Negative probabilities

A new perspective?

Outline of the talk

1. Defensive remarks
2. Entropy of negative ‘probability’ distributions
3. Applications/speculations

A. ***Kolmogorov***: "Axiom III: A **non-negative real** number P(A) is attached to each set A of F. This number P(A) is called the probability of the event A".

R.P. ***Feynman***: "If a physical theory for calculating probabilities yields a **negative probability** for a given situation under certain assumed conditions, we need not conclude the theory is incorrect."

J.S. ***Bell***: "Unfortunately I cannot think of anything intelligent to say about **negative probability**, or indeed about the square circle. [There is a] very recent paper by Feynman…"

A. ***Aspect***: "I know that several smart physicists (among them Richard Feynman) have considered the **negative probabilities** as an issue to the EPR problem. As a simple-minded experimentalist, I can hardly accept such a solution."

"Although I expect that your attitude towards **negative probabilities** is very negative."
"Correct, **Abdus Salam**"

Signed measure

From Wikipedia, the free encyclopedia

In mathematics, **signed measure** is a generalization of the concept of measure by allowing it to have negative values.

Contents [hide]

Definition [edit]

There are two slightly different concepts of a signed measure, depending on whether or not one allows it to take infinite values. In research papers and advanced books signed measures are usually only allowed to take finite values, while undergraduate textbooks often allow them to take infinite values. To avoid confusion, this article will call these two cases "finite signed measures" and "extended signed measures".

Given a measurable space (X, Σ), that is, a set X with a sigma algebra Σ on it, an **extended signed measure** is a function

$$\mu : \Sigma \to \mathbb{R} \cup \{\infty, -\infty\}$$

such that $\mu(\emptyset) = 0$ and μ is sigma additive, that is, it satisfies the equality

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

where the series on the right must converge absolutely, for any sequence $A_1, A_2, \ldots, A_n, \ldots$ of disjoint sets in Σ. One consequence is that any extended signed measure can take $+\infty$ as value, or it can take $-\infty$ as value, but both are not available. The expression $\infty - \infty$ is undefined [1] and must be avoided.

A **finite signed measure** (aka. **real measure**) is defined in the same way, except that it is only allowed to take real values. That is, it cannot take $+\infty$ or $-\infty$.

Finite signed measures form a vector space, while extended signed measures are not even closed under addition, which makes them rather hard to work with. On the other hand, measures are extended signed measures, but are not in general finite signed measures.

Examples [edit]

Consider a nonnegative measure ν on the space (X, Σ) and a measurable function $f: X \to \mathbf{R}$ such that

$$\int_X |f(x)|\, d\nu(x) < \infty.$$

Then, a finite signed measure is given by

$$\mu(A) = \int_A f(x)\, d\nu(x)$$

for all A in Σ.

This signed measure takes only finite values. To allow it to take $+\infty$ as a value, one needs to replace the assumption about f being absolutely integrable with the more relaxed condition

Entropy of negative probability distribution

X

$p_1 = -\delta$

$p_0 = 1+\delta$

$p_0 + p_1 = 1$

$\delta \geqslant 0$

$\longrightarrow$ Gábor J. Székely $\longrightarrow$ $\mathbf{X + Q = Z}$

positive ($\mathbf{Z}$)

positive (yay!) & independent from X ($\mathbf{Q}$)

Entropy of negative probability distribution

X

$p_1 = -\delta$

$p_0 = 1+\delta$

$p_0 + p_1 = 1$

$\delta \geq 0$

+

Q

q_5

q_4

q_3

q_2

q_1

q_0

$$\sum_{k=0}^{\infty} q_k = 1$$

$q_k \geq 0$

=

Z

$\vdots$

z_4

z_3

z_2

z_1

z_0

$$\sum_{k=0}^{\infty} z_k = 1$$

$z_k \geq 0$

Z is a *convolution* of **X** and **Q**

Entropy of negative probability distribution

$$z_0 = p_0 q_0$$

$$z_1 = p_0 q_1 + p_1 q_0$$

$$z_2 = p_0 q_2 + p_1 q_1$$

$$z_3 = p_0 q_3 + p_1 q_2$$

$$z_4 = p_0 q_4 + p_1 q_3$$

$$\vdots$$

$$z_0 = 1$$

$$z_1 = z_2 = z_3 = \dots = 0$$

$$\Downarrow$$

$$q_0 = 1/p_0$$

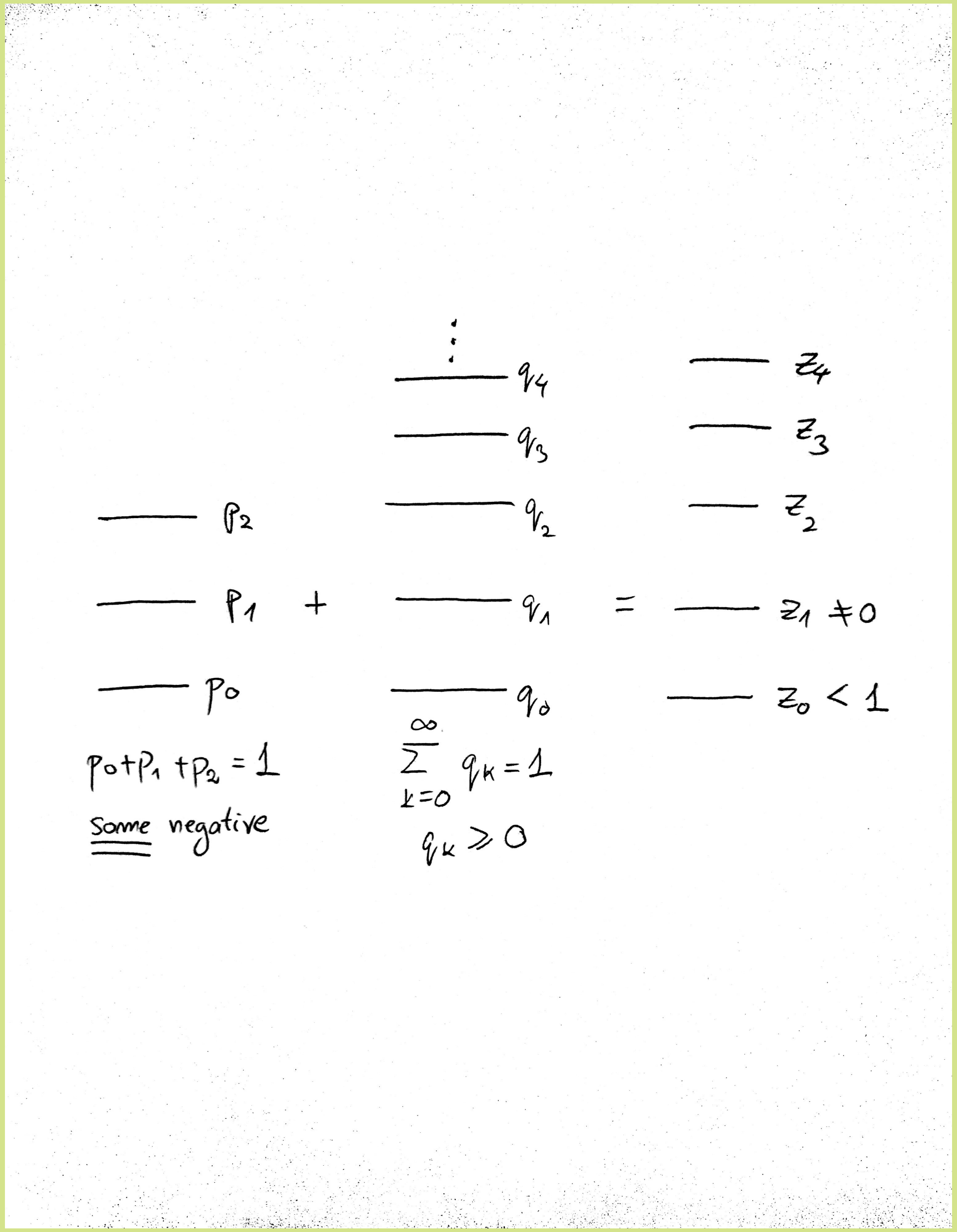
$$\frac{q_{k+1}}{q_k} = \frac{|p_1|}{p_0}$$

$$H(Q) = p_0 \log p_0 + p_1 \log(-p_1)$$

$$H(X) + H(Q) = H(Z) = 0$$

$$H(X) = -H(Q)$$

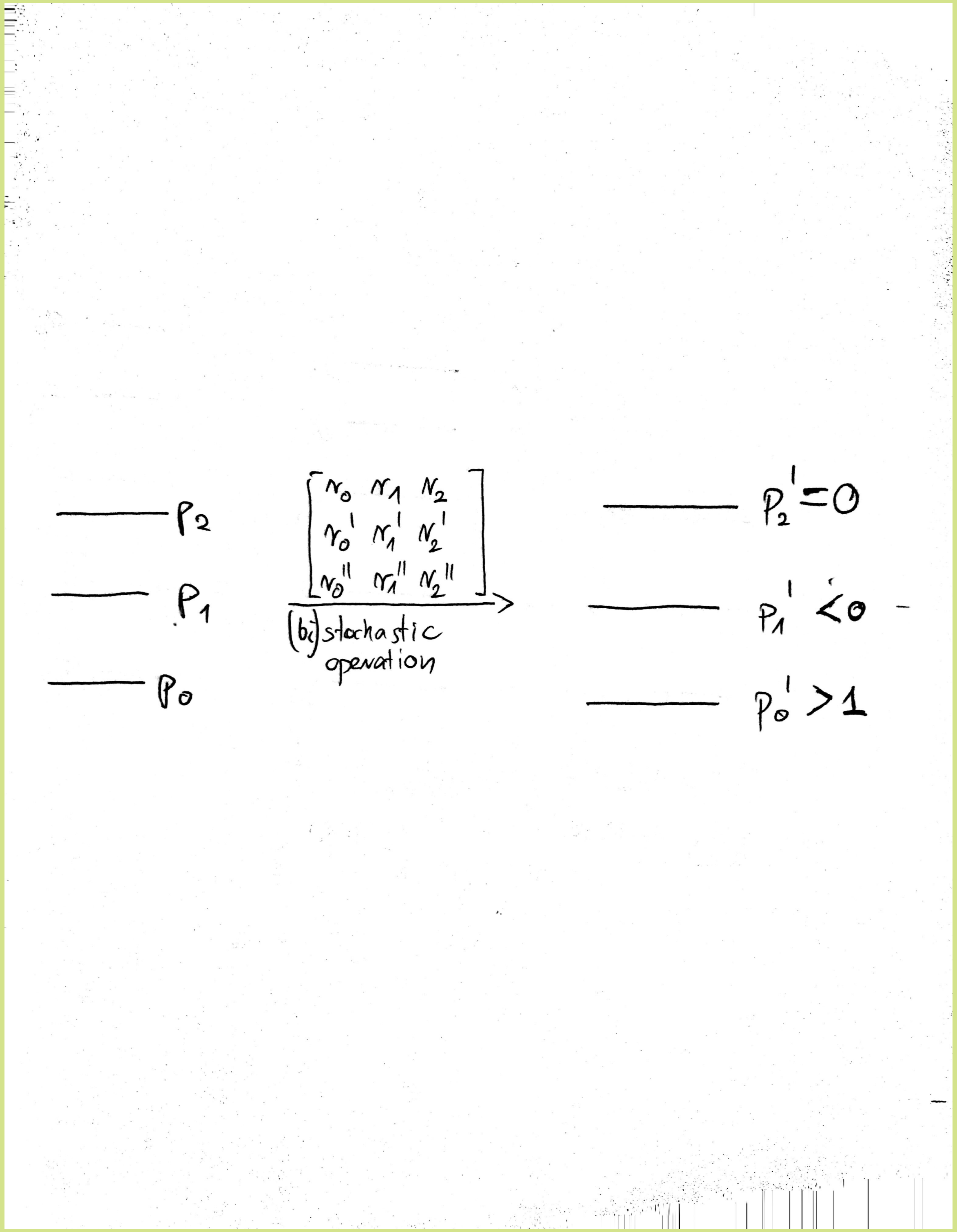
Entropy of negative probability distribution



H(X) + H(Q) = H(Z) > 0

Unappealing definition of H(X)
as H(Z) - H(Q) with min[H(Z)]

Entropy of negative probability distribution



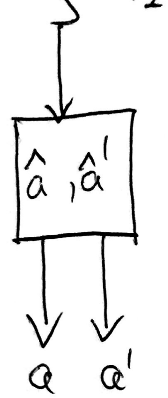
$$H(X) = - H(R) - H(Q)$$

But how to quantify this?

Applications

Why bother?

$$\rho = \tfrac{1}{2}\left(1 + \vec{A}\cdot\vec{\sigma}\right)$$



$$p(++/--) = \frac{|\hat{a}+\hat{a}'|}{4N} \pm \frac{\hat{a}+\hat{a}'}{4}\cdot\vec{A}$$

$$p(+-/-+) = \frac{|\hat{a}-\hat{a}'|}{4N} \pm \frac{\hat{a}-\hat{a}'}{4}\cdot\vec{A}$$

negative but sharp

$$\tilde{p}(++/--) = \frac{|\hat{a}+\hat{a}'|}{4N} \pm \frac{\hat{a}+\hat{a}'}{4N}\cdot\vec{A}$$

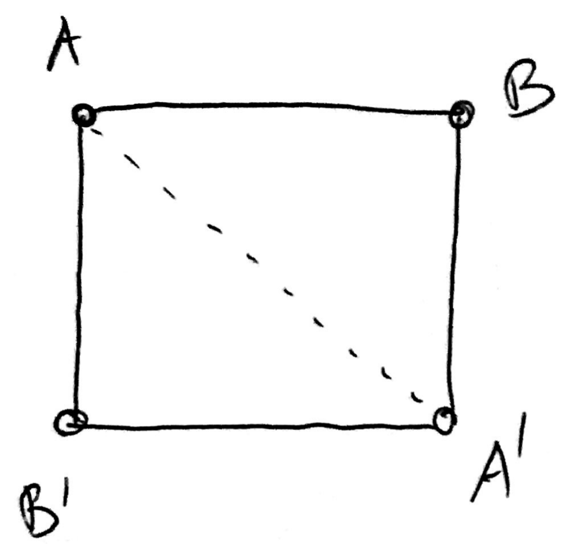
$$\tilde{p}(+-/-+) = \frac{|\hat{a}-\hat{a}'|}{4N} \pm \frac{\hat{a}-\hat{a}'}{4N}\cdot\vec{A}$$

positive but fuzzy

Entropy 'cost' to $p \to \tilde{p}$ could quantify co-measurability 'deficit'

Applications

Why bother?



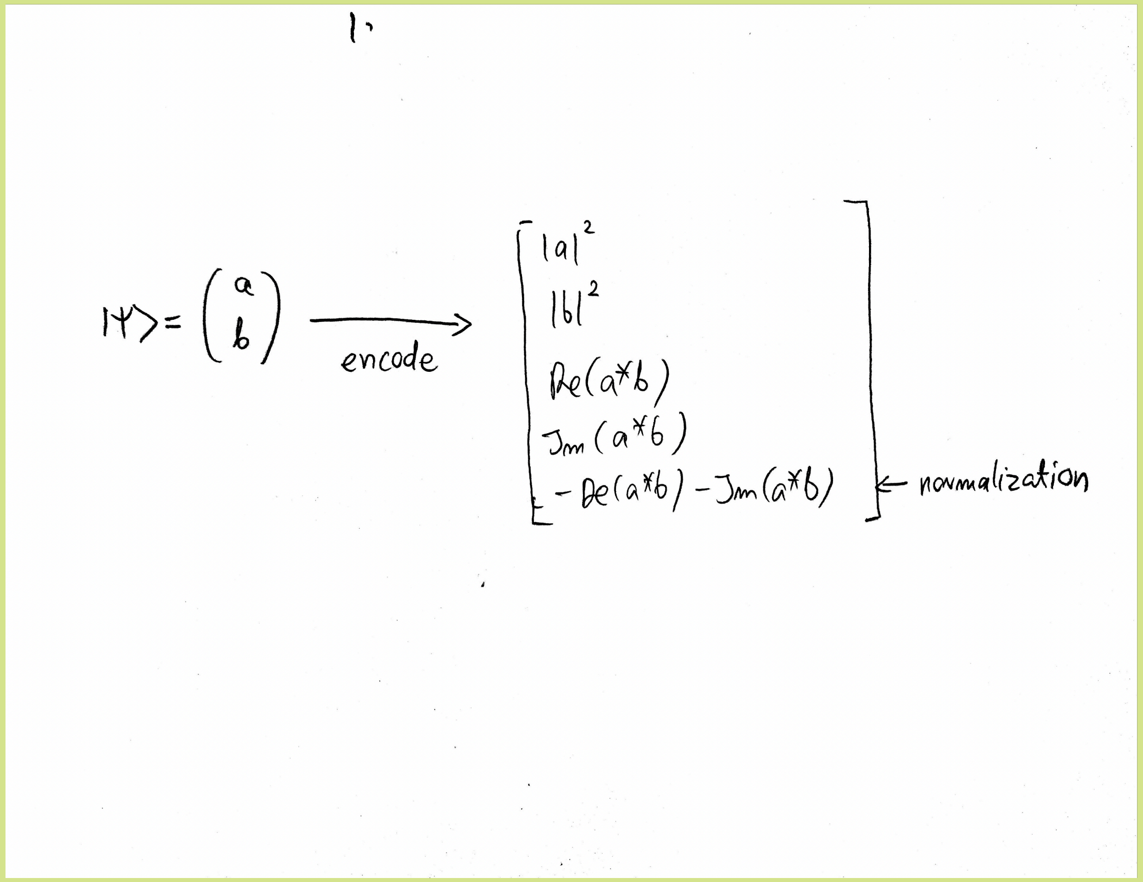
$$d(X,Y) := H(X|Y) + H(Y|X)$$

Bell violation $\Rightarrow d(AA') < 0 \Rightarrow \boxed{H(A,A') < 0}$

Quantifying H(A,A') measures how non-classical it is

Applications

Why bother?



This 'recovers' complete dynamics.
What is deficit of information here?

Centre for Quantum Technologies

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