Proof of the Peres Conjecture for Contextuality

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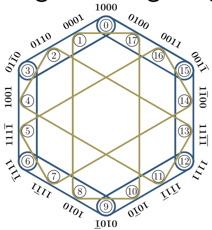

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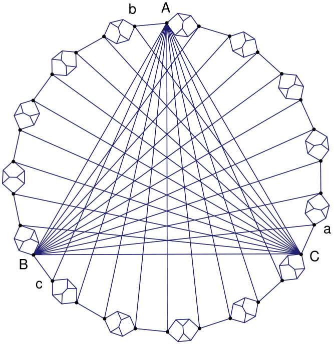
Overview

1. Kochen-Specker theorem and sets of vectors
2. Hardy-type and GHZ-type proofs
3. Proof of the Peres conjecture

The Kochen-Specker theorem



Basic Results

Measurements as events

- Projectors $P = |\psi\rangle\langle\psi|$ can be seen as "yes/no"-questions, corresponding to "click" or "no click" events in a detector.
- Two orthogonal projectors are exclusive: not both anwers are "yes".
- A set of mutually orthogonal projectors forms a context.
- Orthogonal projectors that add up to identity

$$\sum_k P_k = \mathbb{1}_d$$

  form a complete context: One answer has to be "yes"

Kochen-Specker Theorem

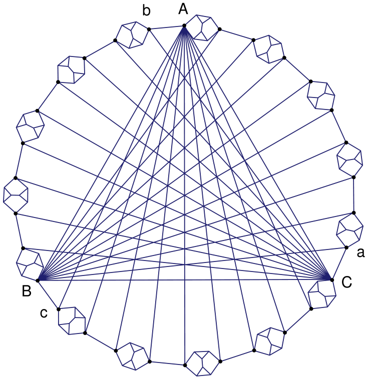
If $d \geq 3$ there is no way to assign to each projector the results "yes" or "no" such that the rules from above are fulfilled.

Historical remarks

- 1960: Ernst Specker states the first version.
  E. Specker, Dialectica 14, 239 (1960)
- 1966: John Bell proves a similar result with additional assumptions.
  J.S. Bell, Rev. Mod. Phys. 38, 447 (1966)
- 1967: Explicite proof from Simon Kochen and Ernst Specker
  S. Kochen, E.P. Specker, J. Math. Mech. 17, 59 (1967)

Proof

Question: How can one prove this?
Idea: Write down a set of projectors and find a contradiction.



Original idea: Take 117 vectors for $d = 3$.

S. Kochen, E.P. Specker, J. Math. Mech. 17, 59 (1967)

A simpler proof

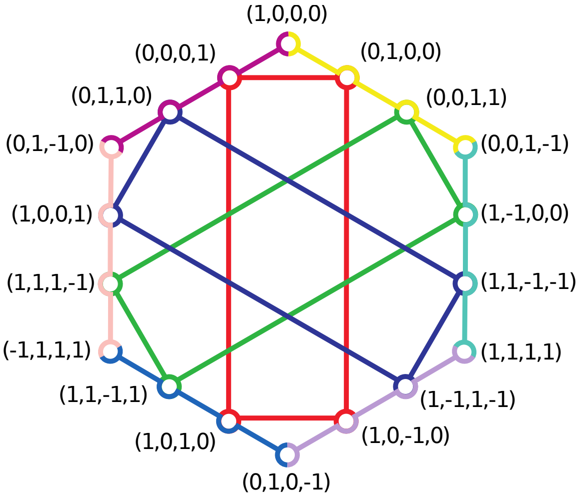
- Consider the following vectors in $d = 4$ (using $010\bar{1} = (0, 1, 0, -1)$) and try to assign "yes" or "no".

$$
\begin{array}{ccccccccc}
1000 & 1111 & 1111 & 1000 & 1001 & 1001 & 111\bar{1} & 111\bar{1} & 100\bar{1} \\
0100 & 11\bar{1}\bar{1} & 1\bar{1}1\bar{1} & 0010 & 0100 & 1\bar{1}1\bar{1} & 1\bar{1}00 & 0101 & 0110 \\
0011 & 1\bar{1}00 & 10\bar{1}0 & 0101 & 0010 & 11\bar{1}\bar{1} & 0011 & 10\bar{1}0 & 11\bar{1}1 \\
001\bar{1} & 001\bar{1} & 010\bar{1} & 010\bar{1} & 100\bar{1} & 0110 & 11\bar{1}1 & 1\bar{1}11 & 1\bar{1}11
\end{array}
$$

- Each column forms a basis or complete set of projectors
  $\Rightarrow$ The number of "yes" is nine.
- Each vector appears twice $\Rightarrow$ The number of "yes" is even.
- Contradiction! The assignment is not possible.
- Is this really the simplest proof?

A. Cabello et al., PLA 212 183 (1996), A. Cabello, Int. J. Quantum Inf. 4, 55 (2006).

Artist's View



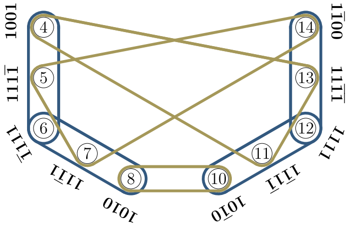
Thanks to Adan!

The Peres Conjecture

In my paper[5] I also stated that 24 was the minimal number number in four dimensions. David was not convinced by my argument. We exchanged several letters on this issue but could not resolve it. Meanwhile I was curious to see the 31 rays in three dimensions. I guessed they were a subset of the same cubic lattice and I did the search with a computer program. The algorithm appears in my book,[6] where there is also an exercise: write a computer program for the four-dimensional case, and check that 24 rays are the minimum number. I never bothered to solve that boring exercise, but two students took up the challenge and found that it was possible to remove *any one* of the 24 rays, and still have a KS set. Michael Kernaghan, in Canada, found a KS set with 20 rays[7] and then Adán Cabello, together with José Manuel Estebaranz and Guillermo García Alcaine in Madrid, found a set of 18 rays.[8] They still hold the world record (probably for ever).

A. Peres, Found. Phys. 33, 1543 (2003).

Hardy, GHZ and finally Peres



Proofs based on QM

Type I: KS sets

Write down a set of projectors and find a contradiction. This does not require any quantum state.

Type II: Hardy & GHZ proofs

- Write down a set of (potentially incomplete) contexts $\{C_k\}$. Denote for the probabilities in some theory $\vec{p}|_C := \sum_{i\in C} p_i$.
- Then, we may have for some NCHV model:

$$\vec{p}|_{C_k} = 1 \text{ for all } k = 1, 2, \ldots, K \;\Rightarrow\; \vec{p}|_{C_0} = 0,$$

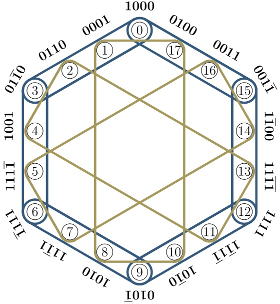
- Hardy-type proof: For QM, we may find a state with:

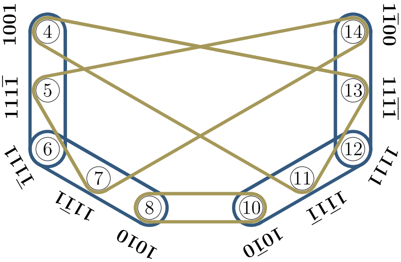
$$\vec{p}|_{C_k} = 1, \forall k = 1, 2, \ldots, K \text{ but } \vec{p}|_{C_0} > 0,$$

- GHZ-type proof: In the same situation, we find for some quantum state $\vec{p}|_{C_0} = 1$, then it is a GHZ-type proof.

S. Abramsky and A. Brandenburger, NJP 13, 113036 (2011).

Example: GHZ from CEG



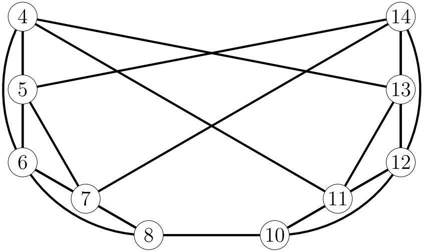


- Right figure: If all three-vector contexts obey $\vec{p}|_{C_k} = 1$, then in a NCHV model one has $\vec{p}|_{C_k} = 0$ for the two-vector context.
- QM Take $|\psi_0\rangle = (1, 0, 0, 0)$. Then for *all* three- and two-vector contexts: $\vec{p}|_{C_k} = 1$.
- This trick is generic!

Example known from: A. Cabello, J. Estebaranz, and G. Garcia-Alcaine, Phys. Lett. A 212, 183 (1996).

Finding the minimal GHZ-type proof

The graph-theoretic approach



- Contextuality scenarios can be encoded into exclusivity graphs.
- Probabilities from NCHV theories: Stable set polytope STAB(G).
- Probabilities from QM: Theta body TH(G).
- Check all 288266 graphs with ≤ 9 vertices with LP and SDPs: There is no GHZ proof with 9 vertices.

A. Cabello, S. Severini, and A. Winter, PRL 112, 040401 (2014). E. Amselem et al., PRL 108, 200405 (2012).

Attacking the Peres Conjecture

Idea

If there were a KS set with 17 vectors, then one could construct a GHZ-type proof with 9 vectors.

- Observation: Small KS sets must contain complete contexts, which are overlapping. That is, they contain bases sharing some states.
- Take a KS set with n vectors for $d = 6$ (as example).
- Take two complete contexts C_1, C_2 with non-empty intersection and $|\psi_0\rangle \in C_1 \cap C_2$.
- We must have $|C_1 \cup C_2| \geq d + 2$, otherwise $C_1 = C_2$.
- Taking system in the state $|\psi_0\rangle$ and removing vectors in $C_1 \cup C_2$ gives a GHZ proof with maximally $n - d + 2 = n - 8$ vectors, so $n \geq 18$.

Conclusion

Results

- There is no GHZ-type contextuality proof with less than 10 vectors.
- There is no KS set with less than 18 vectors.
- This helps to identify minimal contextuality scenarios and study their role for information processing.

Literature

Z.-P. Xu, J.-L. Chen, O. Gühne, Phys. Rev. Lett. 124, 230401 (2020)

Acknowledgements

DFG

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THE ROYAL SOCIETY

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