Homotopical approach to quantum contextuality

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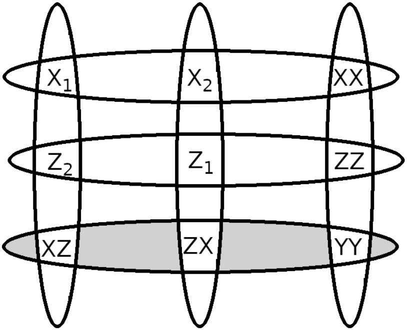
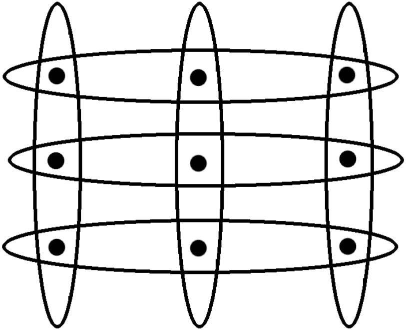
[1]joint with Robert Raussendorf **arXiv:1905.03822**

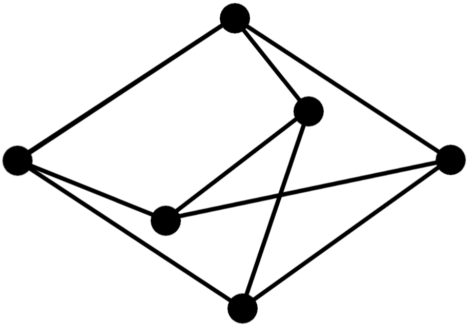


[2]Mermin, "Hidden variables and the two theorems of John Bell".



L: set of labels

$\mathcal{M}$: cover of contexts



Incidence graph[3] $\mathcal{G}$

$$V(\mathcal{G}) = \mathcal{M}, \quad E(\mathcal{G}) = L$$

[3] Arkhipov, "Extending and characterizing quantum magic games".

Arkhipov's theorem

Theorem

An arrangement[4] *is magic if and only if its incidence graph is non-planar.*

[4] Each observable (eigenvalues ± 1) belongs to exactly two contexts.

Arkhipov's theorem

Our goal is to generalize one direction:

Theorem

An arrangement is magic **only if** *its incidence graph is non-planar.*

Quantum realization

Let $d \geq 2$ and $\tau : \mathcal{M} \to \mathbb{Z}_d$ be a function.

- A *quantum realization* is a function

$$T : L \to U(n)$$

  such that

  - each operator satisfies $(T_a)^d = I$,
  - in each context the operators $\{T_a | \, a \in C\}$ pairwise commute,
  - for each context $C \in \mathcal{M}$ the operators satisfy the constraint

$$\prod_{a \in C} T_a^{\pm 1} = \omega^{\tau(C)} \quad \text{where } \omega = e^{2\pi i/d}.$$

Classical realization

- A *classical realization* is a function

$$c : L \to \mathbb{Z}_d$$

such that for each context

$$\prod_{a \in C} \omega^{\pm c(a)} = \omega^{\tau(C)}.$$

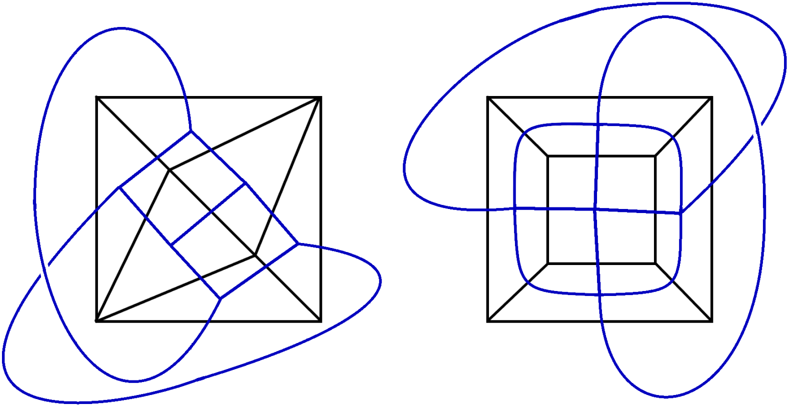
This is a quantum realization with $T_a = \omega^{c(a)}$.

Magic arrangements

Definition

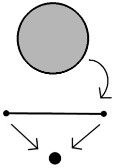
An arrangement is called *magic* if it is quantum realizable (for some τ) but not classically realizable.

Magic arrangements: $K_{3,3}$ and K_5



[5]Okay et al., "Topological proofs of contextuality in qunatum mechanics".

Chain complex



- Define a chain complex $\mathcal{C}_*$

$$\mathbb{Z}_d[\mathcal{M}] \xrightarrow{\partial} \mathbb{Z}_d[L] \xrightarrow{0} \mathbb{Z}_d \quad \text{where} \quad \partial[C] = \sum_{a \in C} \pm[a]$$

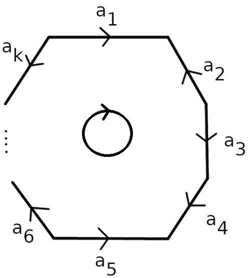
There is an associated cochain complex $\mathcal{C}^*$

$$\mathbb{Z}_d \xrightarrow{0} (\mathbb{Z}_d)^L \xrightarrow{d} (\mathbb{Z}_d)^{\mathcal{M}} \quad \text{where} \quad df(C) = \sum_{a \in C} \pm f(a)$$

$(L, \mathcal{M}, \tau)$ is classically realizable $\Leftrightarrow$ $[\tau] = 0$ in $H^2(\mathcal{C})$.

Topological realization

- A *topological realization* is a connected cell complex X such that
  - $X^1 = L$ and $X^2 = \mathcal{M}$,
  - attaching maps $\phi_C : \partial D^2 \to X^1$



$$\left(\prod_{i=1}^{k} T_{a_i}^{\pm 1} \sim I \right)$$

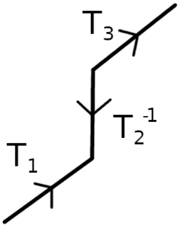
Path operators

- Let p be a path in X traversing a sequence of edges $a_1^{\epsilon(a_1)}, \cdots, a_k^{\epsilon(a_k)}$ where $\epsilon(a_i) = \pm 1$.

  We define a *path operator*

$$T_p = \prod_{i=1}^{k} T_{a_i}^{\epsilon(a_i)}.$$

  If p consists of a single vertex then $T_p = I$.



- Basic properties:
  - $T_{p \cdot q} = T_p T_q$
  - $T_{p^{-1}} = T_p^{-1}$

Combinatorial complexes[6]

- $f : X \to Y$ is called combinatorial if
  - Every 1-cell of X is either mapped onto a 1-cell of Y or collapsed to a 0-cell,
  - Every 2-cell of X is either mapped onto a 2-cell of Y or it is a product cell* that is collapsed to a 1-cell or 0-cell.
- A cell complex is called combinatorial if the attaching map of every cell is combinatorial.

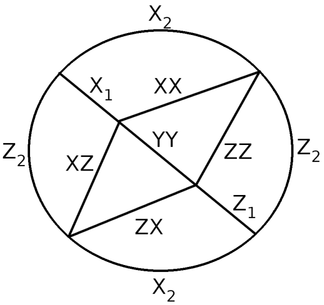
[6] Bogley et al., "Two-dimensional homotopy and combinatorial group theory".

Key observation

Lemma

For a combinatorial map $g : D^2 \to X$ we have

$$T_{g(\partial D^2)} = \omega^{g^*\tau(D^2)}.$$



$$X_2 Z_2 X_2 Z_2 = -1$$

Key observation

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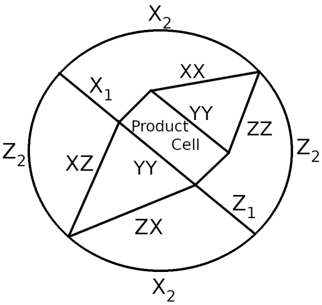


$$X_2 Z_2 X_2 Z_2 = -1$$

Main result

Theorem

An arrangement is magic only if there exists a topological realization with non-trivial fundamental group.

Comparison to Arkhipov's result

- Suppose that $d = 2$ and each observable belongs exactly to two contexts.

  Embed the intersection graph into a closed surface

$$\mathcal{G} \to \Sigma_g$$

- We can take $X = \Sigma_g$ with the dual cell structure

$$\pi_1(X) = 1 \;\Leftrightarrow\; g = 0 \quad (\mathcal{G} \text{ is planar})$$

Idea of the proof

- Show that if τ admits a quantum realization then $[\tau] = 0$.
- By Hurewicz theorem[7]

$$X \simeq S^2 \vee \cdots \vee S^2$$

This decomposes cohomology

$$H^2(X) \cong H^2(S^2) \oplus \cdots \oplus H^2(S^2)$$

[7] Hatcher, *Algebraic topology*.

Idea of the proof

- Let $h : S^2 \to X$ be a map. Show that $h^*\tau(S^2) = 0$.
- h is homotopic[8] to a combinatorial map

$$\Phi : D^2 \to X$$

such that $\Phi(\partial D^2) \simeq *$ in X^1.

Thus there is another combinatorial map

$$\bar{\Phi} : D^2 \to X^1 \subset X$$

such that $\bar{\Phi}(\partial D^2) = \Phi(\partial D^2)$.

[8] Fundamental sequence $0 \to \pi_2(X) \to \pi_2(X, X^1) \xrightarrow{\partial} \pi_1(X^1) \to \pi_1(X) \to 1$

Idea of the proof

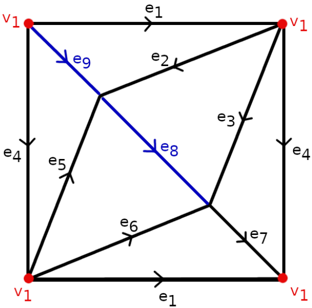
- Homotopy invariance of cohomology

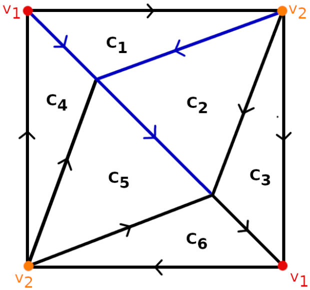
$$h^*\tau = \Phi^*\tau \quad (h \sim \Phi)$$

- Apply the key lemma

$$\omega^{\Phi^*\tau(D^2)} = T_{\partial\Phi(D^2)} = T_{\partial\bar{\Phi}(D^2)} = \omega^{\bar{\Phi}^*\tau(D^2)} = I.$$

Torus vs Projective plane





$a_1 = e_1$

$a_2 = e_2 \cdot e_9^{-1}$

$a_3 = e_3 \cdot e_8^{-1} \cdot e_9^{-1}$

$a_4 = e_4$

$a_5 = e_5 \cdot e_9^{-1}$

$a_6 = e_6 \cdot e_8^{-1} \cdot e_9^{-1}$

$a_7 = e_9 \cdot e_8 \cdot e_7.$

$a_1 = e_1 \cdot e_2 \cdot e_9^{-1}$

$a_3 = e_9 \cdot e_2^{-1} \cdot e_3 \cdot e_8^{-1} \cdot e_9^{-1}$

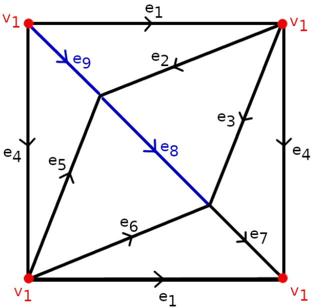
$a_4 = e_9 \cdot e_2^{-1} \cdot e_4$

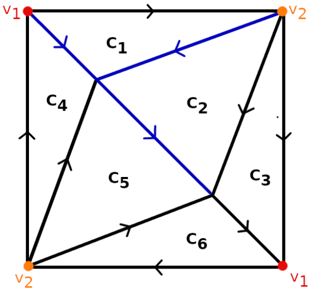
$a_5 = e_9 \cdot e_2^{-1} \cdot e_5 \cdot e_9^{-1}$

$a_6 = e_9 \cdot e_2^{-1} \cdot e_6 \cdot e_8^{-1} \cdot e_9^{-1}$

$a_7 = e_9 \cdot e_8 \cdot e_7.$

Torus vs Projective plane

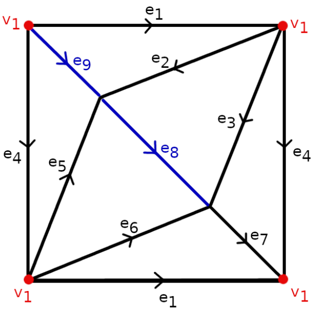


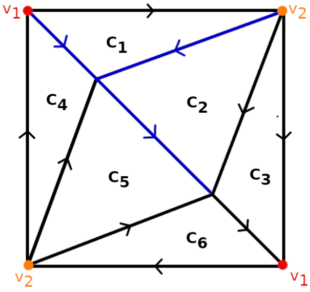


$$
\begin{aligned}
T_1 T_2 &= \omega^{\tau_1} \\
T_2^{-1} T_3 &= \omega^{\tau_2} \\
T_4 T_7^{-1} T_3^{-1} &= \omega^{\tau_3} \\
T_5^{-1} T_4^{-1} &= \omega^{\tau_4} \\
T_5 T_6^{-1} &= \omega^{\tau_5} \\
T_6 T_7 T_1^{-1} &= \omega^{\tau_6} .
\end{aligned}
$$

$$
\begin{aligned}
T_1 &= \omega^{\tau_1} \\
T_3 &= \omega^{\tau_2} \\
T_4 T_7^{-1} T_3^{-1} &= \omega^{\tau_3} \\
T_4 T_5^{-1} &= \omega^{\tau_4} \\
T_5 T_6^{-1} &= \omega^{\tau_5} \\
T_6 T_7 T_1 &= \omega^{\tau_6} .
\end{aligned}
$$

Torus vs Projective plane





$$[T_1, T_4] = \omega^{\sum_{i=1}^{6} \tau_i}$$

$$(T_4)^2 = \omega^{-\tau_1 + \sum_{i=2}^{6} \tau_i}$$

$$\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$$

$$\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$$

Projective representation

- Sending p to T_p gives a group homomorphism

$$T : \pi_1(X) \to U(n)/\langle \omega \rangle$$

Theorem

If an arrangement has a topological realization X such that $\pi_1(X)$ is a finite group whose order is coprime to d then the arrangement is non-magic.

For more arXiv:1905.03822...