Acyclicity and Vorob'ev's theorem

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The essence of contextuality

- Not all properties may be observed at once.
- Jointly observable properties provide **partial snapshots**.

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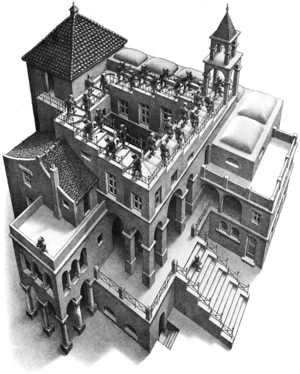

M. C. Escher, *Ascending and Descending*

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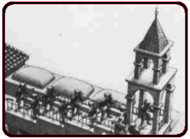

Local consistency

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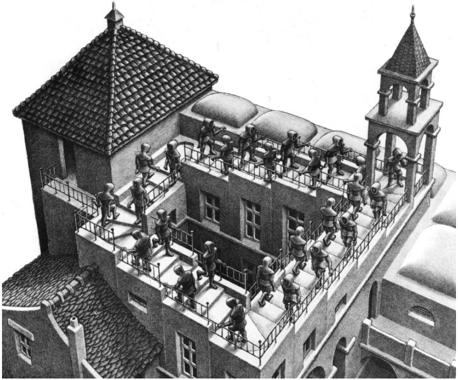

Local consistency *but* **Global inconsistency**

A recurring theme

- Non-locality and contextuality
- Relational databases
- Contraint satisfaction
- . . .

Vorob'ev's theorem

Vorob'ev (1962)

'Consistent families of measures and their extensions'

- In the context of game theory.
- Consider a collection of variables
- and distributions on the joint values of some variables.
- These distributions are pairwise consistent.

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In our language:

For which measurement scenarios is it the case that any no-signalling (no-disturbing) behaviour is non-contextual?

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- Necessary and sufficient condition: **regularity**!

Relational databases

Codd (1970): Relational model of data

- Information is organised into tables (relations).
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- Information is organised into tables (relations).
- Columns of each table are labelled by attributes
- Entries: a row with a value for each attribute of a table
- A database consists of a set of such tables, each with different attributes
- Database schema: blueprint of a database specifying attributes of each table and type of information: $\mathcal{S} = \{A_1, \ldots, A_n\}$
- Database instance: snapshot of the contents of a database at a certain time, consisting of a relation instance (i.e. a set of entries) for each table: $\{R_A\}_{A \in \mathcal{S}}$.

Relational databases

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- It is **totally consistent** if it has a universal relation instance: T on attributes $\bigcup \mathcal{S}$ with $\forall A \in \mathcal{S}.\ \ T|_A = R_A$

Dictionary

Databases	**Empirical models**
attributes	measurements
domain of attribute	outcome value of measurement
relation schema	set of compatible measurements
database schema	measurement scenario
tuple / entry	joint outcome

Dictionary

relation instance	distribution on joint outcomes
database instance	empirical model
projection	marginalisation
projection consistency	no-signalling condition
universal instance	global distribution
total consistency	locality / non-contextuality

An analogous question

For which database schemata does pairwise projection consistency imply total consistency?

- Necessary and sufficient condition: **acyclicity**.
- Acyclic database schemes extensively studied in late 70s / early 80s
- Many equivalent characterisations . . .

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For which database schemata does pairwise projection consistency imply total consistency?

- Necessary and sufficient condition: **acyclicity**.
- Acyclic database schemes extensively studied in late 70s / early 80s
- Many equivalent characterisations . . .
- **Turns out to be equivalent to Vorob'ev's condition!**

Commonalities

- In both instances, the same condition characterises situations where local consistency implies global consistency (LC $\implies$ GC)
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- In both instances, the same condition characterises situations where local consistency implies global consistency (LC $\implies$ GC)
- i.e. situations in which contextuality *cannot* arise.
- What are the essential ingredients for such a characterisation to hold?

Overview of the talk

- Setting the stage
- The condition: acyclicity
- Sufficiency: acyclicity implies (LC $\implies$ GC)
- Necessity: (LC $\implies$ GC) implies acyclicity

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- Acyclicity and topology
- Comparison with other work
- An interesting application

Setting the stage

Abstract simplicial complexes

- Combinatorial objects describing a particularly simple kind of space
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An **abstract simplicial complex** on a set of vertices V is a family Σ of finite subsets of V such that:

- it contains all the singletons: $\forall v \in V.\ \{v\} \in \Sigma$.
- it is downwards closed: $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ implies $\tau \in \Sigma$.

Data over simplicial complexes

We consider a functor $\mathcal{F} : \mathcal{P}(V)^{\mathsf{op}} \longrightarrow \mathsf{Set}$:

- for each $\alpha \subseteq V$, a set $\mathcal{F}(\alpha)$.
  Elements $s \in \mathcal{F}(\alpha)$ are called (local) sections.
  Elements $g \in \mathcal{F}(V)$ are the global sections.

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$$\begin{aligned} \mathcal{F}(\alpha) &\longrightarrow \mathcal{F}(\beta) \\ s &\longmapsto s|_\beta \end{aligned}$$

We think of $\mathcal{F}(\alpha)$ as specifying the kind of information that can be associated to the set of variables/measurements/attributes $\alpha \subseteq V$.

E.g. $\mathcal{F}(\alpha) = \{0,1\}^\alpha$ (deterministic assignments, functions $\alpha \longrightarrow \{0,1\}$)
$\mathcal{F}(\alpha) = \mathsf{Distr}(\{0,1\}^\alpha)$ (prob. distr. on joint assignments)
$\mathcal{F}(\alpha) = \mathcal{P}(\{0,1\}^\alpha)$ (subsets joint assignments)

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We want to know under which conditions acyclicity is the answer:

$$\Sigma \text{ is acyclic} \\ \Updownarrow \\ \text{every compatible family of } \mathcal{F} \text{ for } \Sigma \text{ is extendable}$$

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$\Downarrow \quad \Uparrow$

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The condition: acyclicity

Acyclicity I

Generalising from graphs.

- A naïve approach (cycles as closed paths) does not capture the appropriate notion
- Instead, use the definition in terms of biconnectedness:
  - A graph G is biconnected if it is connected and removing any vertex does not disconnect it.
  - A cycle in G forms a nontrivial biconnected subgraph of G.
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- For simplicial complexes:
  - An articulation set for Σ is a set $A = \sigma_1 \cap \sigma_2$ for $\sigma_1 \neq \sigma_2 \in \Sigma$ s.t. $\Sigma|_{V \setminus A}$ has more connected components than Σ.
  - Σ is **biconnected** if it is connected and has no articulation set
  - Σ is **acyclic** if it has no induced subcomplex that is nontrivial and biconnected
  - Equivalently, if every nontrivial, connected, induced subcomplex has an articulation set.

Acyclicity II

An easier, more algorithmic description.

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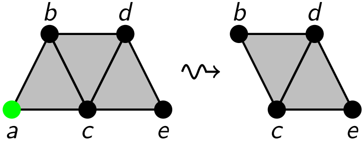

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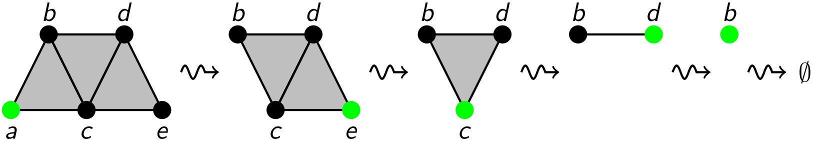

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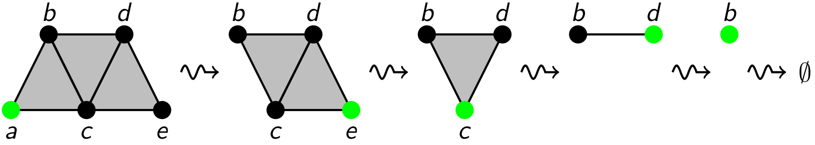
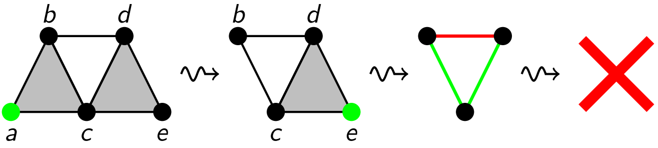


- Σ not acyclic: Graham reduction fails.



Sufficiency: acyclicity implies (LC $\implies$ GC)

Glueing two sections

- Let $s_1 \in \mathcal{F}(\alpha_1)$ and $s_2 \in \mathcal{F}(\alpha_2)$.
  - s_1 and s_2 are **compatible** if

$$s_1|_{\alpha_1 \cup \alpha_2} = s_2|_{\alpha_1 \cup \alpha_2}$$

  - s_1 and s_2 are **strongly compatible** if there is a $t \in \mathcal{F}(\alpha_1 \cup \alpha_2)$ such that

$$t|_{\alpha_1} = s_1 \qquad \text{and} \qquad t|_{\alpha_2} = s_2$$

- $\mathcal{F}$ is glueable if any two compatible sections are strongly compatible
  Glueing map:

$$g_{\alpha_1 \alpha_2} : \mathcal{F}(\alpha_1) \times_{\mathcal{F}(\alpha_1 \cap \alpha_2)} \mathcal{F}(\alpha_2) \longrightarrow \mathcal{F}(\alpha_1 \cup \alpha_2)$$

  (cf. Flori–Fritz's gleaves)

Examples

- Probability distributions $F(\alpha) = \mathrm{Distr}(O^{\alpha})$
  - Given compatible distributions p_{α_1} and p_{α_2}
  - Take $A := \alpha_1 \backslash \alpha_2$, $B := \alpha_1 \cap \alpha_2$, $C := \alpha_2 \backslash \alpha_1$.
  - So we have p_{AB} and p_{BC} with

$$\sum_{\mathbf{x} \in O^A} P_{A,B}(A, B \mapsto \mathbf{x}, \mathbf{y}) = \sum_{\mathbf{z} \in O^C} P_{B,C}(B, C \mapsto \mathbf{y}, \mathbf{z})$$

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  - Define an extension

$$P(A, B, C \mapsto \mathbf{x}, \mathbf{y}, \mathbf{z}) := \begin{cases} \frac{P_{A,B}(A,B \mapsto \mathbf{x},\mathbf{y})\, P_{B,C}(B,C \mapsto \mathbf{y},\mathbf{z})}{P_B(B \mapsto \mathbf{y})} & \text{if } P_B(B \mapsto \mathbf{y}) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Examples

- Relational databases:
  - R_1 on attributes A_1, R_2 on attributes A_2
  - Define the natural join $R_1 \bowtie R_2$ on $A_1 \cup A_2$:

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- More generally:
  - Both of these are examples of distributions
  - $\langle \mathbb{R}_{\geq 0}, +, \cdot, 0, 1 \rangle$: probability
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- Flori–Fritz: metric spaces

- Logic: Robinson Joint Consistency Theorem
  - Let T_i be a theory over the language L_i, with $i \in \{1, 2\}$. If there is no sentence ϕ in $L_1 \cap L_2$ with $T_1 \vdash \phi$ and $T_2 \vdash \neg\phi$, then $T_1 \cup T_2$ is consistent.

Vorobˈev's theorem: sufficiency of acyclicity

Let $\mathcal{F} : \mathcal{P}(V)^{\mathsf{op}} \longrightarrow \mathsf{Set}$ be gluable and Σ a simplicial complex on vertices V. If Σ is acyclic, then any compatible family of $\mathcal{F}$ for Σ is extendable to a global section.

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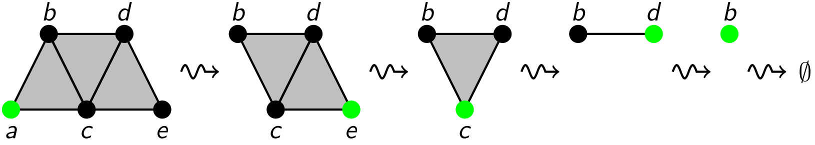

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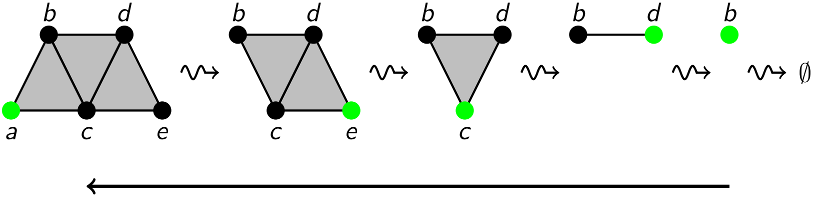

then construct a global distribution by glueing

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Σ is acyclic if and only if for all $\sigma \in \Sigma$ $\mathrm{lk}_\Sigma(\sigma)$ is contractible to a disjoint union of points.

Comparison

Related work in quantum literature

- Cf. Budroni–Morchio
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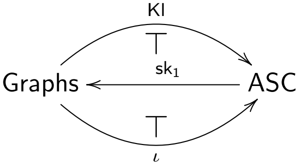

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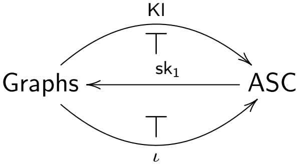

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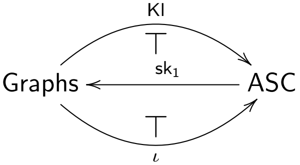

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  - $\mathsf{Kl}(G)$ is acyclic iff G is chordal

An interesting consequence

Monogamy and average macroscopic locality

- Average macro correlations from micro models are local
  (Ramanathan, Paterek, Kay, Kurzyński & Kaszlikowski 2011: multipartite quantum models)
- Monogamy of violation of Bell inequalities
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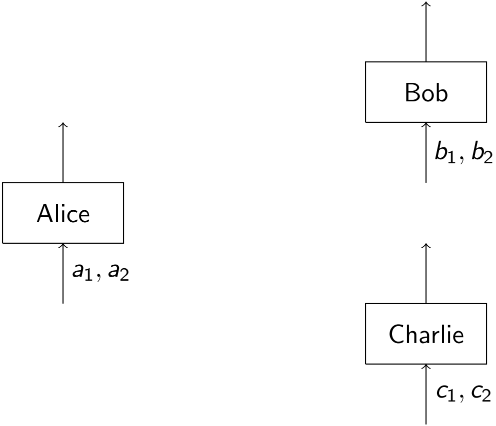
- a structural explanation related to Vorob'ev's theorem

- Let us look at a simple illustrative example.

Monogamy of non-locality

Given a Bell inequality $\mathcal{B}(-,-,) \leq R$,



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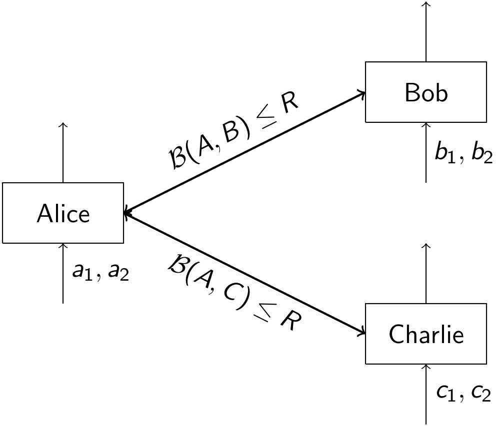

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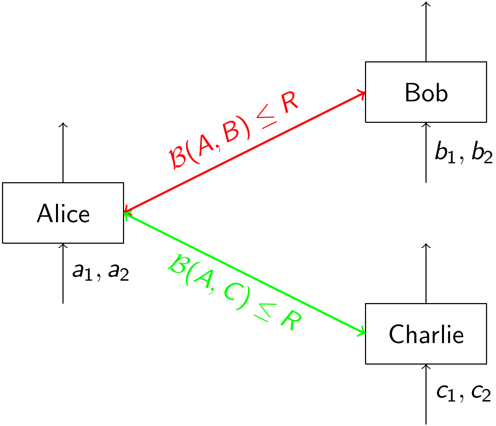

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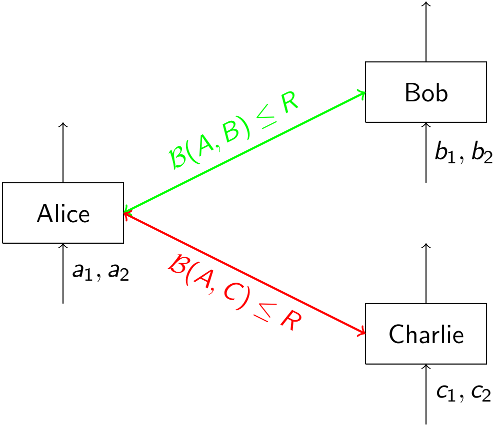

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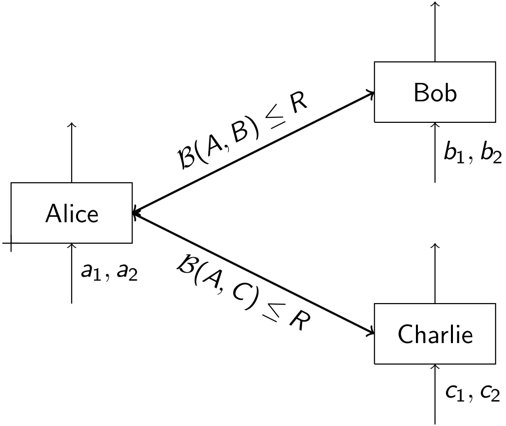

Monogamy relation: $\mathcal{B}(A,B) + \mathcal{B}(A,C) \leq 2R$

Macroscopic average behaviour: tripartite example

- We regard sites B and C as forming one 'macroscopic' site, M, and site A as forming another.
- In order to be 'lumped together', B and C must be symmetric/of the same type: the symmetry identifies the measurements $b_1 \sim c_1$ and $b_2 \sim c_2$, giving rise to 'macroscopic' measurements m_1 and m_2.

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- Given an empirical model $p(a_i, b_j, c_k = x, y, z)$, the 'macroscopic' average behaviour is a bipartite model (with two macro sites A and M) given by the following average of probabilities of the partial models:

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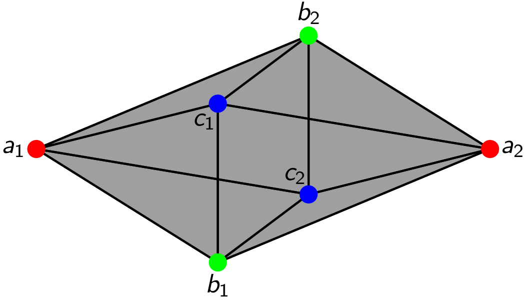
The average model p_{a_i,m_j} **satisfies a bipartite Bell inequality** if and only if in the microscopic model Alice is **monogamous** with respect to violating it with Bob and Charlie.

Structural Reason



- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.

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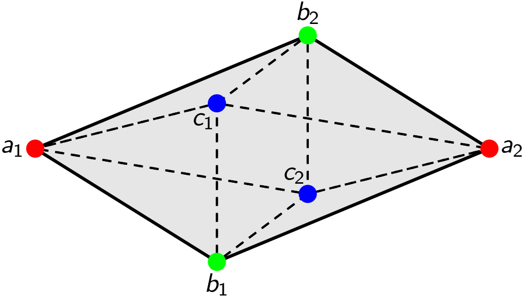

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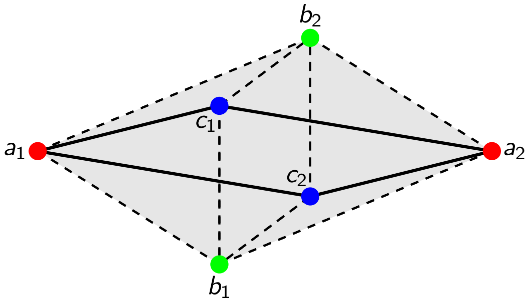

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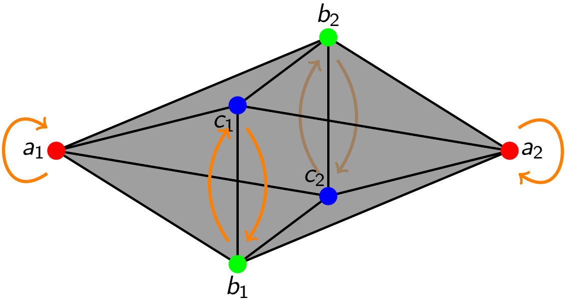

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- We identify B and C: $b_1 \sim c_1$, $b_2 \sim c_2$.
- The macro scenario arises as a quotient.

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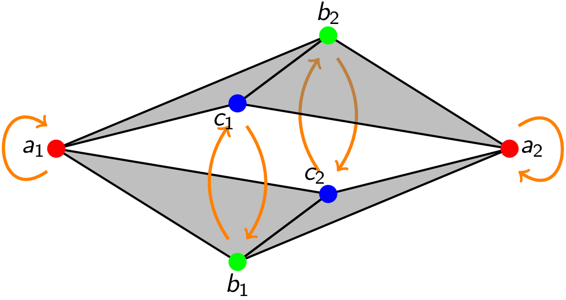

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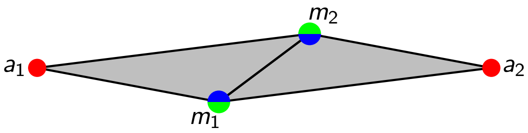

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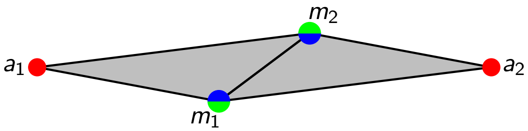

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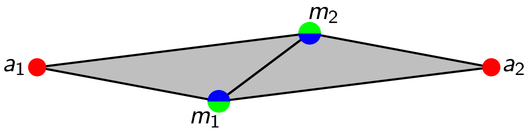

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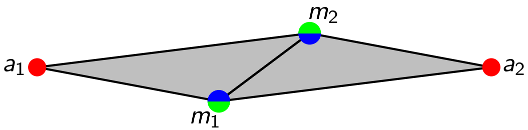

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- Therefore, no matter from which micro model p_{a_i, b_j, c_k} we start, the averaged macro correlations p_{a_i, m_j} are local.

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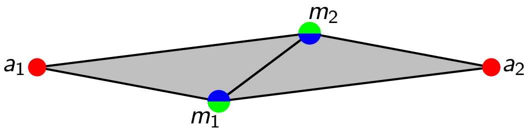

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- In particular, they satisfy any Bell inequality.
- Hence, the original tripartite model also satisfies a monogamy relation for any Bell inequality.

Questions...

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