Causal Model-Based Contextuality

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Overview

- Extending contextuality to inconsistently connected systems
- Multimaximal couplings of contextuality-by-default (CbD; Dzhafarov & Kujala, 2017)
  - Physical interpretation? Implications for theory?
- Probabilistic causal models (Cavalcanti, 2018)
  Generalizes original hidden-variables approach (Bell, 1964; Fine, 1982; Einstein, Podolsky, & Rosen, 1935; Kochen & Specker, 1967)
- Prove equivalence of 3 definitions of extended contextuality, including CbD's

Setup (adapted from Kujala et al., 2015)

- **Measurement system** $M = \left\{M_q^c : q \prec c\right\}$
  - Observables (contents) $\mathcal{Q} = \{q\}$
  - Contexts $\mathcal{C} = \{c\}$
  - Outcome space $\mathcal{O}_q$ for each q (assumed finite)
  - Measurements M_q^c for $q \prec c$
  - Distributions μ_c on $\prod_{q \prec c} \mathcal{O}_q$ for $M^c = \left\{M_q^c : q \prec c\right\}$
- **Standard contextuality**
  - Consistently connected: $M_q^c \sim M_q^{c'}$ for all $c, c' \succ q$
  - No global distribution on $\prod_{q \in \mathcal{Q}} \mathcal{O}_q$ compatible with all μ_c

Contextuality with Inconsistent Connectedness

- Motivation (Kujala et al., 2015; Dzhafarov & Kujala, 2015, 2016)
  - Difficult to avoid in experimental practice
  - Generally present in finite samples
  - Essential to certain systems of interest
- CbD approach (Kujala et al., 2015; Dzhafarov & Kujala, 2016, 2017)
  - Influence of context beyond direct influence
  - $M_q^c, M_q^{c'}$ differ more than mandated by their distributions
  - **No-conspiracy principle** (Cervantes & Dzhafarov, 2018): Direct influences do not cancel out in empirical distributions

Present Contribution

- Define direct influence within probabilistic causal models
  - Theory-dependent, distinct from inconsistent connectedness
- Formalize no-conspiracy principle
  - No **crossed influences**: Opposite direct influences for different latent states of system
- Prove equivalence of three forms of contextuality (formalizing 1 & 2)
  1. Modeling full system requires more direct influence than modeling each observable
  2. Modeling system requires violating no-conspiracy principle
  3. CbD definition based on multimaximal couplings
  - All reduce to standard contextuality for consistently connected systems

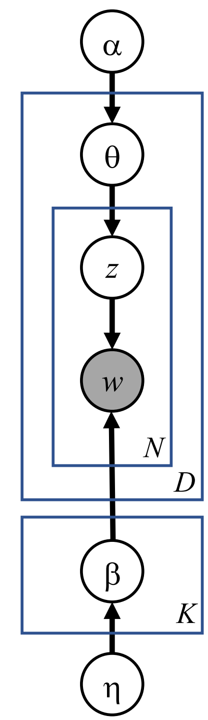
Probabilistic Causal Models

- Powerful tool from statistics, machine learning, psychology (Jordan, 1999; Pearl, 2000; Tenenbaum et al., 2011)

  $\mathcal{X} = \{X_i\} \quad \forall i, Pa(X_i) \subset \mathcal{X}$

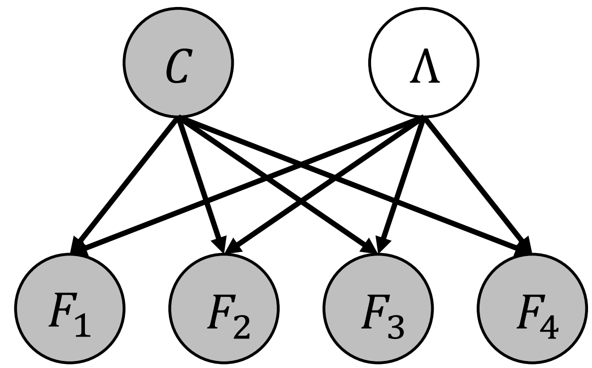
  $\Pr[\mathcal{X}] = \prod_i \Pr[X_i | Pa(X_i)]$

  - Acyclic directed graph, $X_j \in Pa(X_i)$
  - Joint distribution factors into causal dependencies
  - Includes observed and latent (hidden) variables
- Natural extension of hidden-variables models
  - Valuable framework for studying contextuality (Cavalcanti, 2018)



LDA: Blei, Ng, & Jordan (2003)

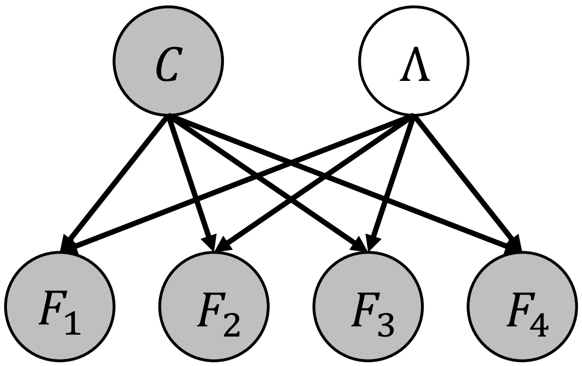
Canonical Causal Models

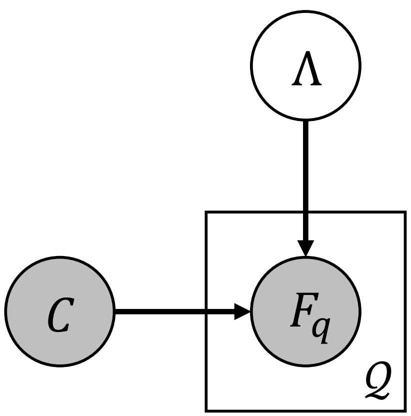


- Context, latent state, and observables
  - $\mathcal{M} = \left(C, \Lambda, \{F_q\}\right)$
  - $Pa(\Lambda) = Pa(C) = \emptyset$
  - $Pa\left(F_q\right) = \{C, \Lambda\}$
  - C treated as index variable
- Deterministic observables (Fine, 1982)
  - $F_q(c, \lambda) \in \mathcal{O}_q$
  - No loss of expressive power: push stochasticity into Λ
- Model $\mathcal{M}$ for a system M (cf. couplings in CbD; Kujala et al., 2015)
  - $\Pr\left[\{F_q : q \prec c\} \middle| C = c\right] = \mu_c$ for all c

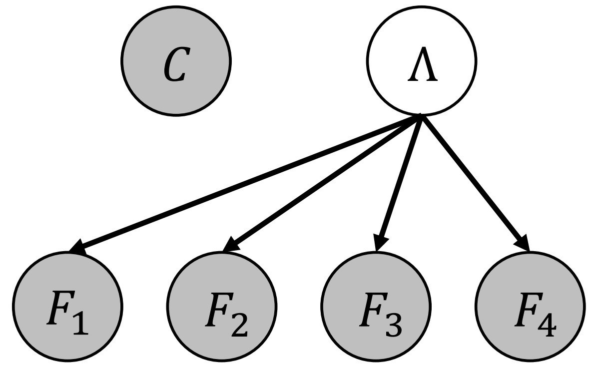
Canonical Causal Models

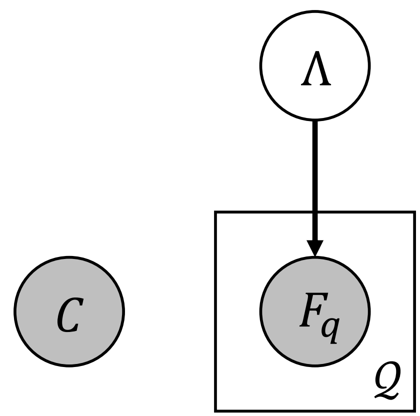
General





Context-free





Canonical Causal Models and Standard Contextuality

- **Proposition 1**. For any measurement system M, there exists a canonical causal model $\mathcal{M}$ that is a model for M.

- **Proposition 2 (Fine, 1982)**. A consistently connected measurement system is noncontextual iff there exists a context-free model of that system.

- **Proposition 3**. A measurement system M is consistently connected iff there exist context-free models for the single-observable subsystems $M_q = \{M_q^c : q \prec c\}$ for all q.

Direct Influence in Causal Models

- **Direct influence:** Given $\mathcal{M}$, q, and $\{c, c'\} \succ q$:

$$\Delta_{c,c'}(F_q) = \Pr\left[\{\lambda: F_q(\lambda, c) \neq F_q(\lambda, c')\}\right]$$

  - Probability of latent state for which context change affects measurement outcome
- **Crossed influences**

$$\Pr\left[\{\lambda: F_q(\lambda, c) = v, F_q(\lambda, c') \neq v\}\right] > 0$$
$$\Pr\left[\{\lambda: F_q(\lambda, c) \neq v, F_q(\lambda, c') = v\}\right] > 0$$

- **Aligned model**
  - No crossed influences, for any q, $v \in \mathcal{O}_q$, $\{c, c'\} \succ q$
  - Formalizes Cervantes & Dzhafarov's (2018) no-conspiracy principle

Example: Popescu-Rohrlich Box

M

$\Pr\left[M_q^c = 1\right]$

	A_1	B_1	A_2	B_2
c_{11}	½	½		
c_{21}		½	½	
c_{22}			½	½
c_{12}	½			½

$\Pr\left[M_q^c = M_{q'}^c\right]$

q \ q'	B_1	B_2
A_1	1	0
A_2	1	1

$\mathcal{M}$

	λ_1	λ_2
$\Pr[\Lambda = \lambda]$	½	½
$F_{A_1}(c_{11}, \lambda)$	1	-1
$F_{A_1}(c_{12}, \lambda)$	1	-1
$F_{B_1}(c_{11}, \lambda)$	1	-1
$F_{B_1}(c_{21}, \lambda)$	-1	1
$F_{A_2}(c_{21}, \lambda)$	-1	1
$F_{A_2}(c_{22}, \lambda)$	-1	1
$F_{B_2}(c_{12}, \lambda)$	-1	1
$F_{B_2}(c_{22}, \lambda)$	-1	1

$\Delta_{c_{11},c_{12}}\left(F_{A_1}\right) = 0$

$\Delta_{c_{11},c_{21}}\left(F_{B_1}\right) = 1$

$\Delta_{c_{21},c_{22}}\left(F_{A_2}\right) = 0$

$\Delta_{c_{12},c_{22}}\left(F_{B_2}\right) = 0$

Model-based Contextuality

- **M-contextuality**
  - Given any model $\mathcal{M}$ for M, there exist $c, c' \succ q$ such that $\Delta_{c,c'}(F_q)$ is greater than necessary
- **M-noncontextuality**
  - $\mathcal{M}$ simultaneously minimizes $\Delta_{c,c'}(F_q)$ for all $c, c' \succ q$
- Formalizes 2nd notion of extended contextuality
  - Modeling full system requires more direct influence than modeling each observable

Model-based Contextuality

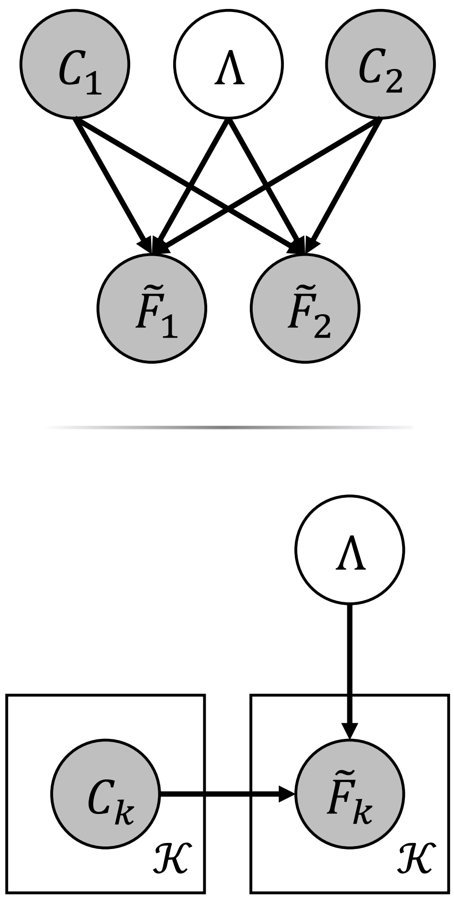
- **Theorem 1**. If M is consistently connected, then M is M-contextual iff it is contextual
  - Follows from Propositions 2 and 3
- **Proposition 4**. Given M and $q \prec c, c'$, the minimum direct influence over all models $\mathcal{M}$ for M is:

$$\min_{\mathcal{M}} \Delta_{c,c'}\left(F_q\right) = 1 - \sum_{v \in \mathcal{O}_q} \min\left\{\Pr\left[M_q^c = v\right], \Pr\left[M_q^{c'} = v\right]\right\}$$

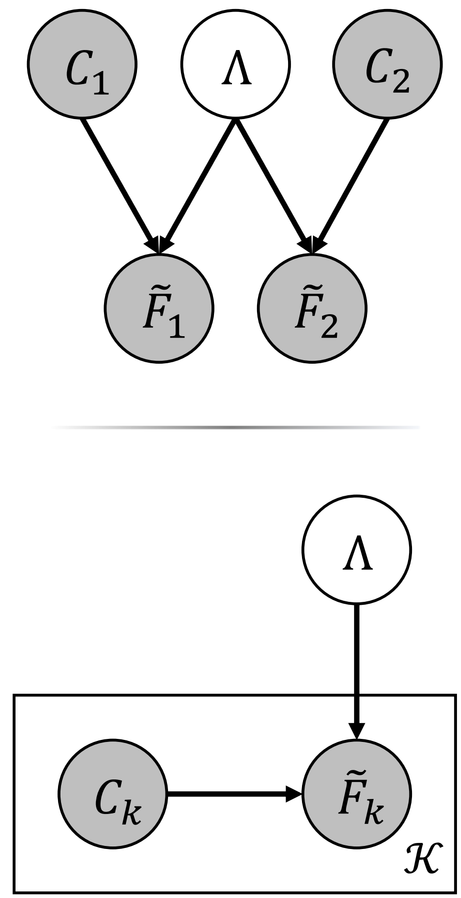
- **Theorem 2**. For any M, M is M-noncontextual iff it admits an aligned model
  - M-contextual iff all models contain crossed influences
  - Crossed influence implies $\Delta_{c,c'}\left(F_q\right)$ can be reduced

Partitionable Systems

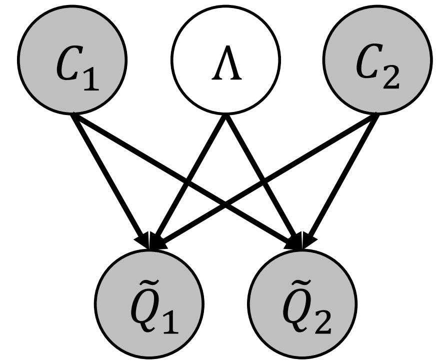
Signaling



No Signaling

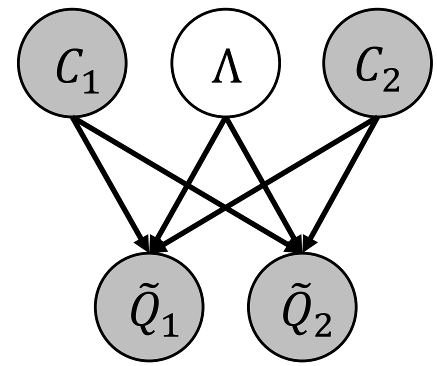


Partitionable Systems



- **Partitionable system**
  - Observers: $\mathcal{K} = \{k\}$
  - Partition of observables: $\mathcal{Q} = \bigsqcup_k \mathcal{Q}_k$
  - Factoring of contexts: $c = (c_k : k \in \mathcal{K})$, $c_k \in \mathcal{Q}_k$
- **Partitioned model**
  - $\widetilde{\mathcal{M}} = \left(\{C_k\}, \Lambda, \{\tilde{F}_k\}\right)$
  - $\tilde{F}_k(\lambda, c) = F_{c_k}(\lambda, c)$
- **Signaling**
  - $\widetilde{\Delta}_{c,c'}\left(\tilde{F}_k\right) = \Pr[\{\lambda : \tilde{F}_k(\lambda, c) \neq \tilde{F}_k(\lambda, c')\}] = \Delta_{c,c'}\left(F_{c_k}\right)$ for $c_k = c'_k$

Partitionable Systems



- Parallel results:
- Any partitionable system admits a partitioned model
- A consistently connected system is noncontextual iff it admits a model with no signaling, $\tilde{F}_k(\lambda, c) = \tilde{F}_k(\lambda, c_k)$
- M is M-noncontextual iff a model $\widetilde{\mathcal{M}}$ minimizes $\widetilde{\Delta}_{c,c'}(\tilde{F}_k)$ for all k, c, c' ($c_k = c'_k$)
- M is M-noncontextual iff it admits a model $\widetilde{\mathcal{M}}$ without crossed signals

M-contextuality and CbD-contextuality

- Probabilistic coupling (Kujala et al., 2015; Thorisson, 2000)
  - Jointly distributed $\left\{S_q^c : q \prec c\right\}$ with $S^c \sim M^c$ for all c
- Multimaximal coupling (Dzhafarov & Kujala, 2017)
  - Simultaneously maximizes $\Pr\left[S_q^c = S_q^{c'}\right]$ for all $q \prec c, c'$
- CbD-contextuality: Non-existence of multimaximal coupling
- **Theorem 3**. M-contextuality is equivalent to CbD-contextuality
  - Translation between models and couplings satisfying $\Delta_{c,c'}\left(F_q\right) = \Pr\left[S_q^c \neq S_q^{c'}\right]$
- **Theorem 4**. A system is CbD-noncontextual iff it admits an aligned model

A Challenge with Many-Valued Observables

- Cannot always simultaneously minimize all direct influences for a single observable, if not binary (Dzhafarov & Kujala, 2017):

	$v = 1$	$v = 2$	$v = 3$
M_1^1	0	½	½
M_1^2	½	0	½
M_1^3	½	½	0

  - M-contextual and CbD contextual
  - Remains so if dichotomized
- Violates intuition that contextuality has to do with what measurements are or are not made

A Challenge with Many-Valued Observables

- Single measure of direct influence for each observable?
  - $\Delta\left(F_q \middle| C\right) = \frac{1}{2}\sum_{c,c' \succ q} \Delta_{c,c'}\left(F_q\right)$
  - Violates intuition of monotonicity
- Definition of contextuality satisfying all of these axioms?
  - **Monotonicity** (Dzhafarov & Kujala, 2017): Subsytems of any noncontextual system are noncontextual
  - **Coarse-graining** (Dzhafarov & Kujala, 2017): Relabeling values of an observable in a non-contextual system, $\mathcal{O} \rightarrow \mathcal{O}'$, yields a noncontextual system
  - **Completeness**: A system with only one observable, or with all observables measured in every context, is noncontextual

Conclusions

- Extends hidden-variable approaches (Cavalcanti, 2018)
- Physical/theoretical interpretation of CbD-contextuality
  - Formalizes Cervantes & Dzhafarov's no-conspiracy principle
- Value in translation, beyond specific definitions
  - Might facilitate solution to problem of complete systems
- Naturally suited for real (finite) datasets
  - $\left\{M_q^{c,i}: q \prec c, i \leq n_c\right\}$ – numbers, not random variables
  - Standard model evaluation and inferential statistics
- Distinction between inconsistent connectedness and direct influence
  - Empirical vs. theoretical
  - Atmanspacher & Filk (2019): non-communicating signaling
  - Inescapable: contextuality only meaningful with some exclusions on direct influence (e.g. spacelike separation)