GADGET STRUCTURES IN PROOFS OF THE KOCHEN-SPECKER THEOREM

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Prague, May 2019

R. R., M. Rosicka, K. Horodecki, S. Pironio, M. Horodecki, P. Horodecki. "Gadget structures in proofs of the Kochen-Specker theorem". arXiv: 1807.00113. Accepted in Quantum.

OUTLINE

- Kochen-Specker proofs. Orthogonality graphs. {0,1}-colorings.
- 01-Gadgets (generalised “bugs”)

- Constructing Kochen-Specker proofs using 01-gadgets
- Constructing statistical KS proofs (a la Yu-Oh) using 01-gadgets

- Pitowsky’s Indeterminacy Principle (“extended Kochen-Specker Theorem”)
- Extended 01-gadgets and constructive proofs of extended KS theorems

- Applications in Randomness Generation
- Conclusions

KOCHEN-SPECKER THEOREM

- Impossible to find a deterministic non-contextual assignment to measurement outcomes (for d > 2)

Outcome probs only take values 0 or 1

Assignment to projectors independent of the context

- Usually consider a set ***S*** of vectors: $\mathcal{S} = \{|v_1\rangle, \ldots, |v_n\rangle\} \subset \mathbb{C}^d$.

- Assignment ***f***: ***S*** → {0,1} also called a {0,1}-colouring of ***S*** obeys

  - $\sum_{|v\rangle \in \mathcal{O}} f(|v\rangle) \leq 1$ for every set $\mathcal{O} \subseteq \mathcal{S}$ of mutually orthogonal vectors;
  - $\sum_{|v\rangle \in \mathcal{B}} f(|v\rangle) = 1$ for every set $\mathcal{B} \subseteq \mathcal{S}$ of d mutually orthogonal vectors.

- KS proofs: finite sets of vectors that do not admit any {0,1}-colouring.

S. Kochen and E. P. Specker. "The problem of hidden variables in quantum mechanics". Journal of Mathematics and Mechanics 17, 59 (1967).

ORTHOGONALITY GRAPHS

- Orthogonality graph: Represent each vector $|v_i\rangle$ by a vertex v_i of the graph. Connect any two vertices v_1 and v_2 by an edge if $\langle v_1|v_2\rangle = 0$.

- Graph is {0,1}-colorable if there exists an assignment ***f***: *V(G)* → {0,1} that obeys

  - $\sum_{v\in Q} f(v) \leq 1$ for every clique $Q \subset V(G)$;
  - $\sum_{v\in Q_{\max}} f(v) = 1$ for every maximum clique $Q_{\max} \subset V(G)$.

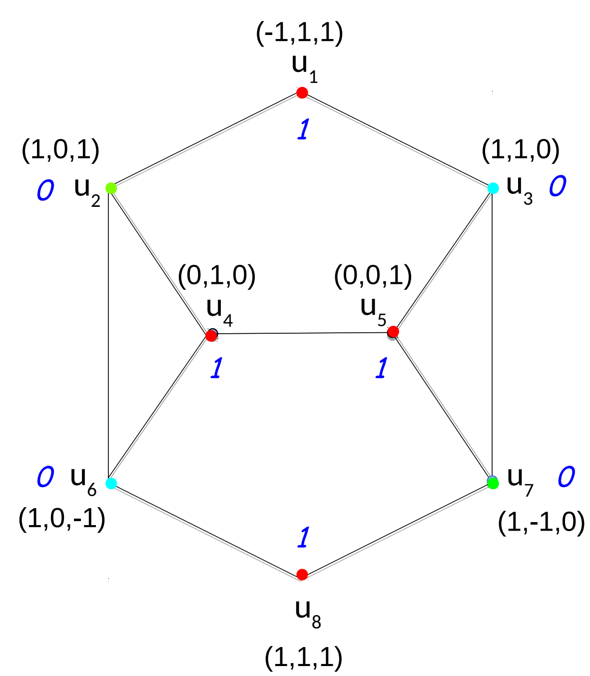
- KS: exist finite vector sets ***S*** such that their ort. graphs $\boldsymbol{G_S}$ are not {0,1}-colorable.

L. Lovasz, M. Saks and A. Schrijver. Linear Algebra and its Applications. 4, 114/115, 439 (1987).

A. Cabello, S. Severini and A. Winter. arXiv: 1010.2163 (2010). Phys. Rev. Lett. 112 040401 (2014).

ORTHOGONAL REPRESENTATIONS

- Given graph G, an orthogonal representation of G in dimension d is a set of vectors ***S*** in **C**ᵈ obeying the orthogonality conditions $v_1 \sim v_2 \Rightarrow \langle v_1 | v_2 \rangle = 0$
- $d(G) \geq \omega(G)$ denotes the minimum dimension of an orthogonal representation of G
- Faithful Orthogonal Representation: $v_1 \sim v_2 \Leftrightarrow \langle v_1 | v_2 \rangle = 0$ and $v_1 \neq v_2 \Leftrightarrow |v_1\rangle \neq |v_2\rangle$
- $d^*(G)$ denotes the minimum dimension of a faithful ort. repn. of G.



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A. Cabello, S. Severini and A. Winter. arXiv: 1010.2163 (2010). Phys. Rev. Lett. 112 040401 (2014).

FINDING KS SETS IS HARD

- Given a graph, finding an ort. repn. can be expressed as a system of polynomial equations, but efficient numerical methods still lacking.

- Additionally, deciding if a given graph admits a {0,1}-coloring is NP-complete.

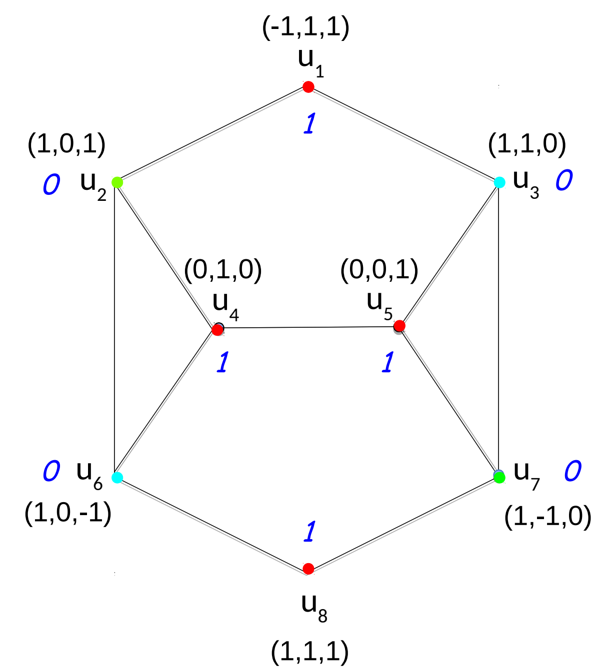
- Consequentemente, finding KS sets in arbitrary dimensions is difficult.

F. Arends. Masters thesis. University of Oxford (2009).

F. Arends, J. Ouaknine, and C. W. Wampler. "On searching for small Kochen-Specker vector systems". Proc. WG vol. 6986 of Springer LNCS (2011).

01-GADGETS (GENERAL "BUGS")

- Bug: Set S_C of 8 vectors such that two particular non-orthogonal vectors $|u_1\rangle$ and $|u_8\rangle$ cannot be assigned value 1 in any {0,1}-coloring



- Define a 01-gadget as any set of vectors with this property, i.e.

Definition 1. *A 01-gadget in dimension d is a $\{0,1\}$-colorable set $\mathcal{S}_{gad} \subset \mathbb{C}^d$ of vectors containing two distinguished vectors $|v_1\rangle$ and $|v_2\rangle$ that are non-orthogonal, but for which $f(|v_1\rangle) + f(|v_2\rangle) \leq 1$ in every $\{0,1\}$-coloring f of $\mathcal{S}_{gad}$.*

S. Kochen and E. P. Specker. "The problem of hidden variables in quantum mechanics". Journal of Mathematics and Mechanics 17, 59 (1967).

R. K. Clifton. "Getting Contextual and Nonlocal Elements-of-Reality the Easy Way". American Journal of Physics 61: 443 (1993).

Cabello, A and G. Garcia-Alcaine. "Bell-Kochen-Specker Theorem for any finite dimension n \geq 3". J. Phys. A: Math. And Gen. 29, 1025 (1996).

A. Cabello, J. R. Portillo, A. Solis, K. Svozil. "Minimal true-implies-false and true-implies-true sets of propositions in noncontextual hidden variable theories". Phys. Rev. A 98, 012106 (2018).

01-GADGETS

- 01-Gadgets can also be defined in terms of graphs and act as "virtual edges" in colouring

Definition 2. *A 01-gadget in dimension d is a $\{0,1\}$-colorable graph G_{gad} with faithful dimension $d^*(G_{gad}) = \omega(G_{gad}) = d$ and with two distinguished non-adjacent vertices $v_1 \nsim v_2$ such that $f(v_1) + f(v_2) \leq 1$ in every $\{0,1\}$-coloring f of G_{gad}.*

- We identify the role of 01-gadgets in KS proofs and give new constructions of KS sets based on them.

Theorem 1. *For any Kochen-Specker graph G_{KS}, there exists a subgraph $G_{gad} < G_{KS}$ with $\omega(G_{gad}) = \omega(G_{KS})$ that is a 01-gadget. Moreover, given a 01-gadget G_{gad}, one can construct a KS graph G_{KS} with $\omega(G_{KS}) = \omega(G_{gad})$.*

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CONSTRUCTING KS PROOFS USING 01-GADGETS

- Suppose we have constructed a 01-gadget between any two given vectors.
- We may then construct multiple new KS proofs using several graph frustrations as in the figure below

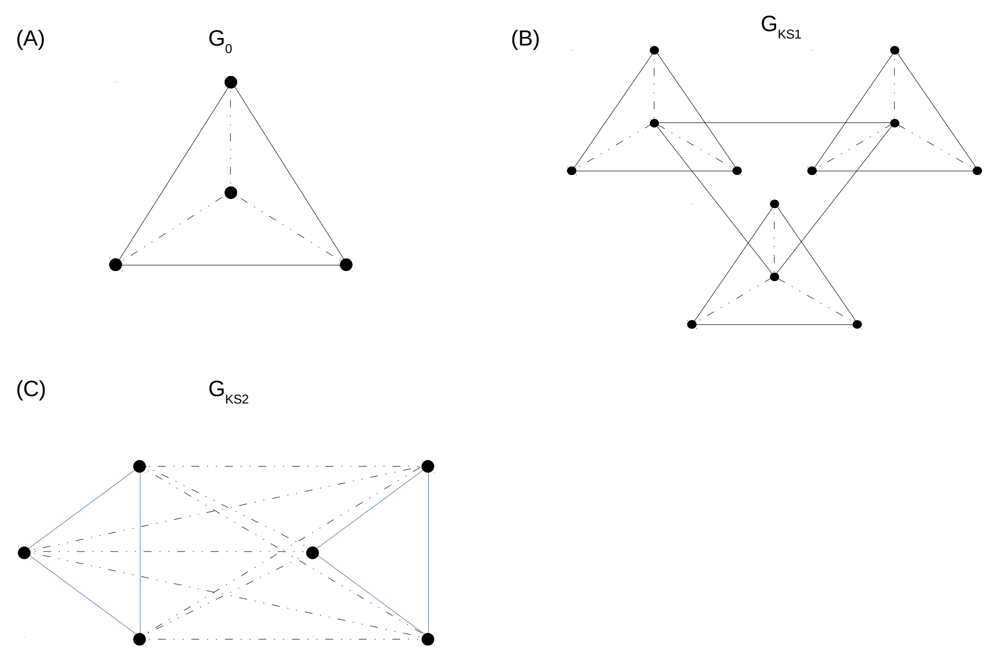


FIG. 4: Graphs with the dashed edges denoting 01-gadgets. (a) In any $\{0,1\}$-coloring of the graph G_0, the central vertex is necessarily assigned value 0. (b) Three copies of G_0 with the central vertices forming a basis in $\mathbb{C}^3$ so that the resulting graph G_{KS1} forms a Kochen-Specker proof. (c) Another proof of the KS theorem G_{KS2} is obtained by connecting every pair of vectors in two bases by a 01-gadget.

CONSTRUCTING MINIMAL 01-GADGETS

- Qn: Construct minimal 01-gadgets between any two given vectors
- Our candidate solution: uses 43 vectors.

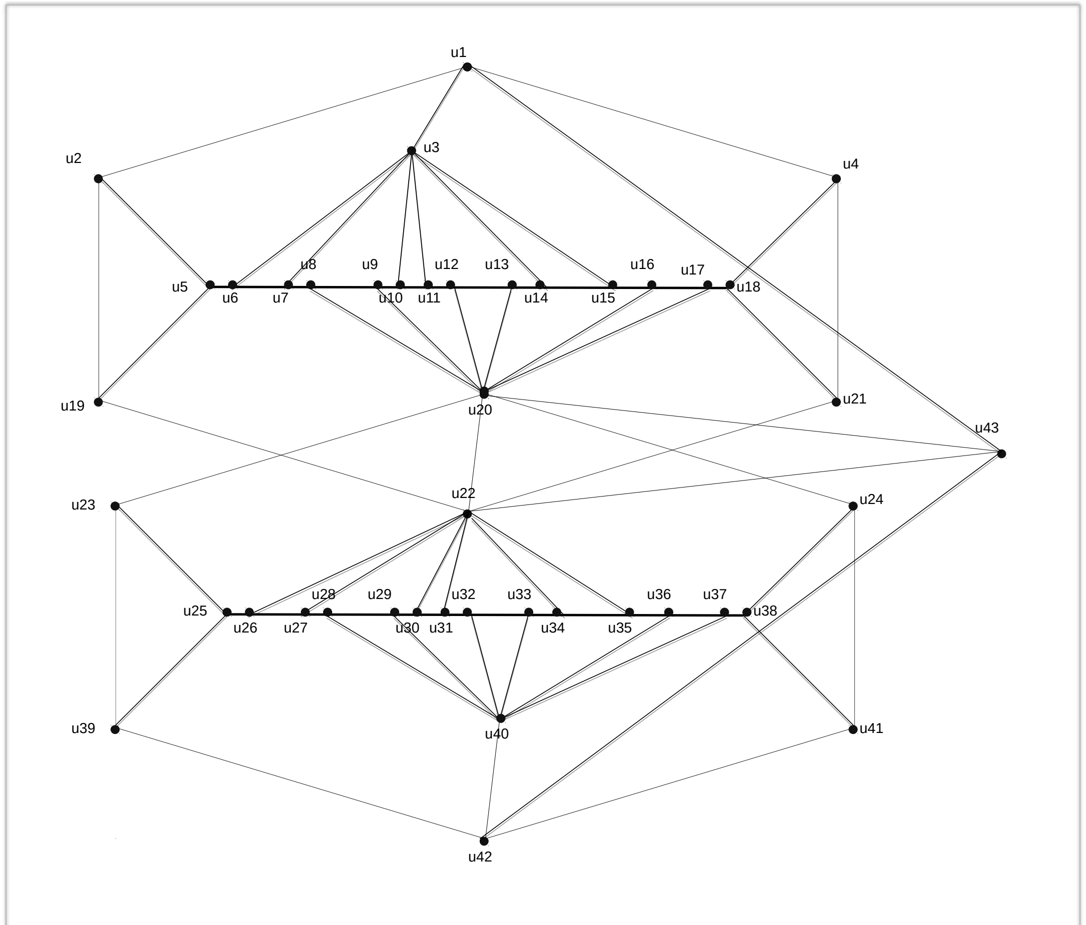


FIG. 3: The 43 vertex 01-gadget used in the proof of Theorem 2.

Theorem 2. *Let $|v_1\rangle$ and $|v_2\rangle$ be any two distinct non-orthogonal vectors in $\mathbb{C}^d$ with $d \geq 3$. Then there exists a 01-gadget in dimension d with $|v_1\rangle$ and $|v_2\rangle$ being the two distinguished vertices.*

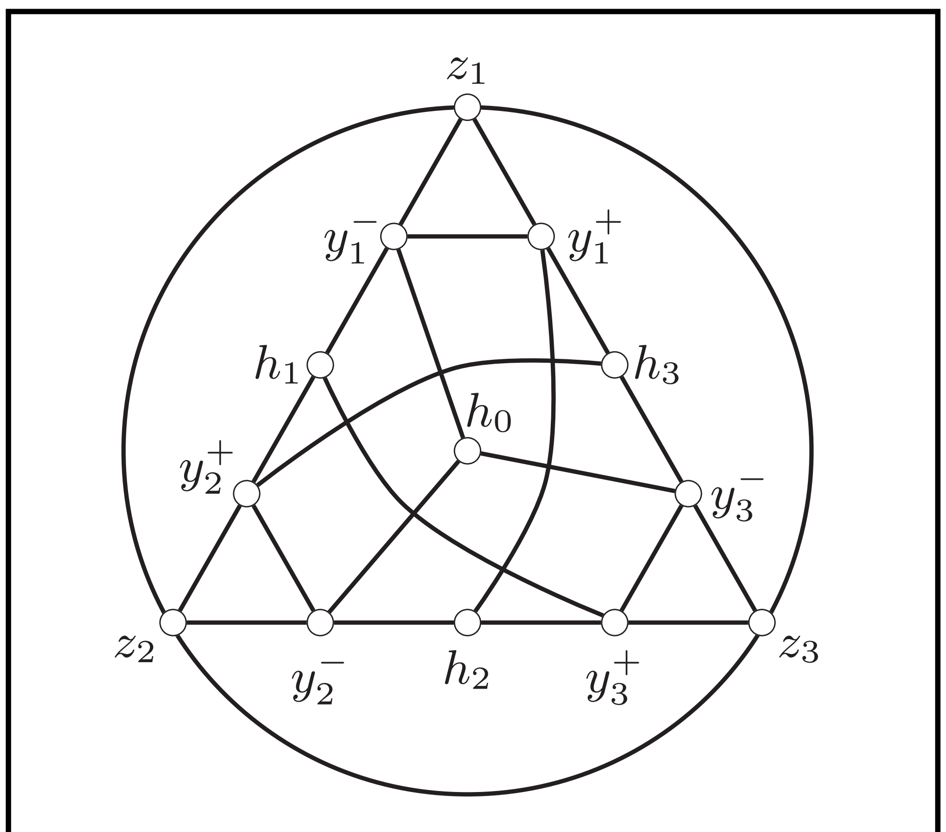
$$\begin{aligned}
&|u_1\rangle = (1,0,0)^T;\;\; |u_2\rangle = (0,1,-1)^T;\;\; |u_3\rangle = (0,1,0)^T;\\
&|u_4\rangle = (0,y,1)^T;\;\; |u_5\rangle = (2x,1,1)^T;\;\; |u_6\rangle = (-1,0,2x)^T;\\
&|u_7\rangle = (-2x,0,-1)^T;\;\; |u_8\rangle = (x,1,-2x^2)^T;\\
&|u_9\rangle = (2x^3,2x^2,1+x^2)^T;\\
&|u_{10}\rangle = (-(1+x^2),0,2x^3)^T;\\
&|u_{11}\rangle = (2x^3,0,1+x^2)^T;\\
&|u_{12}\rangle = (x(1+x^2),1+x^2,-2x^4)^T;\\
&|u_{13}\rangle = (2x^5,2x^4,(1+x^2)^2)^T;\\
&|u_{14}\rangle = (-(1+x^2)^2,0,2x^5)^T;\\
&|u_{15}\rangle = (2x^5,0,(1+x^2)^2)^T;\\
&|u_{16}\rangle = (x(1+x^2)^2,(1+x^2)^2,-2x^6)^T;\\
&|u_{17}\rangle = (2x^7,2x^6,(1+x^2)^3)^T;\\
&|u_{18}\rangle = (-x(1+y^2),-1,y)^T;\\
&|u_{19}\rangle = (1,-x,-x)^T;\;\; |u_{20}\rangle = (1,-x,0)^T;\\
&|u_{21}\rangle = (1,-x,xy)^T;\;\; |u_{22}\rangle = (x,1,0)^T;
\end{aligned} \tag{12}$$

with

$$y = \frac{(1+x^2)^3 + \sqrt{(1+x^2)^6 - 16x^{14}(1+x^2)}}{4x^8}. \tag{13}$$

STATISTICAL PROOFS OF THE KS THEOREM (A LA YU-OH)

- Yu & Oh proposed an alternative proof of the KS Theorem without the use of KS vector sets!
- The 13-vector Yu-Oh set is {0,1}-colorable but admits a state-independent non-contextuality inequality.



$$
\begin{array}{lll}
y_1^- = (0,1,-1) & h_1 = (-1,1,1) & z_1 = (1,0,0) \\
y_2^- = (1,0,-1) & h_2 = (1,-1,1) & z_2 = (0,1,0) \\
y_3^- = (1,-1,0) & h_3 = (1,1,-1) & z_3 = (0,0,1) \\
y_1^+ = (0,1,1) & h_0 = (1,1,1) & \\
y_2^+ = (1,0,1) & & \\
y_3^+ = (1,1,0) & &
\end{array}
$$

S. Yu and C. H. Oh. "State-independent proof of Kochen-Specker theorem with 13 rays". Phys. Rev. Lett. 108, 030402 (2012).

CONSTRUCTING STATISTICAL KS PROOFS USING 01-GADGETS

- How to construct new (and minimal) statistical KS proofs using 01-gadgets?
- First, note

Lemma 3. *Let $|u_i\rangle$, for $i = 1, \ldots, d+1$ be the unit vectors denoting the vertices of a d-dimensional simplex embedded in $\mathbb{R}^d$. Then*

$$\sum_{i=1}^{d+1} |u_i\rangle\langle u_i| = \frac{d+1}{d}\mathbb{I}. \tag{17}$$

- Consider the set of d+1 vertices of the regular d-simplex in $\mathbf{R}^d$ with additionally 01-gadgets connecting every pair of them (this is possible from Theorem 2)
- Then, this set of vectors admits the following state-independent NC inequality

$$\boxed{\sum_{i=1}^{d+1} f(|u_i\rangle) \leq 1}$$

- While the simplex is minimal, any set of vectors such that $\sum_i |\langle\psi|u_i\rangle|^2 > 1$ for all $|\psi\rangle \in \mathbb{C}^d$ can be used in the construction.

MINIMAL 01-GADGETS?

- Recently, we found a more clever 40 vector construction of a 01-gadget between arbitrary vectors.
- Question of minimality still open.

$|u_1\rangle = (1,-1,0)^T;\ \ |u_2\rangle = (1,1,1)^T;$
$|u_3\rangle = (1,1,0)^T;\ \ |u_4\rangle = (1,1,b)^T;$
$|u_5\rangle = (-2,1,1)^T;\ \ |u_6\rangle = (1,-1,3)^T;$
$|u_7\rangle = (3,-3,-2)^T;\ \ |u_8\rangle = (2,0,3)^T;$
$|u_9\rangle = (-3,0,2)^T;\ \ |u_{10}\rangle = (-2,2,-3)^T;$
$|u_{11}\rangle = (3,-3,-4)^T;\ \ |u_{12}\rangle = (4,0,3)^T;$
$|u_{13}\rangle = (-3,0,4)^T;\ \ |u_{14}\rangle = (-4,4,-3)^T;$
$|u_{15}\rangle = (3,-3,-8)^T;\ \ |u_{16}\rangle = (8,0,3)^T;$
$|u_{17}\rangle = (-3,0,8)^T;\ \ |u_{18}\rangle = (-8,4+\sqrt{7},-3)^T;$
$|u_{19}\rangle = (0,1,-1)^T; |u_{20}\rangle = (0,1,0)^T;$
$|u_{21}\rangle = (0,-3+8b,-16-3b)^T;\ \ |u_{22}\rangle = (1,0,0)^T;$
$|u_{23}\rangle = (1,0,-1)^T;\ \ |u_{24}\rangle = (2-\sqrt{2},0,1)^T;$
$|u_{25}\rangle = (1,-2,1)^T;\ \ |u_{26}\rangle = (0,1,2)^T;$
$|u_{27}\rangle = (0,2,-1)^T;\ \ |u_{28}\rangle = (1,-1,-2)^T;$
$|u_{29}\rangle = (1,-1,1)^T;\ \ |u_{30}\rangle = (0,1,1)^T;$
$|u_{31}\rangle = (0,1,-1)^T;\ \ |u_{32}\rangle = (-1,1,1)^T;$
$|u_{33}\rangle = (-1,1,-2)^T;\ \ |u_{34}\rangle = (0,2,1)^T;$
$|u_{35}\rangle = (0,1,-2)^T;\ \ |u_{36}\rangle = (2,-2,-1)^T;$
$|u_{37}\rangle = (1,-1,4)^T;\ \ |u_{38}\rangle = (-2-\sqrt{2},6-\sqrt{2},2)^T;$
$|u_{39}\rangle = |u_2\rangle;\ \ |u_{40}\rangle = |u_3\rangle;\ \ |u_{41}\rangle = (1,1,-2+\sqrt{2})^T;$
$|u_{42}\rangle = |u_1\rangle; |u_{43}\rangle = (0,0,1)^T;$

LOGICAL INDETEREMINACY PRINCIPLE (EXTENDED KS THEOREM)

- Pitowsky introduced an "Extended Kochen-Specker Theorem" that he termed the "Logical Indeterminacy Principle".
- The principle concerns arbitrary [0,1]-prob. assignments rather than {0,1}-colorings
- $f : \mathcal{S} \to [0,1]$ is a [0,1]-assignment satisfying the KS rules. For example, the Born rule $f(|v_i\rangle) = |\langle\psi|v_i\rangle|^2$ is a valid [0,1]-assignment.

Theorem 3 ([23]). *Let $|v_1\rangle$ and $|v_2\rangle$ be two non-orthogonal vectors in $\mathbb{C}^d$ with $d \geq 3$. Then there is a finite set of vectors $S \subset \mathbb{C}^d$ with $|v_1\rangle, |v_2\rangle \in S$ such that for any $[0,1]$-assignment, it holds that $f(|v_1\rangle), f(|v_2\rangle) \in \{0,1\}$ if and only if $f(|v_1\rangle) = f(|v_2\rangle) = 0$.*

- For any two non-orthogonal vectors, in any [0,1]-assignment at least one of the probabilities must be strictly between 0 and 1, unless they are both 0.

I. Pitowsky. *"Infinite and finite Gleason's theorems and the logic of indeterminacy". Journal of Mathematical Physics 39, 218 (1998).*

A. Abbott, C. S. Calude and K. Svozil. "A variant of the Kochen-Specker theorem localising value-indefiniteness". J. Math. Phys. 56, 102201 (2015).

LOCALISING VALUE INDEFINITENESS

Theorem 3 ([23]). *Let $|v_1\rangle$ and $|v_2\rangle$ be two non-orthogonal vectors in $\mathbb{C}^d$ with $d \geq 3$. Then there is a finite set of vectors $S \subset \mathbb{C}^d$ with $|v_1\rangle, |v_2\rangle \in S$ such that for any $[0,1]$-assignment, it holds that $f(|v_1\rangle), f(|v_2\rangle) \in \{0,1\}$ if and only if $f(|v_1\rangle) = f(|v_2\rangle) = 0$.*

- Corollary: if $f(|v_1\rangle) = 1$ then $f(|v_2\rangle) \neq 0, 1$ showing that one can localise the value-indefiniteness of quantum observables.
- Important for an application of contextuality in randomness generation.
- The $f(|v_2\rangle) \neq 0, 1$ is a good hash function.
- Outcome v_2 is random in any general probabilistic theory.

A. A. Abbott, C. S. Calude and K. Svozil. "A variant of the Kochen-Specker theorem localising value-indefiniteness". J. Math. Phys. 56, 102201 (2015).

A. A. Abbott, C. S. Calude, J. Conder and K. Svozil. "Strong Kochen-Specker theorem and incomputability of quantum randomness". Phys. Rev. A 86(6), 062109 (2012).

A. A. Abbott, C. S. Calude, and K. Svozil. "A Quantum Random Number Generator Certified by Value Indefiniteness". Mathematical Structures in Computer Science 24 (3) e240303 (2014).

EXTENDED 01-GADGETS

- To provide constructive proofs of the extended KS theorem, we will need special kind of gadgets that we term 'extended 01-gadgets'.
- Recall the definition of 01-gadgets

Definition 1. *A 01-gadget in dimension d is a $\{0,1\}$-colorable set $\mathcal{S}_{gad} \subset \mathbb{C}^d$ of vectors containing two distinguished vectors $|v_1\rangle$ and $|v_2\rangle$ that are non-orthogonal, but for which $f(|v_1\rangle) + f(|v_2\rangle) \leq 1$ in every $\{0,1\}$-coloring f of $\mathcal{S}_{gad}$.*

- Extended 01-gadgets are generalisations of 01-gadgets with {0,1}-coloring replaced by [0,1]-assignment and $f(|v_1\rangle) + f(|v_2\rangle) \leq 1$ replaced by $f(|v_1\rangle) + f(|v_2\rangle) < 2$
- For example, the bug is both a 01-gadget and an extended 01-gadget.

CONSTRUCTING EXTENDED 01-GADGETS

- We use ext. 01-gadgets to give constructive proofs of the extended KS theorem.
- First, let us construct extended 01-gadgets between any two given vectors.

Theorem 4. *Let $|v_1\rangle$ and $|v_2\rangle$ be any two distinct non-orthogonal vectors in $\mathbb{C}^d$ with $d \geq 3$. Then there exists an extended 01-gadget in dimension d with $|v_1\rangle$ and $|v_2\rangle$ being the two distinguished vertices.*

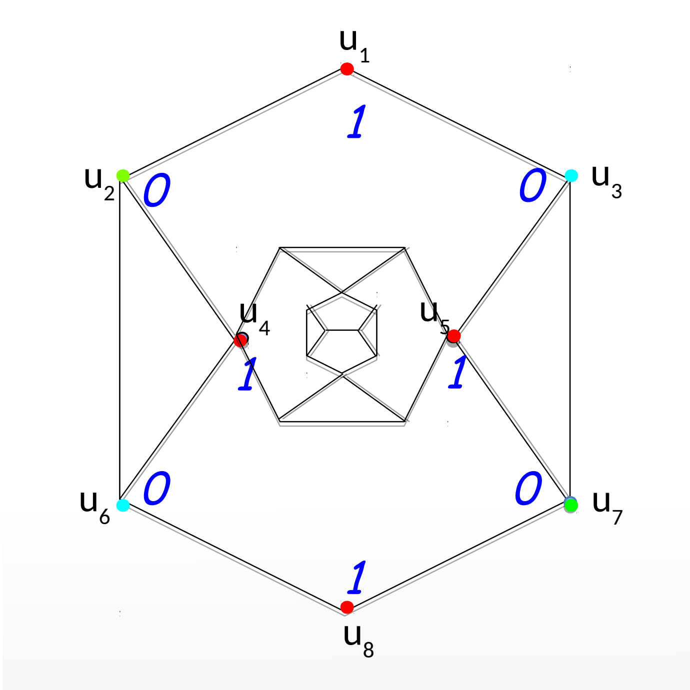


FIG. 5: An iterative construction of an extended 01-gadget for which the two distinguished vertices u_1 and u_8 are such that in the limit of large number of iterations k, $|\langle u_1^{(k)}|u_8^{(k)}\rangle| \in [0, 1[$.

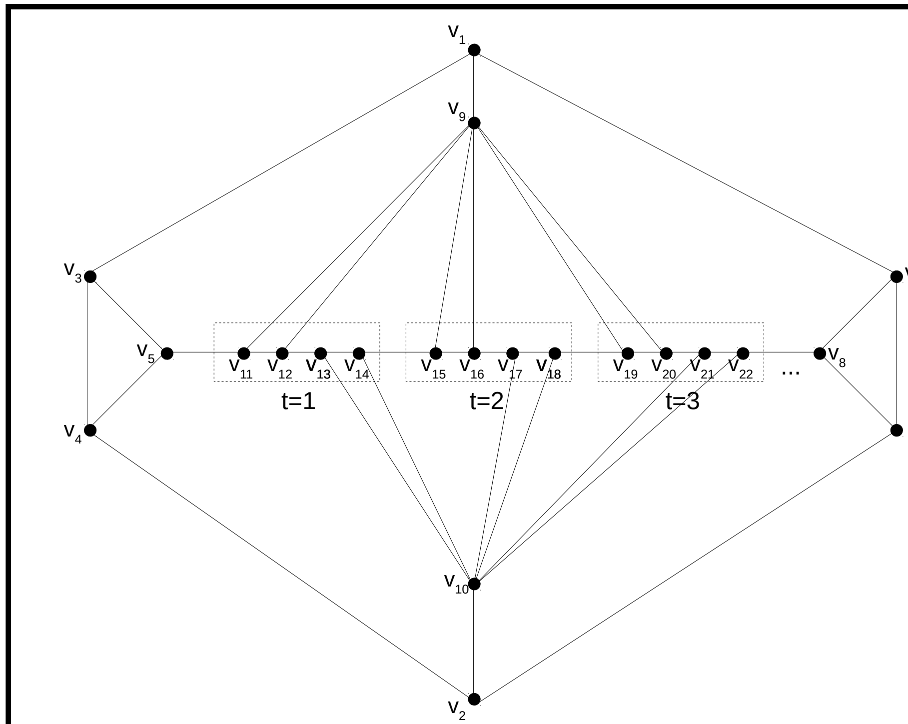


FIG. 6: An alternative construction of an extended 01-gadget for which the two distinguished vertices v_1 and v_2 are such that in the limit of large number t of the repeating unit of four vectors, $|\langle v_1|v_2\rangle|$ can take any value in $[0, 1[$.

CONSTRUCTING PROOFS OF THE EXTENDED KS THEOREM

- How to construct proofs of the extended KS theorem using extended 01-gadgets

$|v_3\rangle = |v_1\rangle \times |v_2\rangle$ $|v_4\rangle$ orthogonal to $|v_1\rangle$ and in the plane $\text{span}(|v_1\rangle, |v_2\rangle)$

$|v_5\rangle$ orthogonal to $|v_2\rangle$ and in the plane $\text{span}(|v_1\rangle, |v_2\rangle)$

Connect vectors using extended 01-gadgets as in the figure to obtain the proof

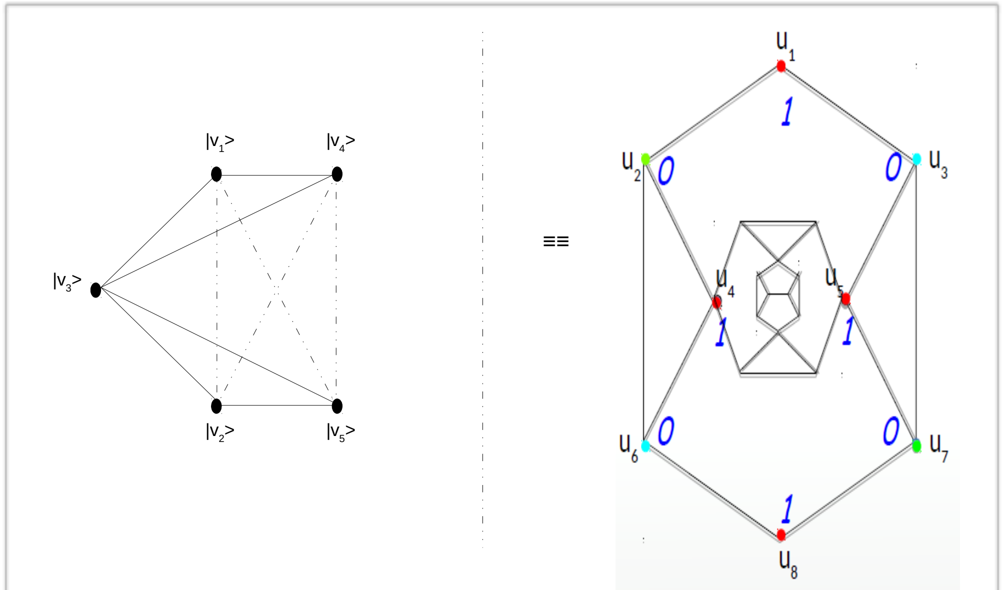


FIG. 7: A constructive proof of the extended Kochen-Specker theorem 3 using the extended 01-gadgets. Given vectors $|v_1\rangle, |v_2\rangle \in \mathbb{C}^d$, we obtain vector $|v_3\rangle \perp \text{span}(|v_1\rangle, |v_2\rangle)$ and two other vectors $|v_4\rangle, |v_5\rangle$ in the plane $\text{span}(|v_1\rangle, |v_2\rangle)$ with the orthogonality relations indicated in the left figure. Dashed edges between two vertices indicate an extended 01-gadget from Theorem 4 with the two vertices being distinguished.

COMPUTATIONAL COMPLEXITY OF CHECKING KS COLORABILITY

- Forbidden subgraphs in particular dimensions:

- It turns out that checking {0,1}-colorability of graphs without these forbidden subgraphs is still NP-complete

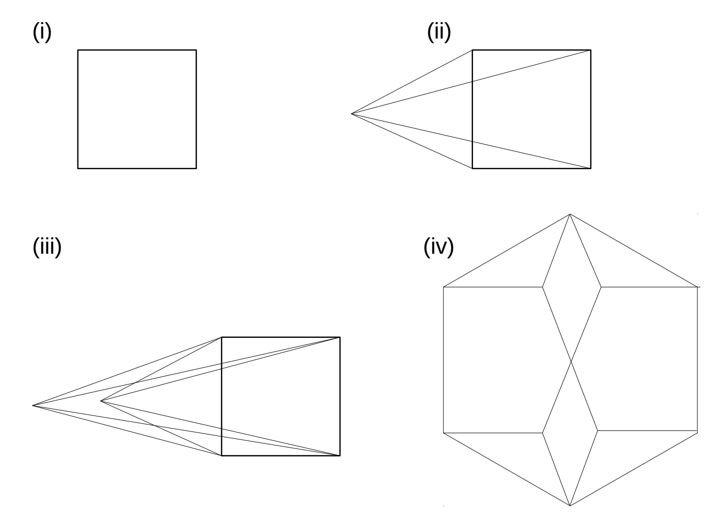


FIG. 8: Examples of forbidden subgraphs in dimensions $3, 4$ and 5. Graph (i) is the square graph which is not faithfully realizable in $\mathbb{C}^3$ as explained in the text. Graph (iv) is the graph from [14] which was verified to be not faithfully realizable in dimension three despite being square-free. Graph (ii) is not faithfully realizable in $\mathbb{C}^4$, which can be seen as arising from the fact that the induced square subgraph is not faithfully realizable in $\mathbb{C}^3$ and the additional vertex being adjacent to all vertices of the square, the vector corresponding to this vertex occupies an orthogonal subspace to that spanned by the square. Graph (iii) is similarly not realizable in $\mathbb{C}^5$ this time owing to the presence of two vertices (which themselves cannot be represented by identical vectors) that are adjacent to all the vertices of the square. It is clear that the construction can be extended to higher dimensions.

Theorem 5 (see also [15]). *Checking $\{0,1\}$-colorability of $\{G_{fbd}\}$-free graphs is NP-complete.*

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APPLICATIONS OF CONTEXTUALITY: RANDOMNESS GENERATION

- Device-independent application: thru' link with Hardy paradoxes (Prof. Pawel Horodecki's talk).
- Suppose Alice and Bob measure according to the Clifton bug configuration and check for perfect correlations.
- In any no-signaling theory:

$$p_1 + p_2 \leq 1, \quad p_1 + p_6 \leq 1, \quad p_4 + p_5 \leq 1, \quad p_7 + p_8 \leq 1, \quad p_3 + p_2 \leq 1$$

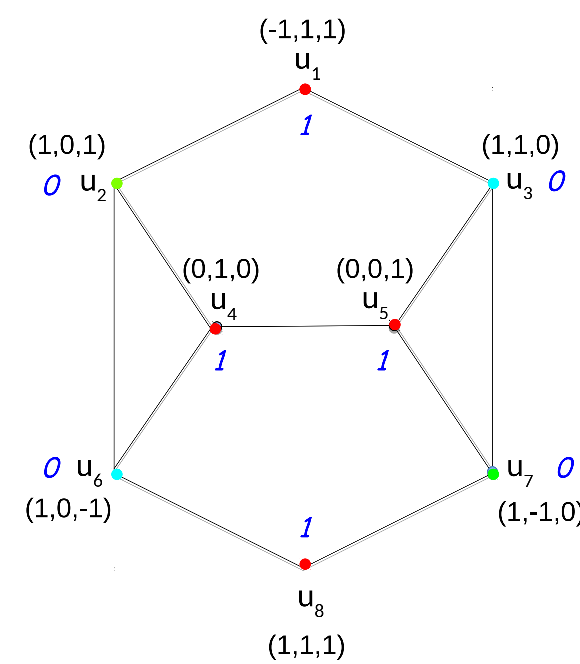
$$p_2 + p_3 + p_4 = 1, \quad p_5 + p_6 + p_7 = 1$$

- Translates to:

$$p(A_1 = 1) + p(B_8 = 1) \leq \frac{3}{2}.$$

$$\frac{1}{27} \leq p(A_1 = 1, B_8 = 1) \leq \frac{3}{4}$$

- This i-o combination can be employed in a DI protocol.



R. Ramanathan, F.G.S.L. Brandao, K. Horodecki, M. Horodecki, P. Horodecki and H. Wojewodka. "Randomness Amplification under minimal fundamental assumptions on the devices". Phys. Rev. Lett. 117, 230501 (2016).

APPLICATIONS OF CONTEXTUALITY: RANDOMNESS GENERATION

- Hardy paradoxes turn out to have interesting applications in DI protocols for randomness amplification.
- Crucial parameter controlling noise: prob. Of Hardy output.
- The 01-gadgets can be used to construct Hardy paradoxes with the prob. Of Hardy output in the entire range (0,1].

Proposition 9. *There exist Hardy paradoxes for the maximally entangled state* $\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i,i\rangle$ *for all* $d \geq 3$ *with the non-zero probability taking any value in* $(0, \frac{1}{d}]$*. In dimensions four and eight, there exist Hardy paradoxes for the maximally entangled state with the non-zero probability taking any value in* $(0, 1]$*.*

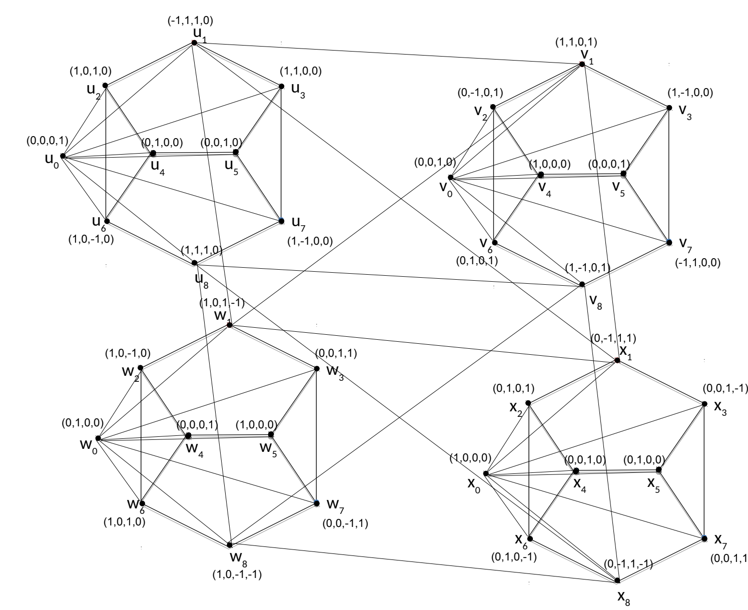


FIG. 11: An illustration of the Hardy paradox construction in dimension four by means of the real orthogonal matrix in Eq.(96). We embed the Clifton graph in dimension four by adding the extra vertex u_0 corresponding to the vector $(0,0,0,1)^T$. We then obtain four copies of the resulting gadget in dimension four by multiplication by the orthogonal unit as in Eq.(96) (not all resulting edges are shown for clarity). The two distinguished vertices u_1 and u_8 give rise in this manner to distinguished measurement bases given as $\{|u_{1/8}\rangle, |v_{1/8}\rangle, |w_{1/8}\rangle, |x_{1/8}\rangle\}$. In the scenario of the gadget constructed in Prop. 8 with $|v_1\rangle$ and $|v_2\rangle$ identical, measurements on a maximally entangled ququart state by Alice and Bob in these respective bases give rise to a Hardy paradox with non-zero probability in $(0,1]$ as explained in the text.

CONCLUSIONS AND OPEN QUESTIONS

- Constructed 01-gadgets and applied them to produce novel KS sets.
- Minimal 01-gadgets? So far: 40 vectors for arbitrary distinguished vectors.
- If you have a non-{0,1}-colorable graph, chances are with 01-gadgets you can convert it into a KS proof.

- Introduced extended 01-gadgets.
- Applied to produce constructive proofs of Pitowsky's Logical Indeterminacy.
- Minimal constructions? Current proofs require large vector sets for arbitrary distinguished vectors.

- Important role of 01-gadgets in Randomness Generation to be explored further.
- Do rational representations of 01-gadgets exist?…