# From Vorob'ev's theorem to monogamy of non-locality and local macroscopic averages

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INTERNATIONAL IBERIAN NANOTECHNOLOGY LABORATORY

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17th May 2021

This talk is based on:

'*On monogamy of non-locality and macroscopic averages*'
RSB, Proceedings QPL 2014, arXiv:1412.8541 [quant-ph].

'*Contextuality in quantum mechanics and beyond*'
RSB, DPhil thesis, University of Oxford, 2015.

# Monogamy and average macroscopic locality

- **Average macroscopic correlations from microscopic models are local**

  For multipartite quantum models:

  '*Local realism of macroscopic correlations*'
  Ramanathan, Paterek, Kay, Kurzyński, Kaszlikowski, Phys. Rev. Lett. 107, 060405, 2011.

- **Monogamy of violation of Bell inequalities**

  For bipartite no-signalling models:

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- Connect and generalise these two results,
- providing a structural explanation related to Vorob'ev's theorem.
- We will mainly consider a simple illustrative example.

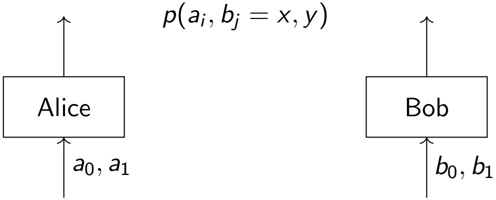
# Related

Related:

‘*Generalized monogamy of contextual inequalities from the no-disturbance principle*’
Ramanathan, Soeda, Kurzyński, Kaszlikowski, Phys. Rev. Lett. 109, 050404, 2012.

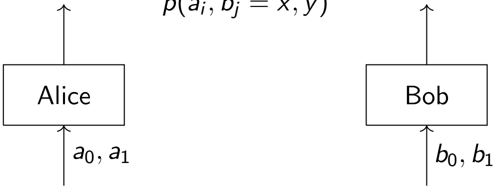
# Monogamy of non-locality

# Non-locality



# Non-locality

$$p(a_i, b_j = x, y)$$




| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $a_0 b_0$ | $1/2$ | 0 | 0 | $1/2$ |
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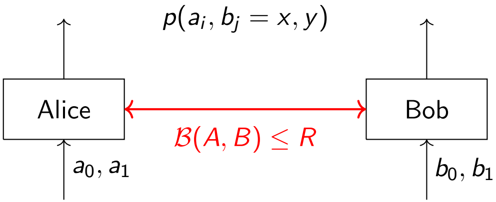


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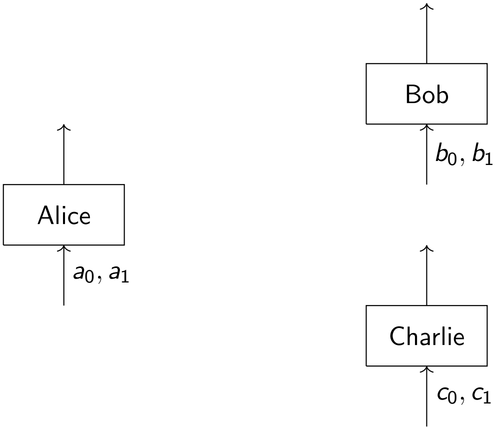

# Monogamy of non-locality

- Empirical model: no-signalling probabilities

$$p(a_i, b_j, c_k = x, y, z)$$

where $x$, $y$, $z$ are possible outcomes.

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  where $x$, $y$, $z$ are possible outcomes.

- Consider the subsystem composed of $A$ and $B$ only, given by marginalisation (in QM, partial trace):

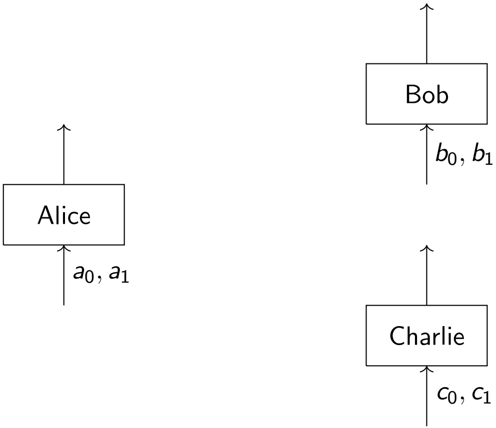
$$p(a_i, b_j = x, y) \;=\; \sum_z p(a_i, b_j, c_k = x, y, z)$$

  This is independent of $c_k$ due to no-signalling.

- Similarly define $p(a_i, c_k = x, z)$. (subsystem consisting of A and C)

# Monogamy of non-locality

Given a Bell inequality $\mathcal{B}(-,-,) \leq R$,



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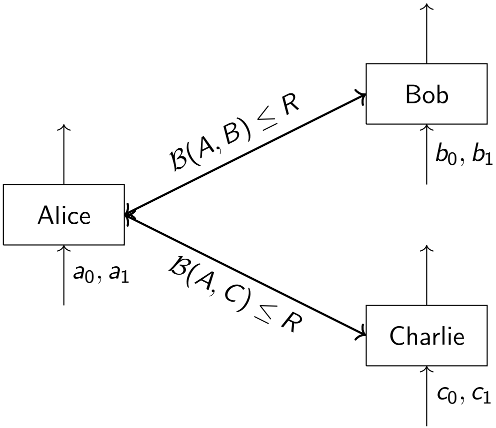

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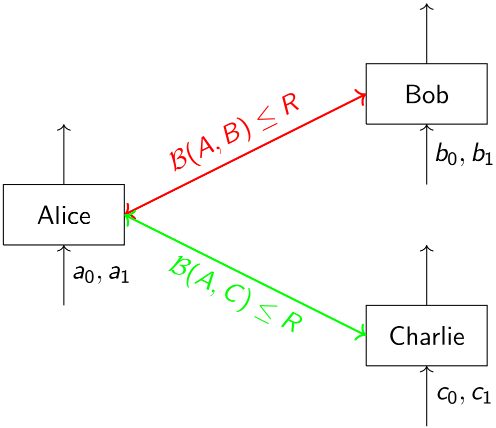

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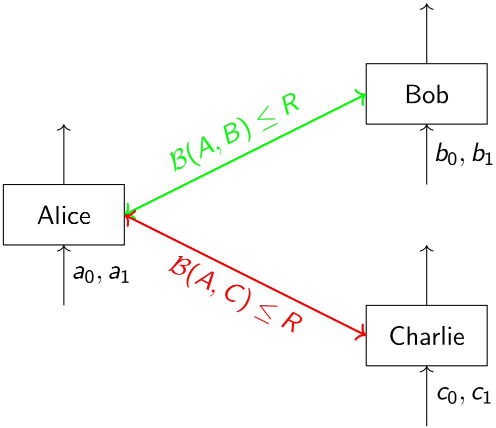

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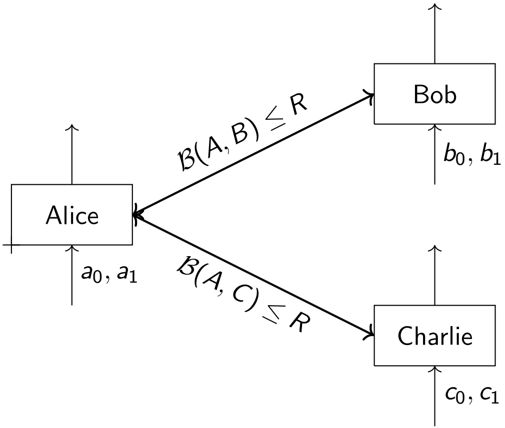


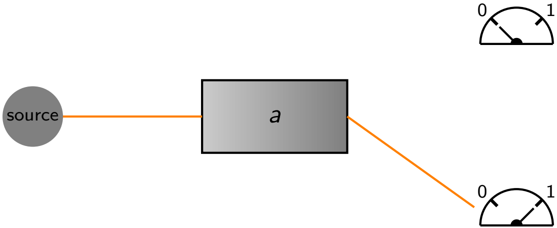

**Monogamy relation:** $\mathcal{B}(A,B) + \mathcal{B}(A,C) \leq 2R$

# Locality of macroscopic averages

# Macroscopic measurements

**(Microscopic) dichotomic measurement**

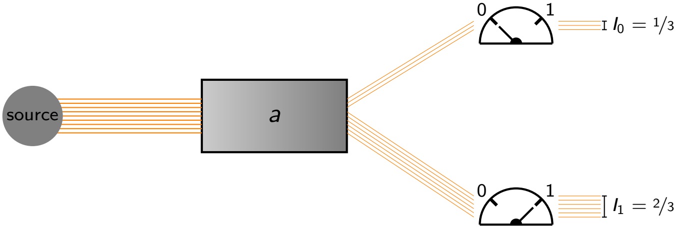
- A single particle is subject to an interaction $a$ and collides with one of two detectors: outcomes 0 and 1.
- The interaction is probabilistic: $p(a = x)$, with $x = 0, 1$.



# Macroscopic measurements

## Macroscopic dichotomic measurement

- Consider beam (or region) of $N$ particles, differently prepared.
- Subject each particle to the interaction $a$: the beam gets divided into 2 smaller beams hitting each of the detectors.



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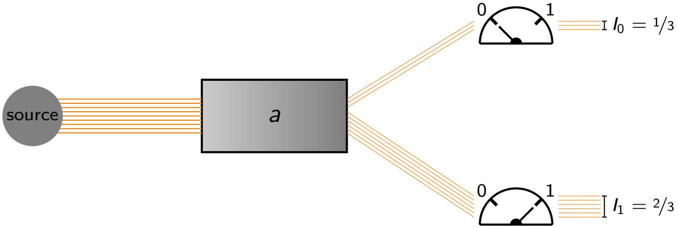

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- Outcome represented by the intensity of resulting beams:
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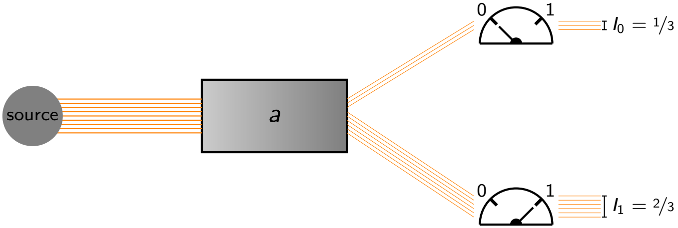
- We are only concerned with the mean, or **expected value**, of such intensities.



# Macroscopic average behaviour

- This mean intensity can be interpreted as the average behaviour among the $N$ particles:

  if we would randomly select one of the particles and subject it to the microscopic measurement $a$, we would obtain outcome $x$ with probability $I_x$:

$$I_x = \frac{1}{N} \sum_{i=1}^{N} p_i(a = x) \ .$$

- The situation is analogous to statistical mechanics, where a macrostate arises as an averaging over an extremely large number of microstates, and hence several different microstates can correspond to the same macrostate.

# Macroscopic average behaviour: multipartite

- **Multipartite macroscopic scenarios**
  - several 'macroscopic' sites consisting of a large number of microscopic sites/particles;
  - several (macro) measurement settings at each site.

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- **Average macroscopic Bell experiment**
  the (mean) values of the macroscopic intensities indicate the behaviour of a randomly chosen tuple of particles: one from each of the beams, or sites.
- We'll show that, as long as there are enough particles (microscopic sites) in each macroscopic site, such average macroscopic behaviour is always local no matter which no-signalling model accounts for the underlying microscopic correlations.

# Macroscopic average behaviour: (toy) tripartite example

- We regard sites $B$ and $C$ as forming one 'macroscopic' site, $M$, and $A$ as forming another.
- In order to be 'lumped together', $B$ and $C$ must be symmetric/of the same type: the symmetry identifies the measurements $b_0 \sim c_0$ and $b_1 \sim c_1$, giving rise to 'macroscopic' measurements $m_0$ and $m_1$.

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- Given a (tripartite) empirical model $p(a_i, b_j, c_k = x, y, z)$, the 'macroscopic' average behaviour is a bipartite model (with two macro sites $A$ and $M$) given by the following average of probabilities of the partial models:

$$p(a_i, m_j = x, y) = \frac{p(a_i, b_j = x, y) + p(a_i, c_j = x, y)}{2}$$

# Example: $W$ state

$Z$ and $X$ measurements on the $W$ state:

| | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| $a_0 b_0 c_0$ | 9 | 1 | 1 | 1 | 1 | 1 | 1 | 9 |
| $a_0 b_0 c_1$ | 8 | 2 | 0 | 2 | 0 | 2 | 8 | 2 |
| $a_0 b_1 c_0$ | 8 | 0 | 2 | 2 | 0 | 8 | 2 | 2 |
| $a_0 b_1 c_1$ | 4 | 4 | 4 | 0 | 4 | 4 | 4 | 0 |
| $a_1 b_0 c_0$ | 8 | 0 | 0 | 8 | 2 | 2 | 2 | 2 |
| $a_1 b_0 c_1$ | 4 | 4 | 4 | 4 | 4 | 0 | 4 | 0 |
| $a_1 b_1 c_0$ | 4 | 4 | 4 | 4 | 4 | 4 | 0 | 0 |
| $a_1 b_1 c_1$ | 0 | 8 | 8 | 0 | 8 | 0 | 0 | 0 |

(every entry should be divided by 24)

# Example: $W$ state

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $a_0 m_0$ | 10 | 2 | 2 | 10 |
| $a_0 m_1$ | 8 | 4 | 8 | 4 |
| $a_1 m_0$ | 8 | 8 | 4 | 4 |
| $a_1 m_1$ | 8 | 8 | 8 | 0 |

(every entry should be divided by 24)

This is **local**!

# Another example model

| | 000 | 001 | 010 | 011 | 100 | 101 | 110 | 111 |
|---|---|---|---|---|---|---|---|---|
| $a_0 b_0 c_0$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_0 b_0 c_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_0 b_1 c_0$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_0 b_1 c_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_1 b_0 c_0$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_1 b_0 c_1$ | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| $a_1 b_1 c_0$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| $a_1 b_1 c_1$ | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |

(every entry should be divided by 4)

# Another example model

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $a_0 b_0$ | 2 | 0 | 0 | 2 |
| $a_0 b_1$ | 2 | 0 | 0 | 2 |
| $a_1 b_0$ | 2 | 0 | 0 | 2 |
| $a_1 b_1$ | 0 | 2 | 2 | 0 |

(divided by 4)

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $a_0 c_0$ | 1 | 1 | 1 | 1 |
| $a_0 c_1$ | 1 | 1 | 1 | 1 |
| $a_1 c_0$ | 1 | 1 | 1 | 1 |
| $a_1 c_1$ | 1 | 1 | 1 | 1 |

(divided by 4)

maximally non-local

local

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $a_0 m_0$ | 3 | 1 | 1 | 3 |
| $a_0 m_1$ | 3 | 1 | 1 | 3 |
| $a_1 m_0$ | 3 | 1 | 1 | 3 |
| $a_1 m_1$ | 1 | 3 | 3 | 1 |

(every entry should be divided by 8)

Again, this is **local**!

# Connecting monogamy and macroscopic averages

# A simple observation

Consider **any** bipartite Bell inequality $\mathcal{B}(-,-) \leq R$ given by coefficients $\alpha(i,j,x,y)$ and bound $R$.

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$$
\begin{aligned}
& \mathcal{B}(A, M) \leq R \\
\Leftrightarrow \\
& \sum_{i,j,x,y} \alpha(i,j,x,y) p(a_i, m_j = x, y) \leq R \\
\Leftrightarrow \\
& \sum_{i,j,x,y} \alpha(i,j,x,y) \frac{p(a_i, b_j = x, y) + p(a_i, c_j = x, y)}{2} \leq R \\
\Leftrightarrow \\
& \sum_{i,j,x,y} \alpha(i,j,x,y) p(a_i, b_j = x, y) + \sum_{i,j,x,y} \alpha(i,j,x,y) p(a_i, c_j = x, y) \leq 2R \\
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& \mathcal{B}(A, B) + \mathcal{B}(A, C) \leq R
\end{aligned}
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\Leftrightarrow \quad & \\
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\end{aligned}
$$

The average model $p_{a_i,m_j}$ **satisfies the Bell inequality** if and only if in the microscopic model Alice is **monogamous** with respect to violating it with Bob and Charlie.

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- Equivalently, the examples satisfied the monogamy relation for **any** Bell inequality.
- This is true for all no-signalling empirical models on the tripartite scenario under consideration, with two measurement settings per site.
- We now give a structural explanation for this...
- . . . which generalises well beyond this particular scenario.

# Vorobʹev’s theorem

# Formalising empirical data

A **measurement scenario** $\mathbf{X} = \langle X, \Sigma, O \rangle$:

- $X$ – a finite set of measurements
- $O = (O_x)_{x \in X}$ – for each $x \in X$ a non-empty set of possible outcomes $O_x$
- $\Sigma$ – an abstract simplicial complex on $X$ faces are called the **measurement contexts**

| A | B | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
|---|---|---|---|---|---|
| $a_0$ | $b_0$ | $--$ | $--$ | $--$ | $--$ |
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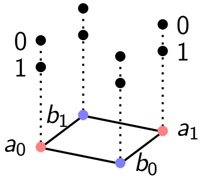

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| family of finite subsets of $X$ such that: |
|---|
| ▶ it contains all the singletons: $\forall x \in X.\ \{x\} \in \Sigma$. |
| ▶ it is downwards closed: $\sigma \in \Sigma$ and $\tau \subseteq \sigma$ implies $\tau \in \Sigma$ . |

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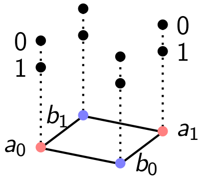

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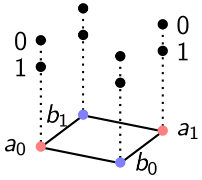

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An **empirical model** $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on $\mathbf{X}$:

- each $e_\sigma \in \mathrm{Prob}\left(\prod_{x \in \sigma} O_x\right)$ is a probability distribution over joint outcomes for $\sigma$.
- *generalised no-signalling* holds: for any $\sigma, \tau \in \Sigma$, if $\tau \subseteq \sigma$,

$$e_\sigma|_\tau = e_\tau$$

(i.e. marginals are well-defined)

| A | B | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
|---|---|---|---|---|---|
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| $a_1$ | $b_0$ | $1/2$ | $0$ | $0$ | $1/2$ |
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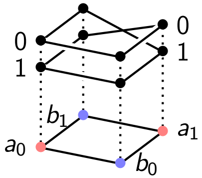

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- $O = (O_x)_{x \in X}$ – for each $x \in X$ a non-empty set of possible outcomes $O_x$
- $\Sigma$ – an abstract simplicial complex on $X$ faces are called the **measurement contexts**

An **empirical model** $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on $\mathbf{X}$:

- each $e_\sigma \in \text{Prob}\left(\prod_{x \in \sigma} O_x\right)$ is a probability distribution over joint outcomes for $\sigma$.
- *generalised no-signalling* holds: for any $\sigma, \tau \in \Sigma$, if $\tau \subseteq \sigma$,

$$e_\sigma|_\tau = e_\tau$$

(i.e. marginals are well-defined)

| A | B | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
|---|---|---|---|---|---|
| $a_0$ | $b_0$ | $1/2$ | $0$ | $0$ | $1/2$ |
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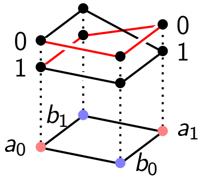
$X = \{a_0, a_1, b_0, b_1\}, \; O_x = \{0, 1\}$

$\Sigma = \downarrow \{ \{a_0, b_0\}, \{a_0, b_1\}, \{a_1, b_0\}, \{a_1, b_1\} \}.$



# Contextuality

An empirical model $e = \{e_\sigma\}_{\sigma \in \Sigma}$ on a measurement scenario $(X, \Sigma, O)$ is **non-contextual** if there is a distribution $d$ on $\prod_{x \in X} O_x$ such that, for all $\sigma \in \Sigma$:

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The import of Bell's and Kochen–Spekker's theorems is that there are behaviours arising from quantum mechanics that are contextual.

# Vorobʹev's theorem

'*Consistent families of measures and their extensions*'
Vorobʹev, Theory Probab. Appl. 7(2), 1962.

- In the context of game theory.
- Consider a collection of variables
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- Necessary and sufficient condition: **regularity** or **acyclicity**!

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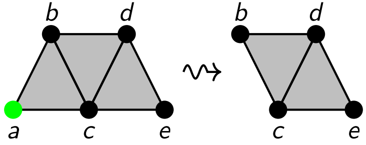

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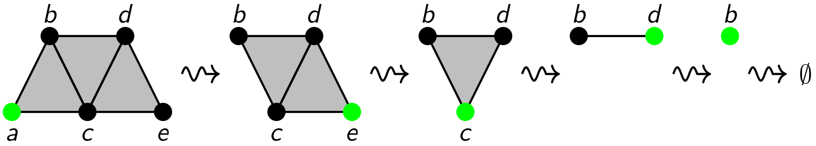

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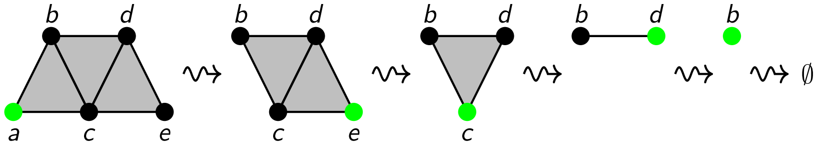
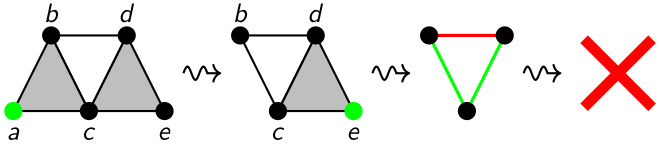



- $\Sigma$ not acyclic: Graham reduction fails.



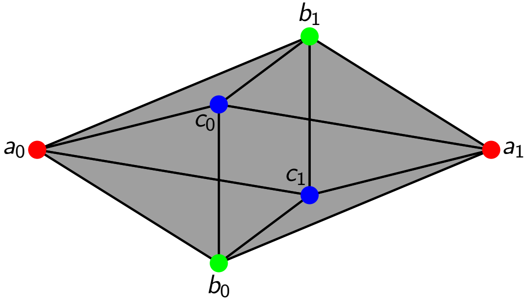
# Vorob'ev's theorem

### Theorem (Vorob'ev 1962, adapted)

*All empirical models on* $\Sigma$ *are extendable iff* $\Sigma$ *is acyclic*

# A structural explanation

# Structural reason

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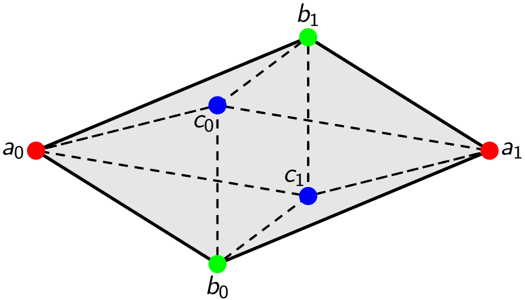


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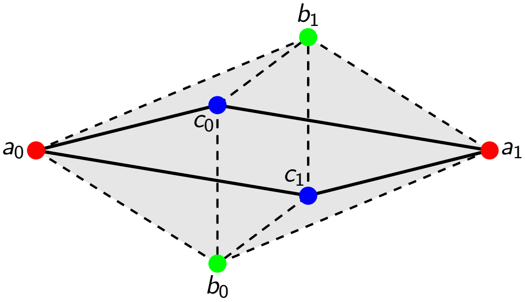


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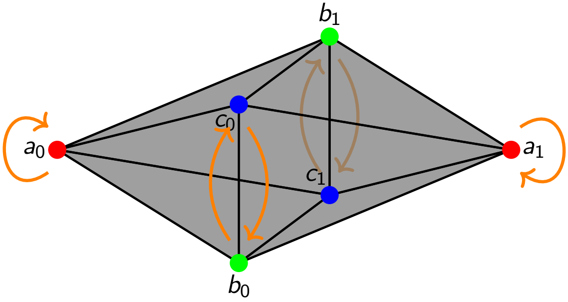


- Measurement scenario: simplicial complex $\mathfrak{D}_2 * \mathfrak{D}_2 * \mathfrak{D}_2$.
- We identify $B$ and $C$: $b_0 \sim c_0$, $b_1 \sim c_1$.
- The macro scenario arises as a quotient.

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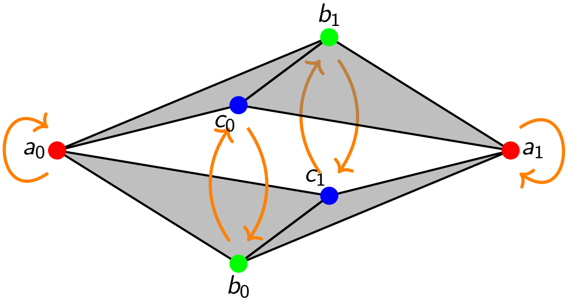


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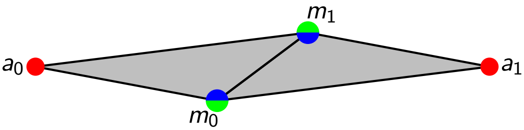


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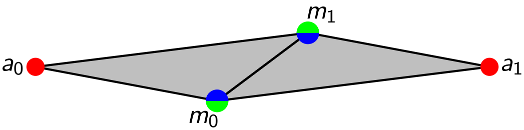

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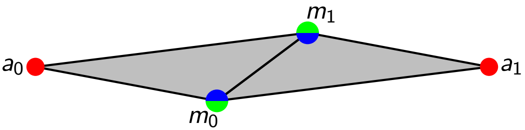


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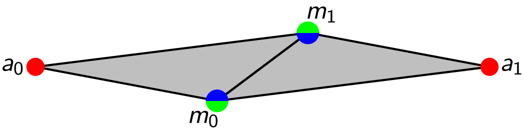


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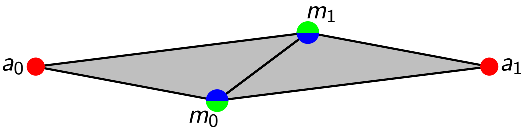


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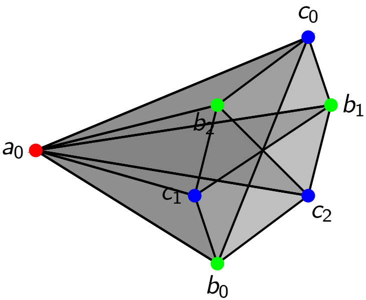
- In particular, they satisfy any Bell inequality.
- Hence, the original tripartite model also satisfies a monogamy relation for any Bell inequality.

# A non-acyclic example

Let $B$ and $C$ have 3 measurement settings: $\mathfrak{D}_2 * \mathfrak{D}_3 * \mathfrak{D}_3 = \mathfrak{D}_2 * \mathfrak{D}_3^{(*2)}$. (only depicted half)



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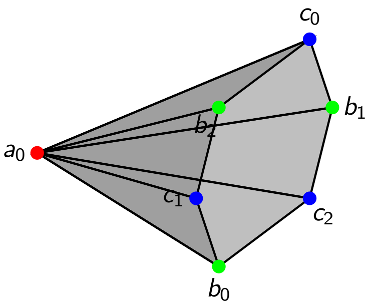

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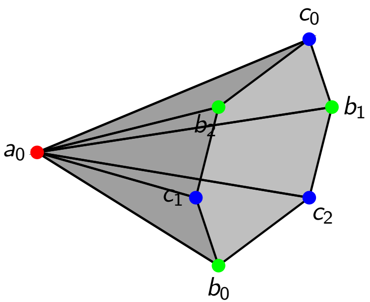


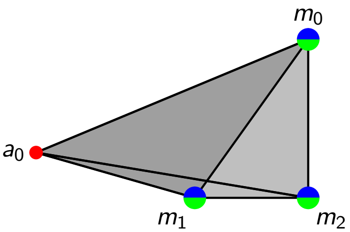

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- Take $r_i$ copies of each site $i$, or $r_i$ micro sites constituting $i$. For a macro site $A$:
  - copies / micro sites: $A^{(1)}, \ldots, A^{(r_1)}$
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- Symmetry by $S_{r_1} \times \cdots \times S_{r_n}$ identifies the copies at each macro site.

$$a_j^{(1)} \sim \cdots \sim a_j^{(r_A)} \quad (\forall j \in \{1, \ldots, k_A\}),$$
$$b_j^{(1)} \sim \cdots \sim a_j^{(r_A)} \quad (\forall j \in \{1, \ldots, k_A\}),$$
etc.

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- one of the sites has a single copy and the condition above is satisfied by all the other sites, i.e. $\exists i_0.\ \left(r_{i_0} = 1 \ \wedge\ \forall i \in \{1, \ldots \widehat{i_0} \ldots, n\}.\ k_i \leq r_i\right)$.

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We get monogamy relations

$$\sum_{m_B=1}^{r_B} \sum_{m_C=1}^{r_C} \cdots \quad \mathcal{B}(A, B^{(m_B)}, C^{(m_C)}, \ldots) \ \leq \ r_B r_C \cdots \ R$$

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  - since the average model, being defined on this quotient scenario, must be **local/non-contextual**.
- The approach is not restricted to multipartite Bell-type scenarios. More generally, we can apply the same ideas to derive monogamy relations for contextuality inequalities.

# Questions...

?