NARODOWE CENTRUM NAUKI

Centre for Quantum Technologies

Contextuality and Physics

Can spin magnitude be conserved in hidden variable models?

Paweł Kurzynski

In collaboration with: W. Laskowski, A. Kolodziejski,
K. F. Pal, J. Ryu, T. Vertesi

Communications Physics 2 , 1–5 (2019)

The problem

- Contextuality – events, projectors, HV assignments

Spin projection +1 along x axis	→	$\pi_{+1,x}$	→	$v(S_x)=+1$
Spin projection -1 along y axis	→	$\pi_{-1,y}$	→	$v(S_y)=-1$
Spin projection +1 along z axis	→	$\pi_{+1,z}$	→	$v(S_z)=+1$

- However, events correspond to different values of physical properties that obey additional laws (not only exclusivity of events)

$$\hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2 = s(s+1)\hat{1}$$

- Hidden variable models usually focus on laws of logics and probability, can additional phisical constraints be incorporated?

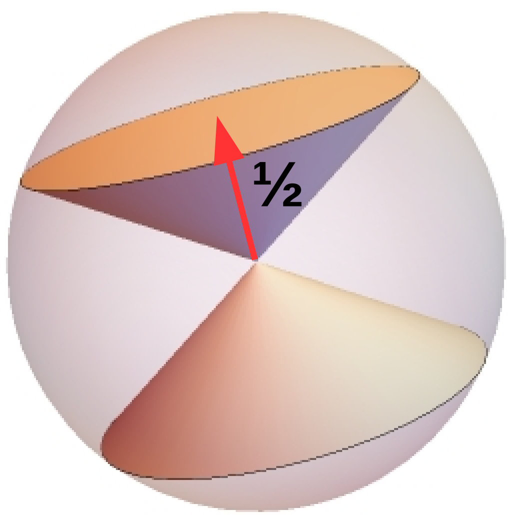
$$\mathbf{s} = \left(v(\hat{S}_x), v\left(\hat{S}_y\right), v(\hat{S}_z)\right)$$

$$\mathbf{s}\cdot\mathbf{s} = v(\hat{S}_x)^2 + v\left(\hat{S}_y\right)^2 + v(\hat{S}_z)^2 = s(s+1)?$$

Conservation of angular momentum

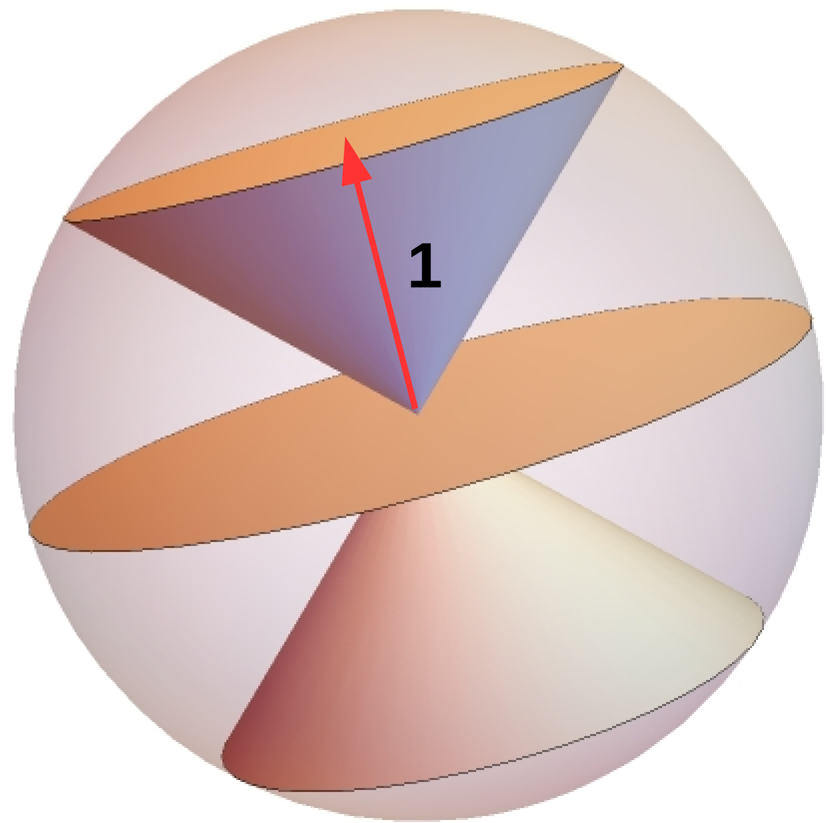
- Spin – angular momentum, vector with a well defined length

S = 1/2



S(S+1) = ¾

S = 1

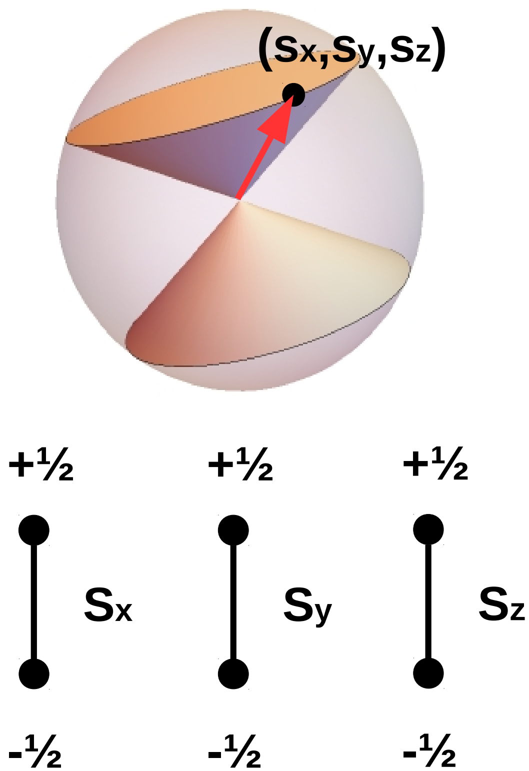


S(S+1) = 2

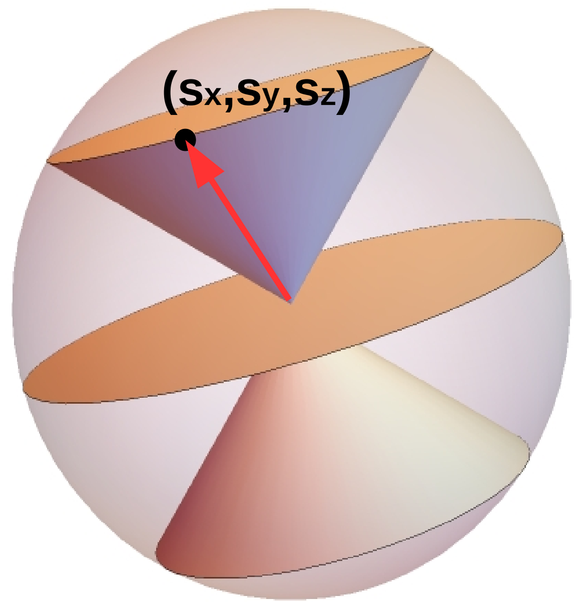
. . .

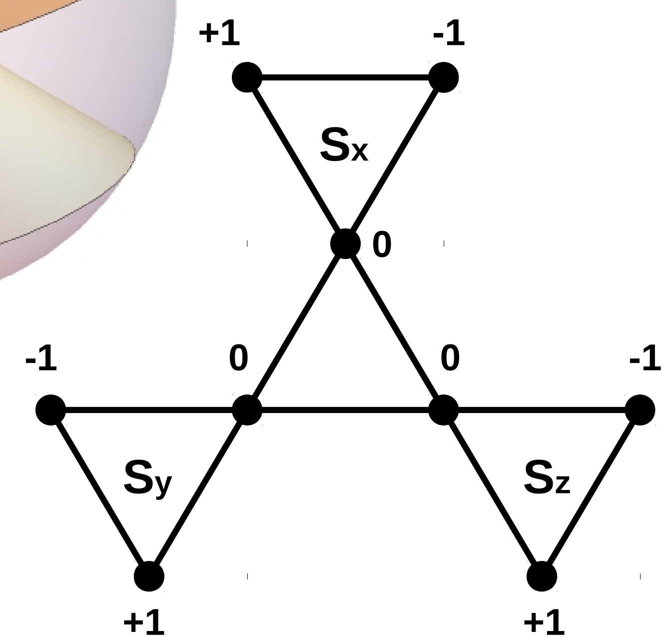
Hidden variable assignment

- Spin measurements along X, Y, Z (KS non-contextual)



(± ½, ± ½, ± ½)





(± 1, ± 1, ± 1)

(± 1, ± 1, 0)

(± 1, 0 , ± 1)

(0 , ± 1, ± 1)

HV and conservation

$$\mathbf{s}\cdot\mathbf{s} = v\left(\hat{S}_x\right)^2 + v\left(\hat{S}_y\right)^2 + v\left(\hat{S}_z\right)^2 = s(s+1)?$$

- S = ½ – yes

  (± ½, ± ½, ± ½) **¾ = ¼ + ¼ + ¼**

- S = 1 – sometimes

  (± 1, ± 1, ± 1) **2 ≠ 1 + 1 + 1**

  (± 1, ± 1, 0) **2 = 1 + 1 + 0**

  (± 1, 0 , ± 1) **2 = 1 + 0 + 1**

  (0 , ± 1, ± 1) **2 = 0 + 1 + 1**

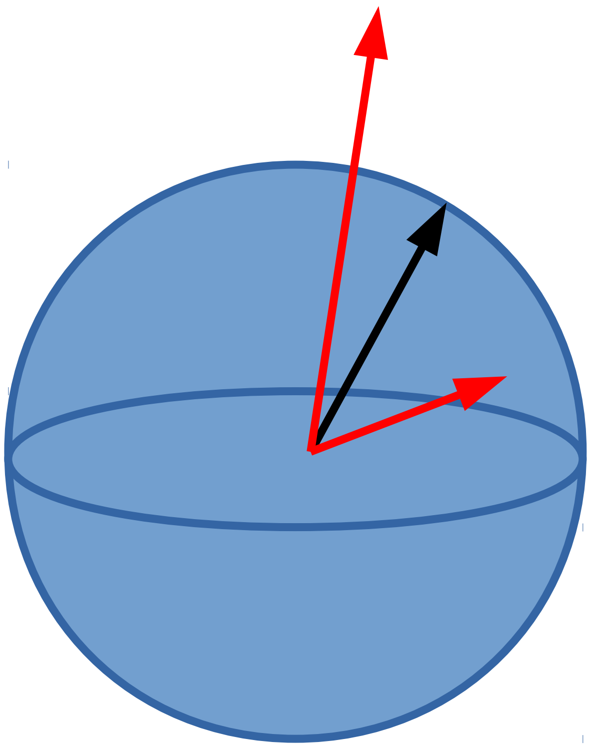
- S = 3/2 – no

  (± 3/2, ± 3/2, ± 3/2) **15/4 ≠ 9/4 + 9/4 + 9/4**

  (± 3/2, ± 3/2, ± 1/2) **15/4 ≠ 9/4 + 9/4 + 1/4**

  (± 3/2, ± 1/2, ± 1/2) **15/4 ≠ 9/4 + 1/4 + 1/4**

  (± 1/2, ± 1/2, ± 1/2) **15/4 ≠ 1/4 + 1/4 + 1/4**



General case

- Similarity to state-dependent/intependent proofs of contextuality
- State-independent proofs – problem of the sum of three squares (Legendre's 3-squares theorem)

$$\mathbf{n = x^2 + y^2 + z^2}$$

$$\mathbf{n \neq 4^a (8\,b + 7)}$$

- Works for half of the half-integer spins and most of integer ones

S = 1/2, 5/2, 9/2, 13/2, ...

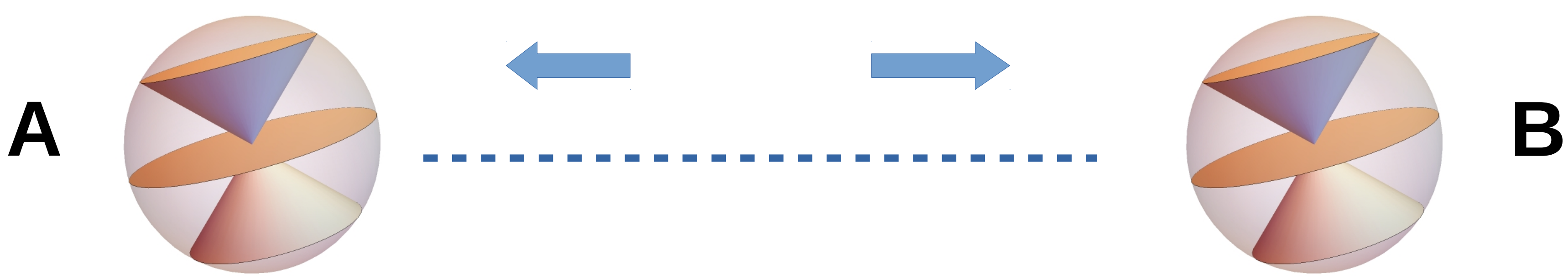
S ≠ 12, 15, 19, 44, 51, ...

State-dependent cases

- How to test it?
- Sometimes easy – (S=2 and projection onto 0)

(± 2, ± 1, ± 1)

- In other cases it is possible in the Bell-like scenario
- A and B share two spin-S particles, each performs one of three measurements: S_x, S_y, S_z



Bell-like scenario

- Bound β derived within HV + conservation model

$$c_{xx}\left\langle \hat{S}_x^{(A)}\hat{S}_x^{(B)}\right\rangle + c_{xy}\left\langle \hat{S}_x^{(A)}\hat{S}_y^{(B)}\right\rangle + \ldots\ + c_{zz}\left\langle \hat{S}_z^{(A)}\hat{S}_z^{(B)}\right\rangle \geq \beta,$$

$$\left\langle \mathbf{S}^{(\mathbf{A})} \cdot \mathbf{C} \cdot \mathbf{S}^{(\mathbf{B})} \right\rangle \geq \beta,$$

$$\mathbf{C} = \begin{pmatrix} c_{xx} & c_{xy} & c_{xz} \\ c_{yx} & c_{yy} & c_{yz} \\ c_{zx} & c_{zy} & c_{zz} \end{pmatrix}$$

Quantum value

- We look for a state that minimizes the product between spins

$$\sum_{j=x,y,z} \left\langle \psi_0 \middle| \hat{S}_j^{(A)} \hat{S}_j^{(B)} \middle| \psi_0 \right\rangle = \left\langle \mathbf{S}^{(\mathbf{A})} \cdot \mathbf{C}_{\mathrm{id}} \cdot \mathbf{S}^{(\mathbf{B})} \right\rangle_{\psi_0} = -s(s+1),$$

- Generalized singlet state (total spin 0, maximally entangled)

$$\left|\psi_0\right\rangle = \frac{1}{\sqrt{2s+1}} \sum_{m=-s}^{s} (-1)^{s-m} |m\rangle \otimes |-m\rangle,$$

Locally rotated singlet state

- B performs euler rotation of his spin

$$|\phi\rangle = \hat{U}^{(B)}\big|\psi_0\big\rangle.$$

$$\sum_{j=x,y,z} \langle\phi|\hat{S}_j^{(A)}\,\hat{U}^{(B)}\hat{S}_j^{(B)}\,\hat{U}^{(B)\dagger}|\phi\rangle = -s(s+1).$$

$$\left\langle \mathbf{S}^{(\mathbf{A})}\cdot\mathbf{C}\cdot\mathbf{S}^{(\mathbf{B})}\right\rangle_{\phi} = -s(s+1),$$

- From now on we assume that C is an orthogonal rotation matrix

HV + conservation bound

- Minimization over allowed vectors **a** and **b**

$$\left\langle \mathbf{S}^{(\mathbf{A})} \cdot \mathbf{C} \cdot \mathbf{S}^{(\mathbf{B})} \right\rangle \geq \beta, \qquad \beta = \min_{\mathbf{a},\mathbf{b}} (\mathbf{a} \cdot \mathbf{C} \cdot \mathbf{b}).$$

- For simplicity we fix C (irrational entries are crucial)

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Bounds and quantum violations

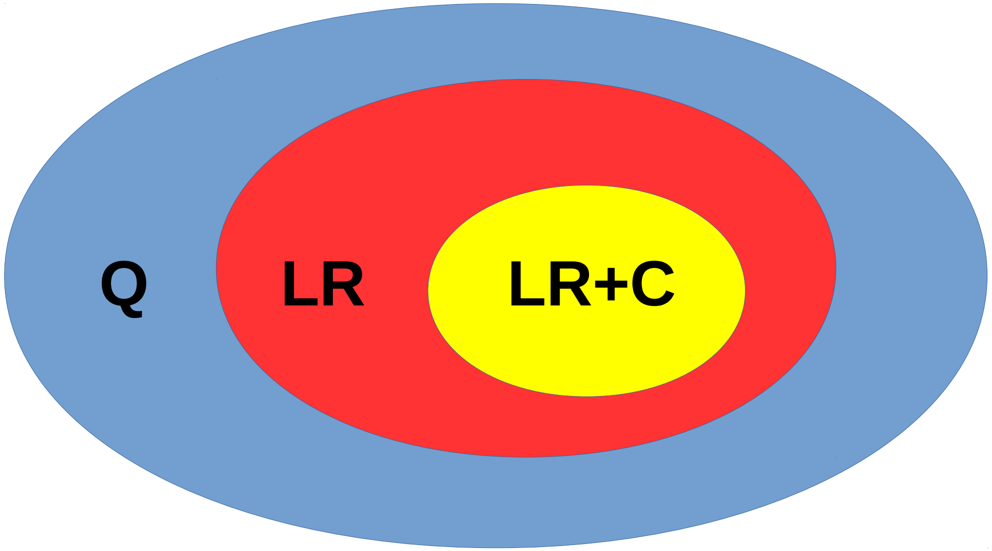
Table 1 Quantum violations of the inequality (4) with *C* given by (9)

s	β and $\bar{\beta}$	$-s(s+1)$
1	$-1-\frac{1}{\sqrt{2}}\approx -1.707$	-2
	$-1-\sqrt{2}\approx -2.414$	
2	$-1+\frac{1}{\sqrt{2}}-4\sqrt{2}\approx -5.949$	-6
	$-4-4\sqrt{2}\approx -9.657$	
3	$-4(1+\sqrt{2})\approx -9.657$	-12
	$-9-9\sqrt{2}\approx -21.730$	
4	$-14\sqrt{2}\approx -19.799$	-20
	$-16-16\sqrt{2}\approx -38.627$	

β denotes the classical HV bound taking into account the conservation of the spin vector magnitude, whereas $\bar{\beta}$ denotes the HV bound without that additional constraint

Conclusions

- Local realism and local realism + conservation



- Implications: in HV theories either spin magnitude is not conserved or HVs do not describe the physical reality, but only provide a deterministic algorithm to predict the outcome of the next measurement