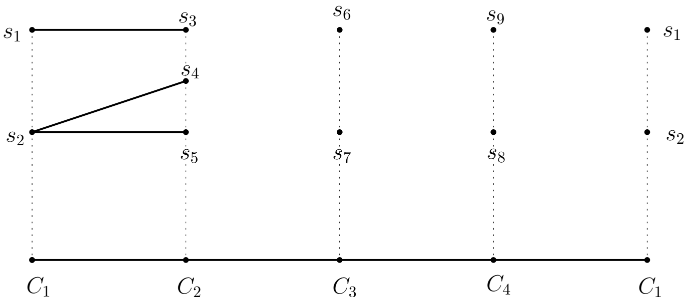

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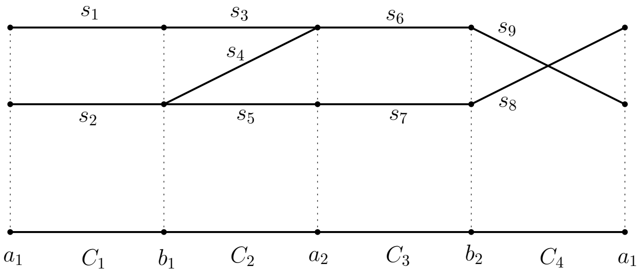

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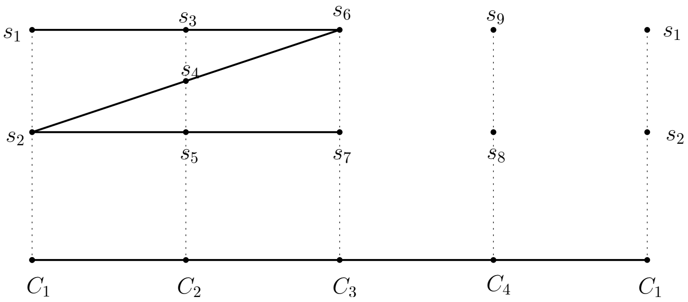

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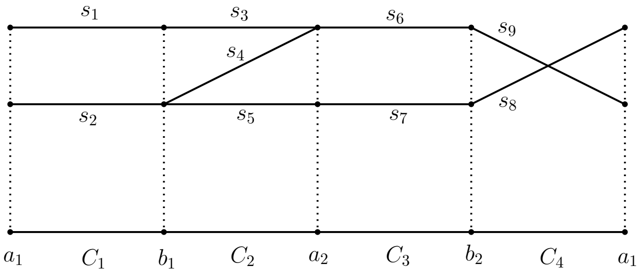

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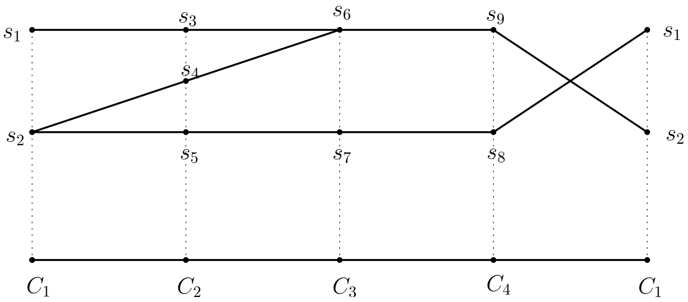

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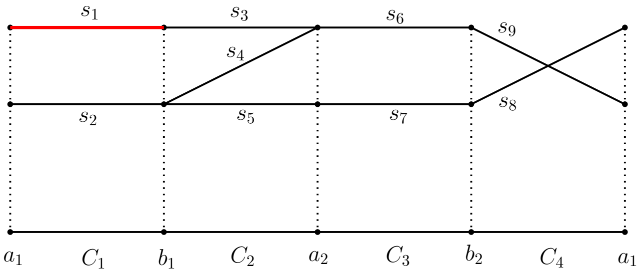

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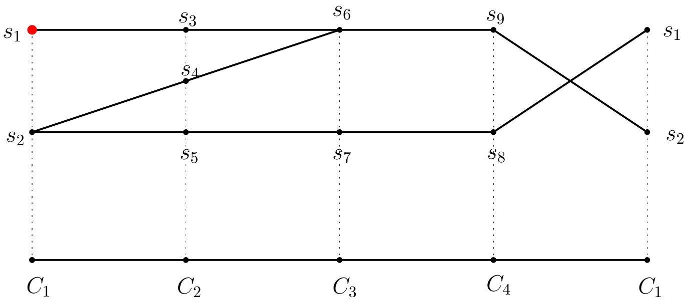

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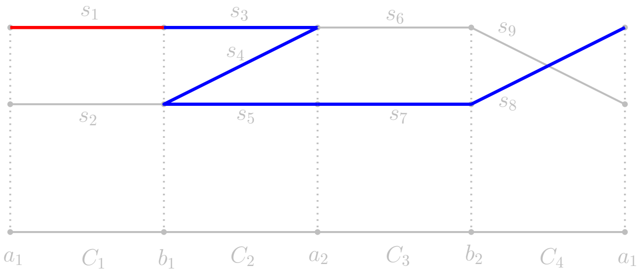

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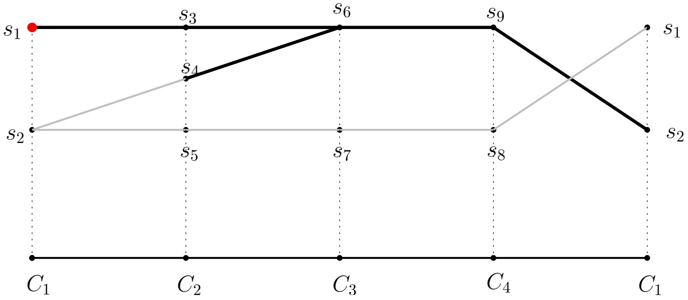

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- Therefore, for cyclic models, cohomology is a complete invariant for contextuality.

Extending the invariant to general models

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where n is the size of the cycle responsible for the contextuality of the model.

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In other words, we strongly suspect the invariant is in fact **universal**.

Examples: solving well-known false positives

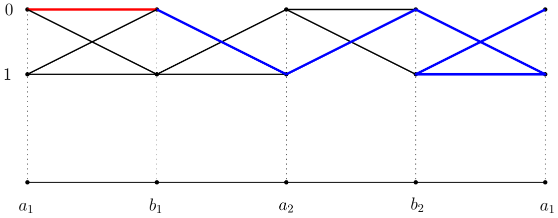

Figure: The Hardy model

Examples: solving well-known false positives

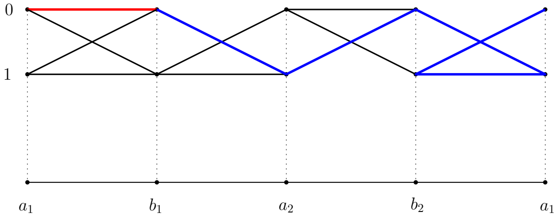

Figure: The Hardy model

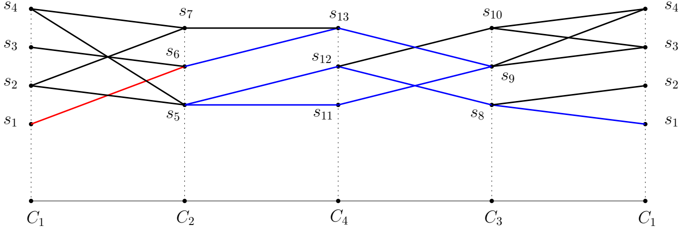

Figure: The first joint model of the Hardy model

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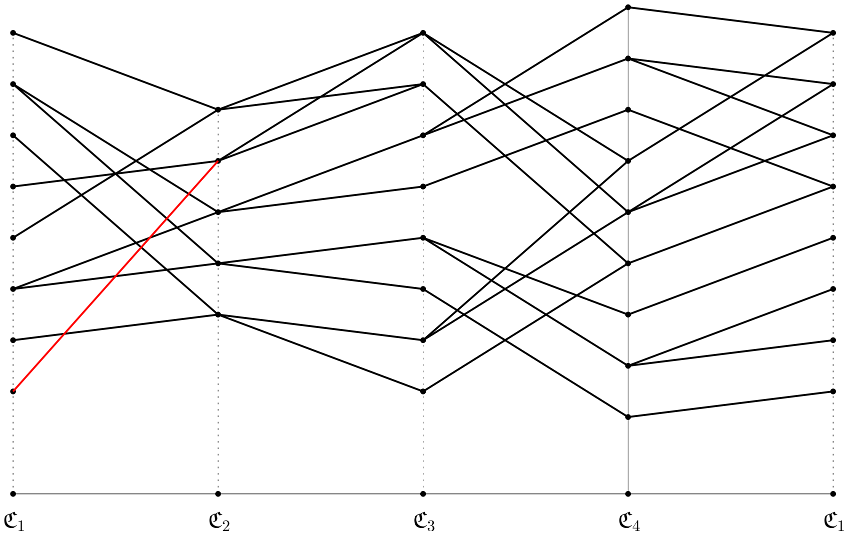
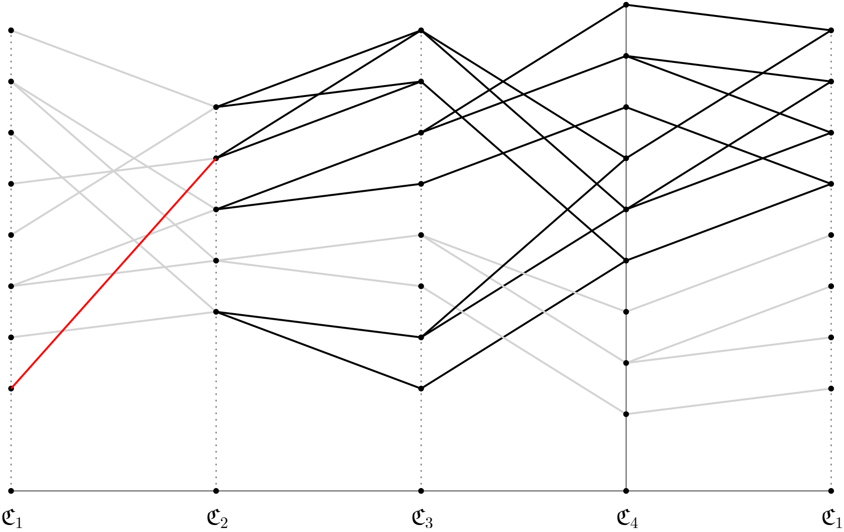


Figure: The third joint model of the Hardy model

Examples: solving well-known false positives



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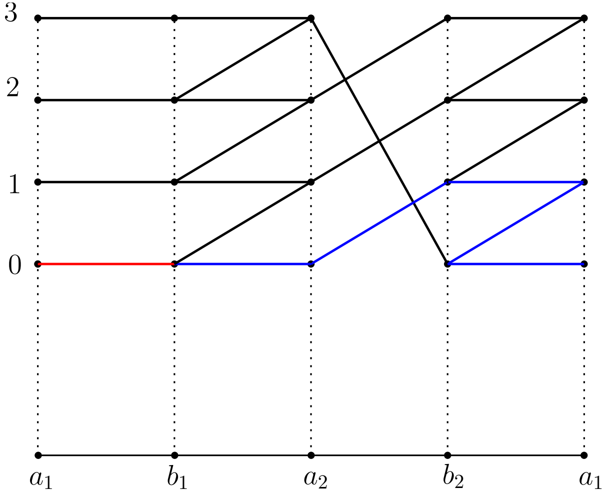

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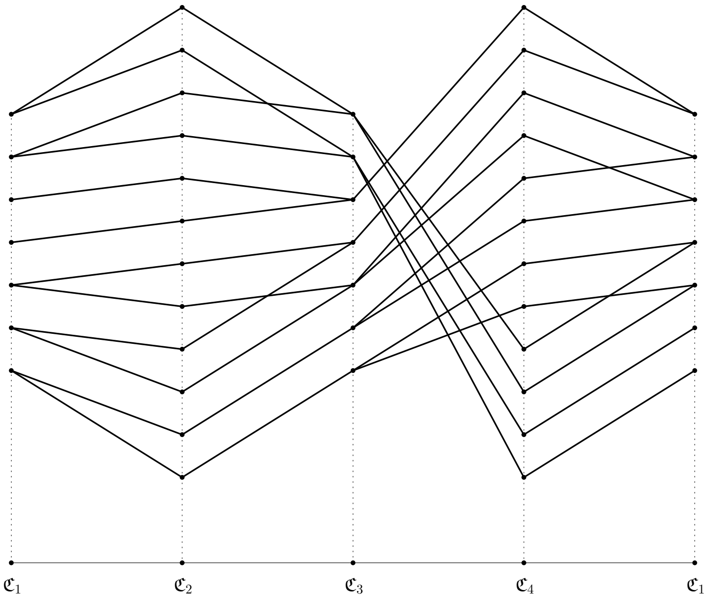

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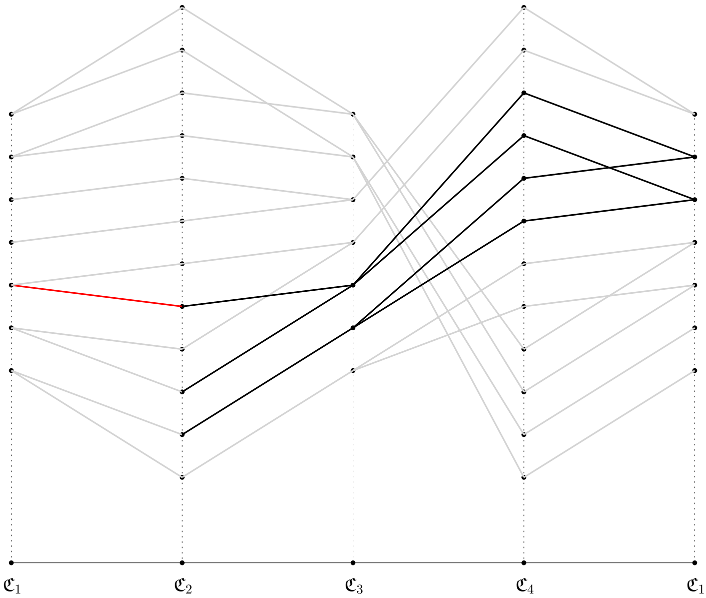

Thank you for your attention! Questions?