Contextuality for transformations

Shane Mansfield

*S. Mansfield and E. Kashefi. "Quantum advantage via sequential transformation contextuality". **arXiv:1801.08150.**

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Contextuality supplies the magic for quantum computing?

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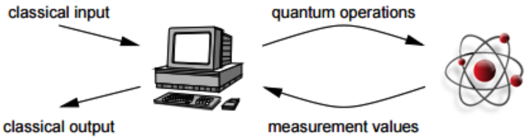

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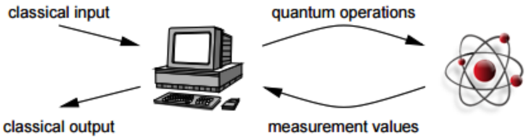

Degree of contextuality relates to degree of advantage in probabilistic computation *(Abramsky, Barbosa, SM)*

$$\varepsilon \geq \mathsf{NCF}(e)\, \nu(f)$$

ε – error; $\mathsf{NCF}(e)$ – classicality; $\nu(f)$ – hardness of task

Problem: single-qubit non-linear protocol

(Dunjko, Kapourniotis, Kashefi)

$$|+\rangle - \boxed{U_a} - \boxed{V_b} - \boxed{W_{a\oplus b}} - \boxed{\sigma_X \text{ meas.}}$$

Classical control ($\oplus L$):

- Classical inputs $a, b \in \mathbb{Z}_2$
- Controls transformations on resource
- Announces meas. outcome $\{+1 \mapsto 0, -1 \mapsto 1\}$
- $o = a \otimes_2 b$

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Quantum resource:

- Prepare qubit in state $|+\rangle$
- Transformations

$$U_0 = V_0 = W_0 = I$$
$$U_1 = V_1 = W_1 = R_z(\pi/2)$$

- Return meas. outcome

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- Issue: contextuality cannot arise with a single qubit!
- So what, if anything, is the non-classical behaviour?

Ontological models*

Usual setting for proving no-go theorems

- Space of ontic states Λ

	Quantum mechanics	Ontological models
Preparation	ρ	$d_\rho \in P(\Lambda)$
Transformation	U	$f_U : \Lambda \to \Lambda$
Measurement	M	$\xi_M : \Lambda \to P(O)$

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- Empirical data $e_{\rho,U,M}$ should be reproduced as

$$e_{\rho,U,M} = \sum_{\lambda \in \Lambda} d_\rho(\lambda)\, \xi_M(f_U(\lambda))$$

- Weighted average over ontic states

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*No implicit assumptions about additional 'features' (cf. Spekkens)

Classicality: Bell-Kochen-Specker

Non-contextuality

- Context: a set of compatible measurements

$$C = \{M_1, \ldots, M_n\}$$

- Ontological representations respect compatibility

$$\xi_C(\lambda) = \prod_{M \in C} \xi_M(\lambda)$$

- . . . and are context-independent; e.g. if $M \in C, C'$

$$\xi_{M^{(C)}} = \xi_{M^{(C')}}$$

Classicality: Spekkens

Non-contextuality of **transformations**

- A context is a convex decomposition of a fixed transformation, e.g.

$$T = \frac{1}{2}U_a + \frac{1}{2}U_A \qquad (C)$$

$$T = \frac{1}{3}U_a + \frac{1}{3}U_b + \frac{1}{3}U_c\,. \qquad (C')$$

- Ontological representations respect convex decompositions, e.g.

$$f_T = \frac{1}{2}f_{U_a} + \frac{1}{2}f_{U_A} = \frac{1}{3}f_{U_a} + \frac{1}{3}f_{U_b} + \frac{1}{3}f_{U_c}$$

- . . . and are context-independent

$$f_{U_a^{(C)}} = f_{U_a^{(C')}}$$

Non-contextuality of **transformations in sequential contexts**

- A context is a sequence $U_n \circ U_{n-1} \circ \cdots \circ U_1$
- Ontological representations respect sequentiality

$$f_{U_n \circ \cdots \circ U_1} = f_{U_n} \circ \cdots \circ f_{U_1}$$

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Classicality III

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Example: Quantum advantage in shallow circuits*

- Contextuality is *necessary* for advantage
- *Sufficient* for a weaker kind of advantage

*Bravyi, Gossett, Kitaev '17

Proposal for generalised non-contextuality

Basic components of an operational physical theory:
preparations (P), transformations (T), measurements (M)

- A context is a composition/combination of basic components
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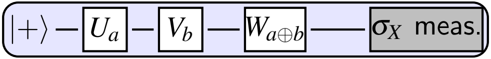
	Components	**Composition**
Locality	M	$\times$
Non-contextuality *(BKS)*	M	$\times$
Measurement NC *(Spekkens)*	M	$\otimes$
Preparation NC *(Spekkens)*	P	$+_\lambda$
Preparation Independence *(PBR)*	P	$\times$
Subsystem Condition *(SM)*	P	$\otimes$
Transformation NC *(Spekkens)*	T	$+_\lambda$
Seq. Transformation NC	T	$\circ$

An appropriate ontology



- Artificial problem: boost $\oplus L \longrightarrow P$
- Trivial without matching restriction on ontological models

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*l*2-ontological models:

Ontic states	$\Lambda = (\mathbb{Z}_2)^n$
Transformations	$f_U(\boldsymbol{\lambda}) = \boldsymbol{\lambda} \oplus \boldsymbol{u}$
Measurements	$\xi_M(\boldsymbol{\lambda}) = \boldsymbol{\lambda} \cdot \boldsymbol{m} \oplus m'$

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(Wlog, project onto relevant copies of $\mathbb{Z}_2$)

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- i.e. Bits, single bit reversible transformations, CNOTs, parity
- Crucially, no C-swaps, or Toffolis, etc.

Parity proof of contextuality

Ontological realisation of the protocol requires the following equations to be satisfied

$$
\begin{aligned}
(\boldsymbol{\lambda} \oplus \boldsymbol{u}_0 \oplus \boldsymbol{v}_0 \oplus \boldsymbol{w}_0) \cdot \mathbf{1} &= 0 \\
(\boldsymbol{\lambda} \oplus \boldsymbol{u}_0 \oplus \boldsymbol{v}_1 \oplus \boldsymbol{w}_1) \cdot \mathbf{1} &= 1 \\
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- System of equations is not jointly satisfiable

- 1st equation: overall number of 1's in $\boldsymbol{\lambda}, \boldsymbol{u}_0, \boldsymbol{v}_0, \boldsymbol{w}_0$ is **even**
- 2nd equation: overall number of 1's in $\boldsymbol{\lambda}, \boldsymbol{u}_0, \boldsymbol{v}_1, \boldsymbol{w}_1$ is **odd**, etc.
- Sum RHS: **odd**
- Sum LHS: **even** (each vector appears even number of times)

Some results

$$|+\rangle - \boxed{U_a} - \boxed{V_b} - \boxed{W_{a\oplus b}} - \boxed{\sigma_X \text{ meas.}}$$

Result 1

DKK protocol $\longrightarrow$ (AvN) sequential transformation contextuality

Cf. *Anders and Browne* protocol using BKS contextuality

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Cf. *Anders and Browne* protocol using BKS contextuality

$l2$-TBQCs: $\oplus L$ control computer with access to a **resource**

- Fixed preparation
- Controlled unitaries
- Fixed 2-outcome measurement

Some results

$$|+\rangle \;-\; \boxed{U_a} \;-\; \boxed{V_b} \;-\; \boxed{W_{a\oplus b}} \;-\!-\; \boxed{\sigma_X \text{ meas.}}$$

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Result 2

Any deterministic non-linear $l2$-TBQC $\longrightarrow$
(strong) sequential transformation contextuality

Cf. *Raussendorf* theorem for BKS contextuality

Quantum advantage for probabilistic computation

(joint with Dan Browne, Lorenzo Catani, Luciana Henaut, Anna Pappa)

$|+\rangle$ — $\boxed{U_a}$ — $\boxed{V_b}$ —— $\boxed{\sigma_X \text{ meas.}}$

- Task: compute $a \otimes_2 b$
- Maximise success probability
- Tsirelson bound for qubits
- Similar for qutrits, with $\otimes_3$, etc.
- Dimensional witness!

	$p^{\text{max}}_{\text{success}}$
bit	0.75
Spekkens toy bit	0.75
stabiliser qubit	0.75
qubit	**0.85**
qutrit	1

Tsirelson bound

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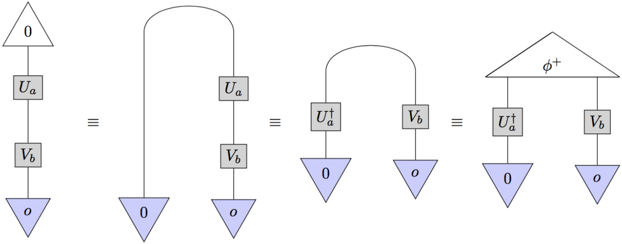

- Correspondence with Tsirelson in CHSH scenario

Empirical models

Empirical model: for each context C, a distribution e_C over possible outcomes

$$e = \{e_C\}$$

Cf. Abramsky and Brandenburger

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E.g. CHSH strategy

context			outcome	
	a	b	$o = 0$	$o = 1$
C_0	0	0	$3/4$	$1/4$
C_1	0	1	$1/4$	$3/4$
C_2	1	0	$1/4$	$3/4$
C_3	1	1	0	1

$$e_{C_0} = (3/4, 1/4)$$
$$e_{C_1} = (1/4, 3/4)$$
$$e_{C_2} = (1/4, 3/4)$$
$$e_{C_3} = (0, 1)$$

Quantifying contextuality

Empirical model: for each context C, a distribution e_C over possible outcomes

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Non-contextual fraction $\mathsf{NCF}(e)$: max ω over all decompositions

$$e = \omega e^{\mathsf{NC}} + (1-\omega)e'$$

s.t. e^{NC} is non-contextual

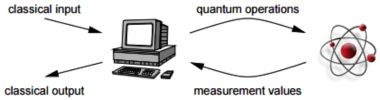
Contextual fraction $\mathsf{CF}(e)$: $1 - \mathsf{NCF}(e)$

- $\mathsf{CF}(e), \mathsf{NCF}(e) \in [0,1]$

Cf. Contextual fraction as a measure of contextuality. *Abramsky, Barbosa, SM*

Advantage, contextuality and hardness

Task: Compute $f : (\mathbb{Z}_2)^r \to \mathbb{Z}_2$ by $l2$-TBQC with resource e



Result 3

Let ε be the failure probability. Then

$$\varepsilon \geq \mathsf{NCF}(e)\,\nu(f)$$

- ε quantifies error
- $\mathsf{NCF}(e)$ quantifies (non)contextuality
- $\nu(f)$ quantifies degree of nonlinearity of f
- Previous results follow from this one

Cf. Contextual fraction as a measure of contextuality. *Abramsky, Barbosa, SM*

Classical erasure

(joint with Dan Browne, Lorenzo Catani, Luciana Henaut, Anna Pappa)

$$|+\rangle - \boxed{U_a} - \boxed{V_b} - \boxed{\sigma_X \text{ meas.}}$$

- Classically, can compute $a \otimes_2 b$ with $l2$-operations and **erasure**

$$U_0 = I \quad U_1 = \mathsf{NOT} \quad V_0 = \mathsf{RESET}_0 \quad V_1 = I$$

- Undesirable for an ontological model

$$f_I \neq I$$

- Expected **erasure cost** per run, averaged over pairs of inputs, to compute a function g, with 1- and 2-bit gates

$$\tilde{\nu}(g)$$

Contextuality-erasure tradeoff

(joint with Dan Browne, Lorenzo Catani, Luciana Henaut, Anna Pappa)

$$|+\rangle - \boxed{U_a} - \boxed{V_b} —— \boxed{\sigma_X \text{ meas.}}$$

Landauer's Principle

Erasure of a bit results in an entropy increase of at least $kT\ln 2$ in the non-information-bearing degrees of freedom of the system

Verification of quantumness

If for n runs, uniformly random inputs, entropy increase ΔS,

$$\bar{\varepsilon} \geq \left[\mathsf{NCF}(e) - \frac{\Delta S}{nkT\ln 2}\right]\tilde{\nu}(g)$$

$$\Delta S \geq \left[1 - \mathsf{CF}(e) - \frac{\bar{\varepsilon}}{\tilde{\nu}(g)}\right] nkT\ln 2$$

$$\mathsf{CF}(e) \geq 1 - \frac{\bar{\varepsilon}}{\tilde{\nu}(g)} - \frac{\Delta S}{nkT\ln 2}$$

Conclusion

- Novel way to be non-classical:
  sequential transformation contextuality
- Quantifiably relates to **quantum advantage** in $l2$-TBQCs
- Results parallel *Anders and Browne, Raussendorf, etc.* for BKS contextuality
- Available to **single qubits**
- Dimensional, irreversibility & quantumness **witnesses**

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Some open questions

- Is it exhibited in less restricted models?
  E.g. of indefinite causal structures
- Where else does it play a role?
  E.g. other single qubit advantages (Knill-Laflamme, Galvão), other informatic tasks, universal QC (similar to Howard et al.)?
- Resource-theoretic treatment
- Experimental tests