Simulations of quantum resources and the degrees of contextuality

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Joint work with Rui Soares Barbosa, Martti Karvonen and Shane Mansfield

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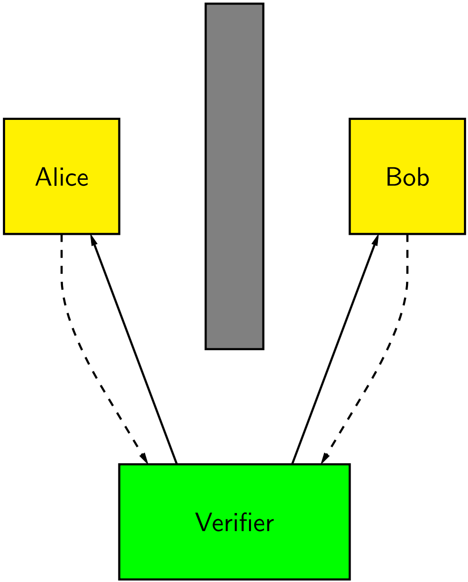
We will give an overview. The talk by Martti will go further into details of results.

Alice-Bob games



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A table of conditional probabilities $p(a, b|x, y)$ defines a **probabilistic strategy** for this game. The **success probability** for this strategy is:

$$1/4[p(a = b|x = 0, y = 0) + p(a = b|x = 0, y = 1) + p(a = b|x = 1, y = 0)$$

$$+p(a \neq b|x = 1, y = 1)]$$

A Strategy for the Alice-Bob game

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Example: The Bell Model

A	B	(0, 0)	(1, 0)	(0, 1)	(1, 1)
0	0	1/2	0	0	1/2
0	1	3/8	1/8	1/8	3/8
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The Bell table exceeds this bound. Since it is **quantum realizable** using an entangled pair of qubits, it shows that quantum resources yield a **quantum advantage** in an information-processing task.

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This translates into 6 linear equations over $\mathbb{Z}_2$:

$$A \oplus B \oplus C = 0 \qquad A \oplus D \oplus G = 0$$
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Of course, the equations are not satisfiable in $\mathbb{Z}_2$!

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Mermin's construction shows that there is a **quantum perfect strategy** for the magic square.

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We will hear more about these ideas in talks by Nadish de Silva and Sivert Aasnaess.

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  Examples: (in)computability, degrees of unsolvability etc.
  Complexity classes.

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In the latter case, we have an **abstract simplicial complex**.

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E.g. $s|_{\{a\}} = \{a \mapsto 0\}$.

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(The coherent relationship is functoriality!)

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Compatibility is a general form of the important physical principle of **No-Signalling**; this general form is also known as **No Disturbance**.

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Thus contextuality arises where we have a family of data which is **locally consistent** but **globally inconsistent**.

Contextuality defined

An empirical model $\{e_C\}_{C \in \mathcal{M}}$ on a measurement scenario $(X, \mathcal{M}, O)$ is **non-contextual** if there is a distribution $d \in \mathsf{Prob}(\prod_{x \in X} O_x)$ such that, for all $C \in \mathcal{M}$:

$$d|_C \;=\; e_C.$$

That is, we can glue all the local information together into a global consistent description from which the local information can be recovered.

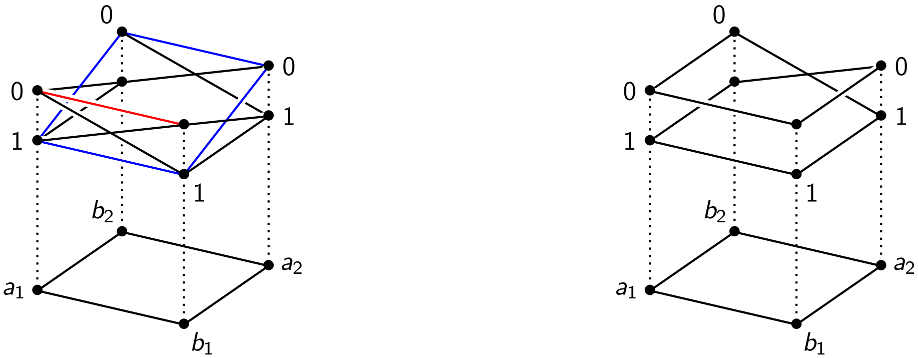
We call such a d a **global section**.

If no such global section exists, the empirical model is **contextual**.

Thus contextuality arises where we have a family of data which is **locally consistent** but **globally inconsistent**.

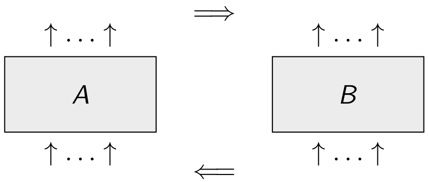
The import of Bell's theorem and similar results is that there are empirical models arising from quantum mechanics which are contextual.

Visualizing Contextuality



The Hardy table and the PR box as bundles

Simulations



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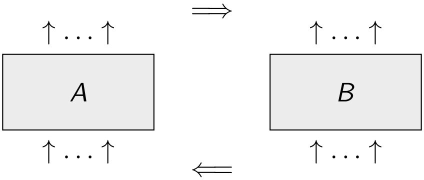

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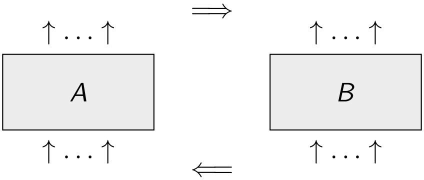

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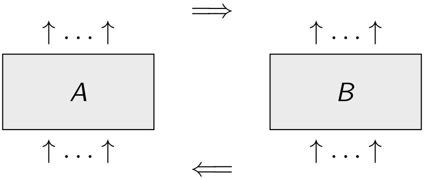

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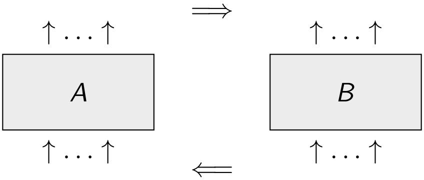

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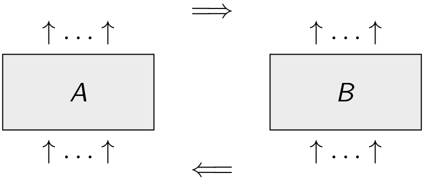

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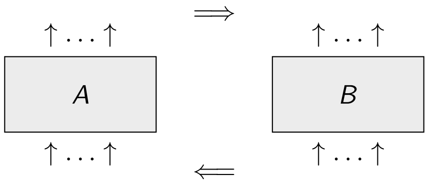

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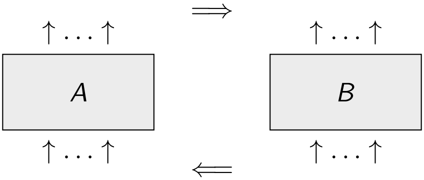

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- Since the map on measurements needn't be surjective, it allows for simulating B using only part of A (**restrictions**).
- Since the map on outcomes needn't be injective, it allows for **coarse-graining** of outcomes.

Simulations formalized

Note the basic character of the fact that mappings of inputs go backward, of outputs forward:

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Note that the map π must be simplicial, since it must map compatible sets of measurements (contexts) in Δ to compatible sets of measurements (contexts) in Σ.

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the push-forward of the probability measure $e_{\pi(C)}$ along the map

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This gives a category **Emp**, with objects $e : (X, \Sigma, O)$, and morphisms $(\pi, h) : e \to e'$ such that $(\pi, h)^*(e) = e'$.

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Formally, we construct a **comonad** MP on the category of empirical models, where $\mathsf{MP}(e : (X, \Sigma, O))$ is the empirical model obtained by taking all measurement protocols over the given scenario.

The MP construction

Intuitively, given a scenario $\mathbf{X} = (X, \Sigma, O)$ we build a new scenario $\mathrm{MP}(\mathbf{X})$, where:

- measurements are the measurement protocols on $\mathbf{X}$
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Given empirical models e and d, a **simulation** of e by d is a map

$$d \otimes c \to e$$

in $\mathbf{Emp}_{\mathrm{MP}}$, the coKleisli category of MP, *i.e.* a map

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The use of the noncontextual model c is to allow for classical randomness in the simulation.

We denote the existence of a simulation of e by d as $d \rightsquigarrow e$, read "d simulates e".

Results

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- Using these normal forms, we obtain one of our main results: we show that the algebraic notion of convertibility coincides with the existence of a simulation morphism.

- We also prove a number of further results, including a form of no-cloning theorem at the abstract level of simulations.

Degrees of Contextuality

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This is just a preliminary observation. Many questions arise, and there are natural variations and refinements.

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This can be formulated as characterizing those scenarios such that every model over them can be simulated by a model over the empty scenario.

More generally, we can ask for conditions on scenarios (X, Σ, O) and (Y, Δ, P) such that every empirical model over (Y, Δ, P) can be simulated by some empirical model over (X, Σ, O).