Measures of Contextuality and Noncontextuality

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Introduction

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$$\max_{(\iota_1,\ldots,\iota_k)\in\{-1,1\}^n:\prod_{i=1}^n \iota_i=-1} \sum_{i=1}^{n} \iota_i \left\langle R_i^i R_{i\oplus 1}^i \right\rangle - n + 2 - \sum_{i=1}^{n} \left| \left\langle R_i^i \right\rangle - \left\langle R_i^{i\ominus 1} \right\rangle \right| > 0.$$

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Couplings

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In particular, for any $\alpha \in I$,

$$R_\alpha \sim S_\alpha.$$

Agglutination property of JD relation

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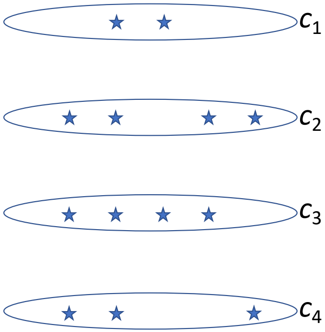
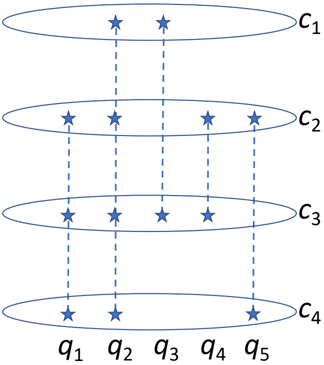
$$\text{If } \mathbb{A} \cap \mathbb{B} \neq \emptyset, \text{ then}$$

$$\mathsf{JD}(\mathbb{A}) \;\&\; \mathsf{JD}(\mathbb{B}) \Longrightarrow \mathsf{JD}(\mathbb{A} \cup \mathbb{B})$$

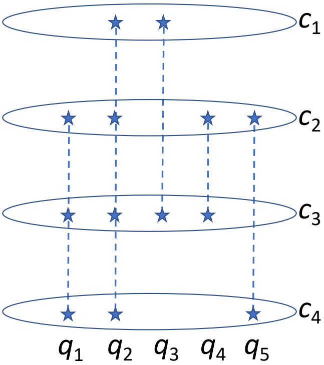
Bunches — contexts



Connections — contents

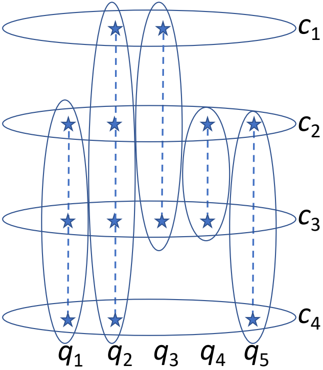


System of random variables



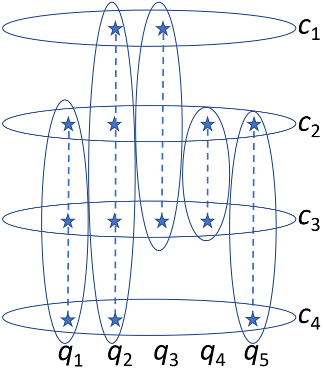
Terminology: (in)consistently connected system

Coupling for connections (multimaximal)



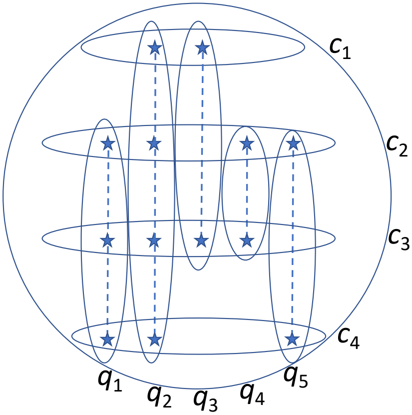
Multimaximal couplings: for any two variables in each connection, the probability of them being equal is maximal possible

Coupling for connections (multimaximal)



With all variables dichotomous,
the multimaximal couplings exists and is unique for every connection

Overall coupling: for bunches and for connection couplings



Overall coupling, respecting bunches and connection couplings: iff such a coupling exists, the system is noncontextual.

A system as a probability vector

	R_2^1	R_3^1			$p^1\left(R_i^1 = \pm 1 : i = 2, 3\right)$
R_1^2	R_2^2		R_4^2	R_5^2	$p^2\left(R_i^2 = \pm 1 : i = 1, 2, 4, 5\right)$
R_1^3	R_2^3	R_3^3	R_4^3		$p^3\left(R_i^3 = \pm 1 : i = 1, 2, 3, 4\right)$
R_1^4	R_2^4			R_5^4	$p^4\left(R_i^4 = \pm 1 : i = 1, 2, 5\right)$

$$\mathbf{p}^*_{(bunch)} = \left(\mathbf{p}^1, \mathbf{p}^2, \mathbf{p}^3, \mathbf{p}^4\right)^\top$$

A system as a probability vector

	V_2^1		
V_1^2	V_2^2	$\cdots$	V_5^2
V_1^3	V_2^3	$\cdots$	
V_1^4	V_2^4	$\cdots$	V_5^4
$p_1 \begin{pmatrix} V_1^j = \pm 1 : \\ j = 2,3,4 \end{pmatrix}$	$p_2 \begin{pmatrix} V_1^j = \pm 1 : \\ j = 1,2,3,4 \end{pmatrix}$	$\cdots$	$p_5 \begin{pmatrix} V_1^j = \pm 1 : \\ j = 2,5 \end{pmatrix}$

$$\mathbf{p}^*_{(connect)} = \left(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4, \mathbf{p}_5\right)^\top$$

A system as a probability vector

$$\mathbf{p}^*_{(full)} = \begin{pmatrix} \mathbf{p}^*_{(bunch)} \\ \mathbf{p}^*_{(connect)} \end{pmatrix}$$

Reduced vector of probabilities

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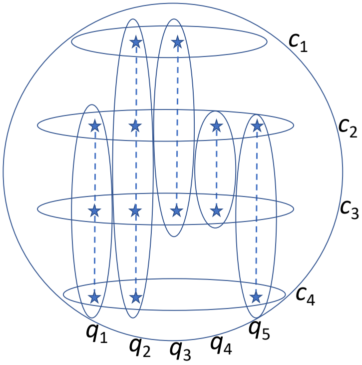
$$\mathbf{p}^*_{(full)} = \begin{pmatrix} \mathbf{p}^*_{(bunch)} \\ \mathbf{p}^*_{(connect)} \end{pmatrix} \longleftrightarrow \begin{pmatrix} \mathbf{p}^*_{\mathbf{low}} \\ \mathbf{p}^*_{\mathbf{bunch}} \\ \mathbf{p}^*_{\mathbf{connect}} \end{pmatrix} = p^*_{reduced}$$



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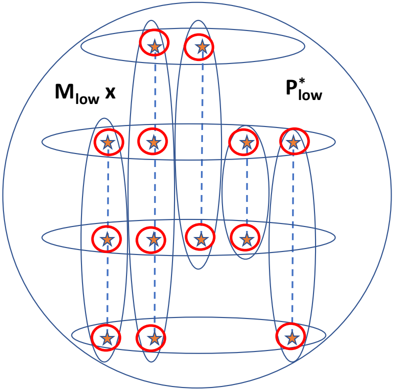

$\mathbf{p_{low}}$ is vector of the 0-marginal and 1-margi

$$e.g., \Pr\left[R_2^1 = 1\right]$$

0-marginal equals 1

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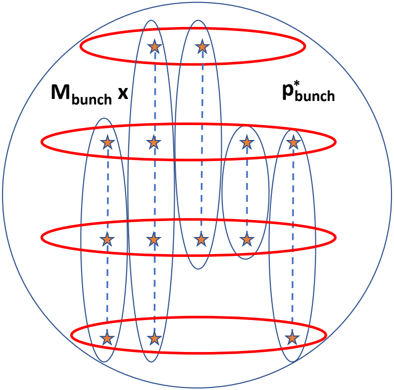

$\mathbf{p_{bunch}}$ is vector of bunch marginals of order >1

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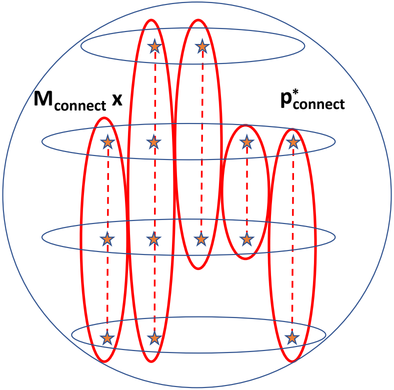

$\mathbf{p}_{\mathbf{connect}}$ is vector of connection marginals of order >1

$$e.g., \Pr\left[V_2^1 = V_2^2 = V_2^3 = 1\right],$$

where V is the multimaximal coupling

Overall coupling's distribution: probability vector

	R_2^1	R_3^1			c^1
R_1^2	R_2^2		R_4^2	R_5^2	c^2
R_1^3	R_2^3	R_3^3	R_4^3		c^3
R_1^4	R_2^4			R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

$\longrightarrow$

	S_2^1	S_3^1			c^1
S_1^2	S_2^2		S_4^2	S_5^2	c^2
S_1^3	S_2^3	S_3^3	S_4^3		c^3
S_1^4	S_2^4			S_5^4	c^4
q_1	q_2	q_3	q_4	q_5	S

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S_1^4	S_2^4			S_5^4	c^4
q_1	q_2	q_3	q_4	q_5	S

$$\mathbf{x} = \Pr\left(S_q^c = \pm 1 : q \prec c\right).$$

Measures of contextuality: Preamble

Fact

A system is noncontextual iff

$$\begin{pmatrix} \mathbf{M}_{\mathbf{low}} \\ \mathbf{M}_{\mathbf{bunch}} \\ \mathbf{M}_{\mathbf{connect}} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{p}^*_{\mathbf{low}} \\ \mathbf{p}^*_{\mathbf{bunch}} \\ \mathbf{p}^*_{\mathbf{connect}} \end{pmatrix},$$

where $\mathbf{x}$ *is vector of probabilities for all possible values of a coupling* S*.*

Measures of contextuality Type I

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1. Consider all $\begin{pmatrix} \mathbf{p}_{\mathbf{low}} \\ \mathbf{p}_{\mathbf{bunch}} \\ \mathbf{p}_{\mathbf{connect}} \end{pmatrix}$ that can be obtained as

$$\begin{pmatrix} \mathbf{p}_{\mathbf{low}} \\ \mathbf{p}_{\mathbf{bunch}} \\ \mathbf{p}_{\mathbf{connect}} \end{pmatrix} = \begin{pmatrix} \mathbf{M}_{\mathbf{low}} \\ \mathbf{M}_{\mathbf{bunch}} \\ \mathbf{M}_{\mathbf{connect}} \end{pmatrix} \mathbf{x},$$

   under the constraint that

$$\mathbf{p}_{\mathbf{low}} = \mathbf{p}^{*}_{\mathbf{low}}$$

   and one of the two,

$$\mathbf{p}_{\mathbf{bunch}} = \mathbf{p}^{*}_{\mathbf{bunch}} \quad or \quad \mathbf{p}_{\mathbf{connect}} = \mathbf{p}^{*}_{\mathbf{connect}}.$$

2. See how close the remaining (free to vary) part can get to the observed/computed one.

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$$\mathbf{p}_{\mathbf{bunch}} = \mathbf{p}^{*}_{\mathbf{bunch}} \quad \Longrightarrow \quad \mathbf{p}_{\mathbf{connect}} \rightsquigarrow \mathbf{p}^{*}_{\mathbf{connect}}.$$

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   and one of the two,

$$\mathbf{p}_{\mathbf{bunch}} \rightsquigarrow \mathbf{p}^*_{\mathbf{bunch}} \quad \Longleftarrow \quad \mathbf{p}_{\mathbf{connect}} = \mathbf{p}^*_{\mathbf{connect}}.$$

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Measures of contextualityType II

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1. Consider some class $\mathbb{Y}$ of vectors that includes all probability distributions $\mathbf{x}$ as a proper subset, and such that

$$\begin{pmatrix} \mathbf{M}_{\mathbf{low}} \\ \mathbf{M}_{\mathbf{bunch}} \\ \mathbf{M}_{\mathbf{connect}} \end{pmatrix} \mathbf{y} = \begin{pmatrix} \mathbf{p}^*_{\mathbf{low}} \\ \mathbf{p}^*_{\mathbf{bunch}} \\ \mathbf{p}^*_{\mathbf{connect}} \end{pmatrix}$$

always has a solution in $\mathbf{y} \in \mathbb{Y}$.

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   always has a solution in $\mathbf{y} \in \mathbb{Y}$.

2. See how close such a $\mathbf{y}$ can get to the set of proper vectors $\mathbf{x}$.

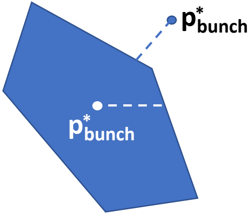
A measure of contextuality (type I) extendable into a measure of noncontextuality

Noncontextuality polytope

$$\mathbb{P} = \left\{ \mathbf{p}_{\mathsf{bunch}} \colon \mathbf{p}_{\mathbf{bunch}} = \mathbf{M}_{\mathbf{bunch}} \mathbf{x} \mid \begin{pmatrix} \mathbf{M}_{\mathbf{low}} \\ \mathbf{M}_{\mathbf{connect}} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{p}^*_{\mathbf{low}} \\ \mathbf{p}^*_{\mathbf{connect}} \end{pmatrix}, \mathbf{x} \geq 0 \right.$$

.

1. Contextuality: L_1-distance between contextual $\mathbf{p}^*_{\mathbf{bunch}}$ (outside) and $\mathbb{P}$.
2. Noncontextuality: L_1-distance between noncontextual $\mathbf{p}^*_{\mathbf{bunch}}$ (inside) and surface of $\mathbb{P}$.



A measure of contextuality (type I) extendable into a measure of noncontextuality

find	minimizing	subject to
$\mathbf{x}$	$\mathbf{1}\cdot\mathbf{d}$	$-\mathbf{d}\leq\mathbf{p}^{*}_{\mathbf{bunch}}-\mathbf{M}_{\mathbf{bunch}}\mathbf{x}\leq\mathbf{d}$
		$\mathbf{x},\mathbf{d}\geq 0$
		$\mathbf{M}_{\mathbf{low}}\mathbf{x}=\mathbf{p}^{*}_{\mathbf{low}}$
		$\mathbf{M}_{\mathbf{connect}}\mathbf{x}=\mathbf{p}^{*}_{\mathbf{connect}}$

$$\mathsf{CNT}_2=\left\Vert \mathbf{p}^{*}-\mathbf{M}\mathbf{x}^{*}\right\Vert ,\mathsf{NCNT}_2=\min_{i=1,\ldots,K}\left\{ \min\left(d_{i}^{*+},d_{i}^{*-}\right)\right\}$$

find	maximizing	subject to
$d_i^{+},d_i^{-},\mathbf{x}$	d_i^{+}	$\mathbf{p_b}^{*}+d_i^{+}\mathbf{e}_i=\mathbf{M}_{\mathbf{bunch}}\mathbf{x}$
		$\mathbf{p_b}^{*}-d_i^{-}\mathbf{e}_i=\mathbf{M}_{\mathbf{bunch}}\mathbf{x}$
		$d_i^{+},d_i^{-},\mathbf{x}\geq 0$
		$\mathbf{M}_{\mathbf{low}}\mathbf{x}=\mathbf{p}^{*}_{\mathbf{low}}$
		$\mathbf{M}_{\mathbf{connect}}\mathbf{x}=\mathbf{p}^{*}_{\mathbf{connect}}$

A measure of contextuality (type I) not extendable into a measure of noncontextuality

Feasibility polytope

$$\mathbb{F} = \left\{ \mathbf{p}_{\mathbf{connect}} : \mathbf{p}_{\mathbf{connect}} = \mathbf{M}_{\mathbf{connect}} \mathbf{x} \mid \begin{pmatrix} \mathbf{M}_{\mathbf{low}} \\ \mathbf{M}_{\mathbf{bunch}} \end{pmatrix} \mathbf{x} = \begin{pmatrix} \mathbf{p}^*_{\mathbf{low}} \\ \mathbf{p}^*_{\mathbf{bunch}} \end{pmatrix}, \mathbf{x} \geq 0 \right\}$$

.

1. Contextuality, CNT_1: L_1-distance between contextual $\mathbf{p}^*_{\mathbf{connect}}$ (outside) and $\mathbb{F}$.
2. Noncontextuality: not defined, noncontextual vectors stick to the surface.

Type II measures of contextuality are not extendable into a measure of noncontextuality

$$\mathbb{P}_{\mathbf{neg}} = \left\{ \mathbf{y} : \begin{pmatrix} \mathbf{M_{low}} \\ \mathbf{M_{bunch}} \\ \mathbf{M_{connect}} \end{pmatrix} \mathbf{y} = \begin{pmatrix} \mathbf{p}^*_{\mathbf{low}} \\ \mathbf{p}^*_{\mathbf{bunch}} \\ \mathbf{p}^*_{\mathbf{connect}} \end{pmatrix} \right\}$$

.

1. Contextuality, CNT_3: $\min_{\mathbf{y}\in\mathbb{Y}} (\mathbf{1} \cdot |\mathbf{y}|) - 1$.
2. Noncontextuality: not defined, noncontextual vectors correspond to $\min_{\mathbf{y}\in\mathbb{Y}} (\mathbf{1} \cdot |\mathbf{y}|) = 1$.

(Generalized) Contextual Fraction is a Type II-like measure

$$\mathbb{Z} = \{\mathbf{z} : \mathbf{M}_{(full)}\mathbf{z} \leq \mathbf{p}^*_{(full)}, \mathbf{z} \geq 0, \mathbf{1} \cdot \mathbf{z} \leq 1\}$$

.

1. Contextuality: CNTF $= 1 - \max_{\mathbf{z} \in \mathbb{Z}} (\mathbf{1} \cdot \mathbf{z})$.
2. Noncontextuality: not defined, noncontextual vectors correspond to $\max_{\mathbf{z} \in \mathbb{Z}} (\mathbf{1} \cdot \mathbf{z}) = 1$.

Conclusion

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- Measures of noncontextuality may be of interest, especially outside QM (but also in QM).

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- Measures of noncontextuality may be of interest, especially outside QM (but also in QM).
- The extendability into a noncontextuality measure may be a selection criterion among otherwise reasonable measures of contextuality.