The Logic of Contextuality

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By contrast, Kochen and Specker represent non-commutativity – and hence **incompability** – by partiality.

This leads to their notion of **partial Boolean algebras**.

Partial Boolean algebras

A partial Boolean algebra A is given by a set (also written A), constants 0, 1, a reflexive, symmetric binary relation $\odot$ on A, read as "commeasurability" or "compatibility", a total unary operation $\neg$, and partial binary operations $\wedge$, $\vee$ with domain $\odot$.

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The key example: $\mathsf{P}(\mathcal{H})$, the projectors on a Hilbert space $\mathcal{H}$.
(A projector on a f.d. Hilbert space is a complex matrix M which is idempotent ($M^2 = M$), and self-adjoint (equal to its conjugate transpose)).
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Morphisms of partial Boolean operations are maps preserving commeasurability, and the operations wherever defined. This gives a category **pBA**.

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We shall call this the **K-S property** of a pBA.

Conditions of impossible experience

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Using this terminology, we can express a (physically) remarkable result from Kochen and Specker as follows:

Theorem

let A be a pba. Then the following are equivalent:

1. *A is K-S.*
2. *For some* ***propositional contradiction*** *$\varphi(\vec{x})$ and assignment $\vec{x} \mapsto \vec{a}$, $A \models \varphi(\vec{a})$.*

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How can the world be this way? Still an ongoing debate, an enduring mystery ...

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Coproducts have a simple direct description. The coproduct $A \oplus B$ of partial Boolean algebras A, B is their disjoint union with 0_A identified with 0_B, and 1_A identified with 1_B. Other than these identifications, no commeasurability holds between elements of A and elements of B.

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More generally, we use this approach to prove the following result, which freely generates from a given partial Boolean algebra a new one where prescribed additional commeasurability relations are enforced between its elements.

Theorem

Given a partial Boolean algebra A and a binary relation $\circledcirc$ on A, there is a partial Boolean algebra $A[\circledcirc]$ such that:

- *There is a* **pBA**-*morphism* $\eta \colon A \longrightarrow A[\circledcirc]$ *such that* $a \circledcirc b \;\Rightarrow\; \eta(a) \odot_{A[\circledcirc]} \eta(b)$.
- *For every partial Boolean algebra B and* **pBA**-*morphism* $h \colon A \longrightarrow B$ *such that* $a \circledcirc b \;\Rightarrow\; h(a) \odot_B h(b)$, *there is a unique homomorphism* $\hat{h} \colon A[\circledcirc] \longrightarrow B$ *such that*

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, "h"'] & A[\circledcirc] \arrow[d, "\hat{h}"] \\ & B \end{tikzcd}$$

This result is proved constructively, by giving proof rules for commeasurability and equivalence relations over a set of syntactic terms generated from A. (In fact, we start with a set of "pre-terms", and also give rules for definedness).

The inductive construction

$$\frac{a \in A}{\imath(a){\downarrow}} \qquad \frac{a \odot_A b}{\imath(a) \odot \imath(b)} \qquad \frac{a \circledcirc b}{\imath(a) \odot \imath(b)}$$

$$\frac{}{0 \equiv \imath(0_A),\ 1 \equiv \imath(1_A),\ \neg\imath(a) \equiv \imath(\neg_A a)}$$

$$\frac{a \odot_A b}{\imath(a) \wedge \imath(b) \equiv \imath(a \wedge_A b),\ \imath(a) \vee \imath(b) \equiv \imath(a \vee_A b)}$$

$$\frac{}{0{\downarrow},\ 1{\downarrow}} \qquad \frac{t \odot u}{t \wedge u{\downarrow},\ t \vee u{\downarrow}} \qquad \frac{t{\downarrow}}{\neg t{\downarrow}}$$

$$\frac{t{\downarrow}}{t \odot t,\ t \odot 0,\ t \odot 1} \qquad \frac{t \odot u}{u \odot t} \qquad \frac{t \odot u,\ t \odot v,\ u \odot v}{t \wedge u \odot v,\ t \vee u \odot v} \qquad \frac{t \odot u}{\neg t \odot u}$$

$$\frac{t{\downarrow}}{t \equiv t} \qquad \frac{t \equiv u}{u \equiv v} \qquad \frac{t \equiv u,\ u \equiv v}{t \equiv v} \qquad \frac{t \equiv u,\ u \odot v}{t \odot v}$$

$$\frac{\varphi(\vec{x}) \equiv_{\mathsf{Bool}} \psi(\vec{x}),\ \bigwedge_{i,j} v_i \odot v_j}{\varphi(\vec{v}) \equiv \psi(\vec{v})} \qquad \frac{t \equiv t',\ u \equiv u',\ t \odot u}{t \wedge u \equiv t' \wedge u',\ t \vee u \equiv t' \vee u'} \qquad \frac{t \equiv u}{\neg t \equiv \neg u}$$

Coequalisers and colimits

A variation of this construction is also useful, where instead of just forcing commeasurability, one forces equality by the additional rule

$$\frac{a \circledcirc a'}{\imath(a) \equiv \imath(a')}$$

This builds a pBA $A[\circledcirc, \equiv]$.

Theorem

Let $h\colon A \longrightarrow B$ be a **pBA***-morphism such that $a \circledcirc a' \;\Rightarrow\; h(a) = h(a')$. Then there is a unique* **pBA***-morphism $\hat{h}\colon A[\circledcirc, \equiv] \longrightarrow B$ such that $h = \hat{h} \circ \eta$.*

This result can be used to give an explicit construction of coequalisers, and hence general colimits, in **pBA**.

An apparent contradiction

BA is a full subcategory of **pBA**. We know from (Heunen and van den Berg) that A is the colimit in **pBA** of its boolean subalgebras. Now let B be the colimit in **BA** of the same diagram D of boolean subalgebras of A and the inclusions between them.

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As such, it is complete and cocomplete, but it also admits the one-element algebra **1**, in which $0 = 1$. Note that **1** does **not** have a homomorphism to **2**.

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In the case of a partial Boolean algebra with the K-S property of not having a homomorphism to **2**, the colimit of its diagram of boolean subalgebras must be **1**.

KS-property and colimits

We can turn this into a theorem:

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A partial Boolean algebra with the K-S property – such as $\mathsf{P}(\mathcal{H})$ – holds this implicitly contradictory information together in a single structure.

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Can we capture the Hilbert space tensor product in logical form?

Question

Is there a monoidal structure $\circledast$ on the category **pBA** such that the functor $\mathsf{P}\colon \mathbf{Hilb} \longrightarrow \mathbf{pBA}$ is **strong monoidal** with respect to this structure, i.e. such that $\mathsf{P}(\mathcal{H}) \circledast \mathsf{P}(\mathcal{K}) \cong \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$?

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A positive answer to this question would offer a complete logical characterisation of the Hilbert space tensor product, and provide an important step towards giving logical foundations for quantum theory in a form useful for quantum information and computation.

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In (Heunen and van den Berg), it is shown that **pBA** has a monoidal structure, with $A \otimes B$ given by the colimit of the family of $C + D$, as C ranges over Boolean subalgebras of A, D ranges over Boolean subalgebras of B, and $C + D$ is the coproduct of Boolean algebras.

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Our Theorem 2 allows us to give an explicit description of this construction using generators and relations.

Proposition

Let A and B be partial Boolean algebras. Then

$$A \otimes B \;\cong\; (A \oplus B)[\odot]$$

where $\odot$ is the relation on the carrier set of $A \oplus B$ given by $\imath(a) \odot \jmath(b)$ for all $a \in A$ and $b \in B$.

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There is a lax monoidal functor $\mathsf{P}\colon \mathbf{Hilb} \longrightarrow \mathbf{pBA}$, which takes a Hilbert space to its projectors, viewed as a partial Boolean algebra, with an embedding $\mathsf{P}(\mathcal{H}) \otimes \mathsf{P}(\mathcal{K}) \longrightarrow \mathsf{P}(\mathcal{H} \otimes \mathcal{K})$ induced by the evident embeddings of $\mathsf{P}(\mathcal{H})$ and $\mathsf{P}(\mathcal{K})$ into $\mathsf{P}(\mathcal{H} \otimes \mathcal{K})$), given by $p \longmapsto p \otimes 1$, $q \longmapsto 1 \otimes q$.

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It is easy to see that this embedding is far from being surjective. For example, if we take $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$, then there are (many) two-valued homomorphisms on $A = \mathsf{P}(\mathbb{C}^2)$, which lift to two-valued homomorphisms on $A \otimes A$. However, by the Kochen–Specker theorem, there is no such homomorphism on $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

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Nevertheless, this result is very suggestive. It poses the challenge of finding a stronger notion of tensor product.

Towards a more expressive tensor product

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An important property satisfied by the rules in Table 1 as applied in constructing $A \otimes B$ is that, if $t{\downarrow}$ can be derived, then $u{\downarrow}$ can be derived for every subterm u of t. This appears to be too strong a constraint to capture the full logic of the Hilbert space tensor product.

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To see why this is an issue, consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$. To ensure in general that they commute, we need the conjunctive requirement that p_1 commutes with q_1, **and** p_2 commutes with q_2.

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However, to show that they are **orthogonal**, we have a disjunctive requirement: $p_1 \perp q_1$ **or** $p_2 \perp q_2$. If we establish orthogonality in this way, we are entitled to conclude that $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are commeasurable, even though (say) p_2 and q_2 are not.

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Indeed, the idea that propositions can be defined on quantum systems even though subexpressions are not is emphasized by Kochen.

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Definition

A partial Boolean algebra A is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are logically exclusive are also commeasurable, i.e. if $\perp \subseteq \odot$.
We write **epBA** for the full subcategory of **pBA** whose objects are partial Boolean algebras satisfying LEP.

Logical exclusivity and transitivity

The logical exclusivity principle turns out to be equivalent to the following notion of transitivity.

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A partial Boolean algebra is said to be **transitive** if for all elements a, b, c, $a \leq b$ and $b \leq c$ implies $a \leq c$.

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As an immediate consequence, any $\mathsf{P}(\mathcal{H})$ satisfies LEP.

A reflective adjunction for logical exclusivity

We can of course form the partial Boolean algebra $A[\perp]$. While the exclusivity principle holds for all its elements in the image of $\eta : A \longrightarrow A[\perp]$, it may fail to hold for other elements in $A[\perp]$.

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However, we can adapt our construction to show that one can freely generate, from any given partial Boolean algebra, a new partial Boolean algebra satisfying LEP.

This LEP-isation is analogous to e.g. the way one can 'abelianise' any group, or use Stone–Čech compactification to form a compact Hausdorff space from any topological space.

Theorem

The category **epBA** *is a reflective subcategory of* **pBA**, *i.e. the inclusion functor* $I\colon \mathbf{epBA} \longrightarrow \mathbf{pBA}$ *has a left adjoint* $X\colon \mathbf{pBA} \longrightarrow \mathbf{epBA}$. *Concretely, to any partial Boolean algebra* A, *we can associate a Boolean algebra* $X(A) = A[\perp]^*$ *which satisfies LEP such that:*

- *there is a homomorphism* $\eta\colon A \longrightarrow A[\perp]^*$;
- *for any homomorphism* $h\colon A \longrightarrow B$ *where* B *is a partial Boolean algebra* B *satisfying LEP, there is a unique homomorphism* $\hat{h}\colon A[\perp]^* \longrightarrow B$ *such that:*

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The proof of this result follows from a simple adaptation of the proof of Theorem 2, namely adding the following rule to the inductive system presented in Table 1:

$$\frac{u \wedge t \equiv u, \; v \wedge \neg t \equiv v}{u \odot v}$$

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Details in paper.

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Is there a “logical” proof?

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