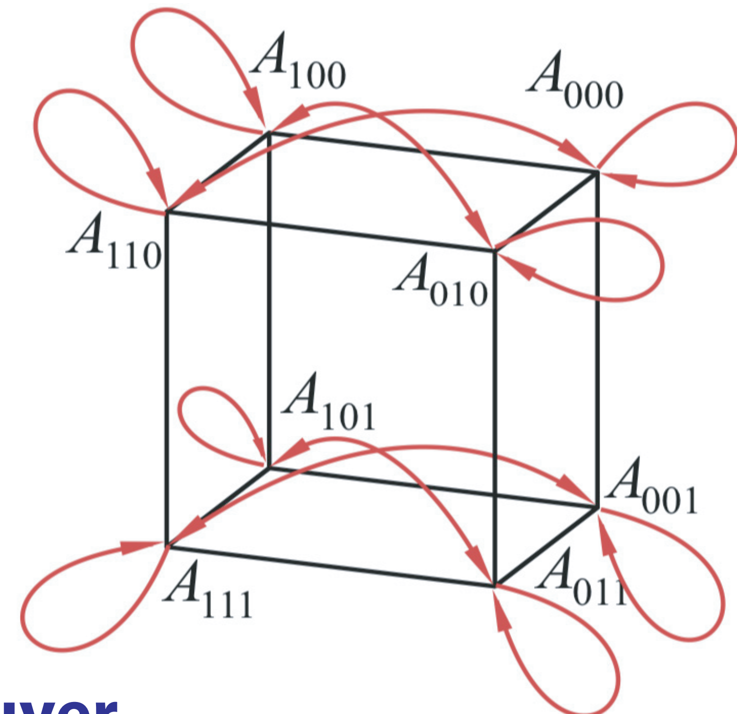
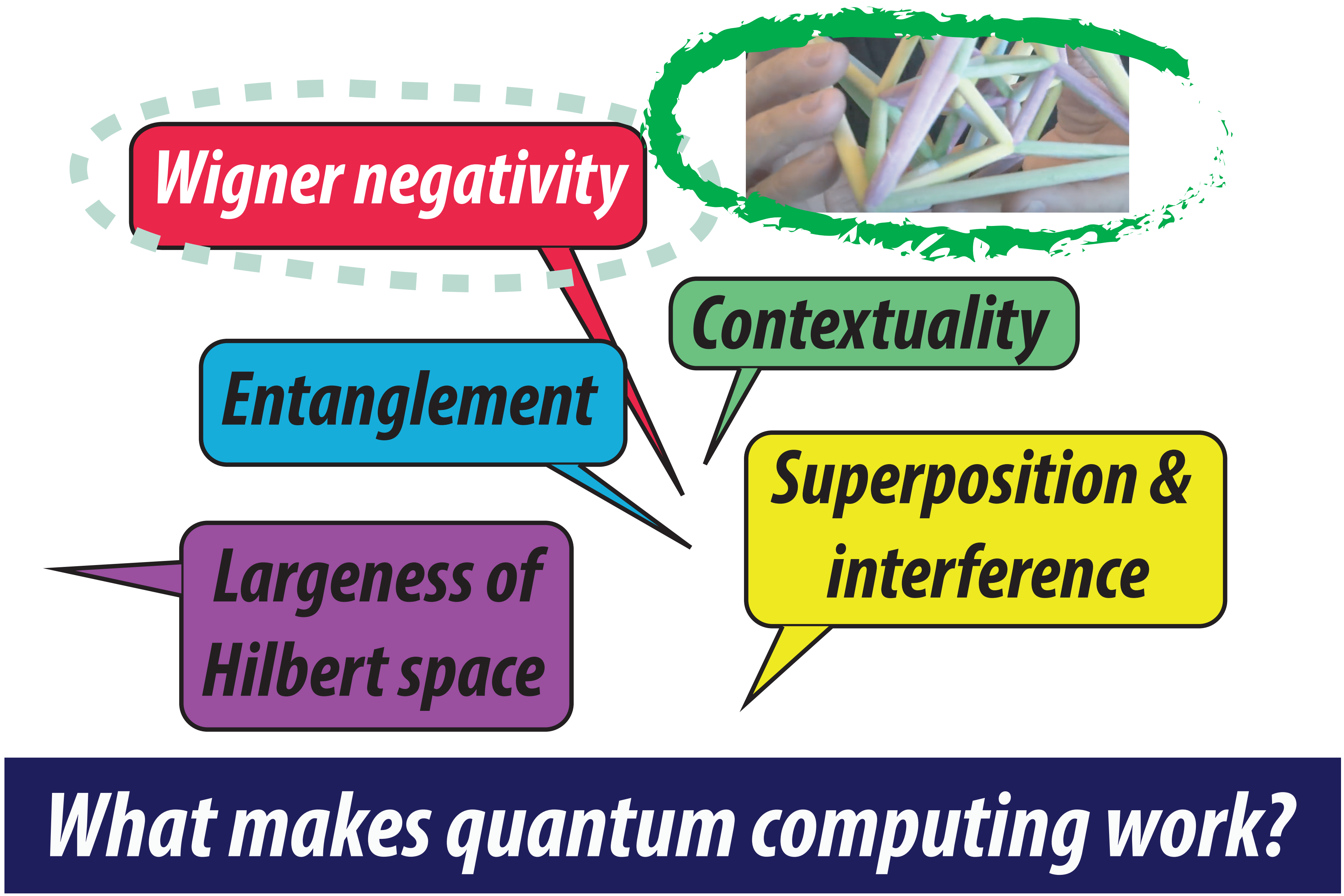
A hidden variable model for universal quantum computation with magic states on qubits



Robert Raussendorf, UBC Vancouver

Workshop QCQMB'21, Zoom, May 2021

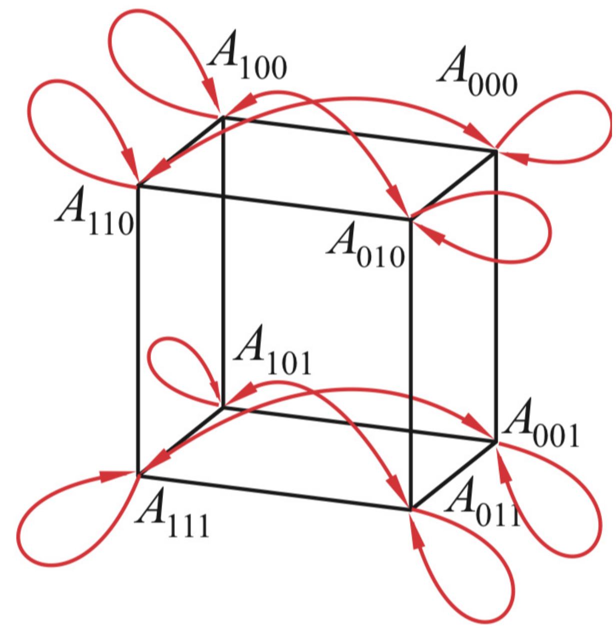
Joint work with Michael Zurel and Cihan Okay, PRL 125 (2020)


Wigner negativity
Contextuality
Entanglement
Superposition & interference
Largeness of Hilbert space
What makes quantum computing work?

Summary of the result

We have constructed a hidden variable model with positive representation for

- All quantum states
- Clifford unitary gates
- Pauli measurements



$\boxed{\text{Those operations suffice for universal quantum computation.}}$

Need for negativity?

Theorem 2. *A quasi-probability representation of quantum theory must have negativity in either its representation of states or measurements (or both).*

[*] C. Ferrie, Rep. Prog. Phys. 74, 116001 (2011).

Need for negativity?

Theorem 2. *A quasi-probability representation of quantum theory must have negativity in either its representation of states or measurements (or both).*

Mind the assumptions:

- Above theorem requires a unique quasiprobability function for each state, ours not.
- Above theorem considers all measurements, ours only Pauli measurements; however, the latter suffice for universal QC.

[*] C. Ferrie, Rep. Prog. Phys. 74, 116001 (2011).

Outline

1. A short history of the subject

   - Our phenomenological triangle:

   - The trouble with qubits

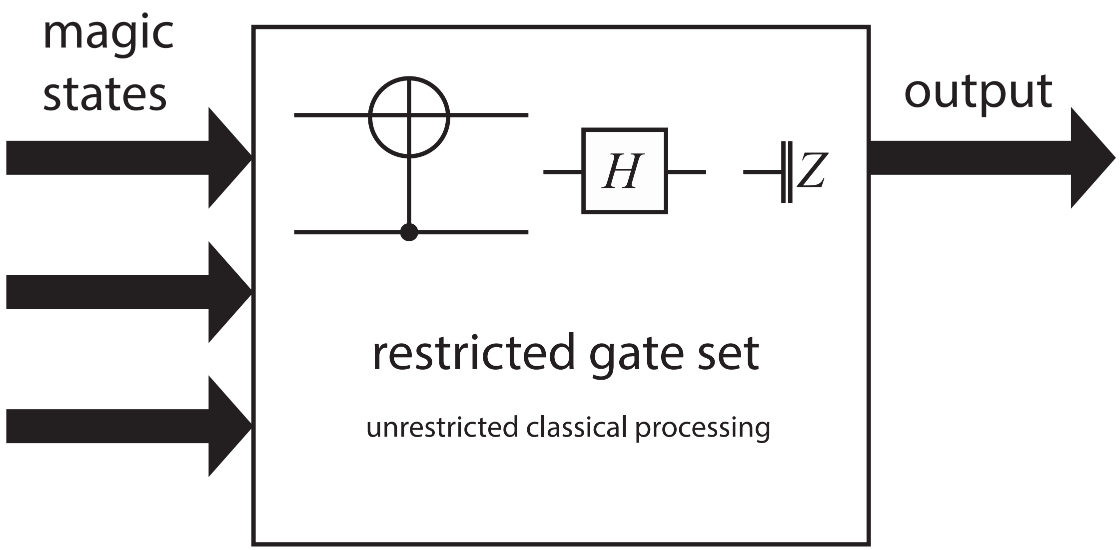
2. A hidden variable model for QM in finite dimensions

   (a) Model and result

   (b) Compatibility: PBR theorem, KS theorem

   (c) Classical simulation of quantum computation

Quantum Computation with magic states

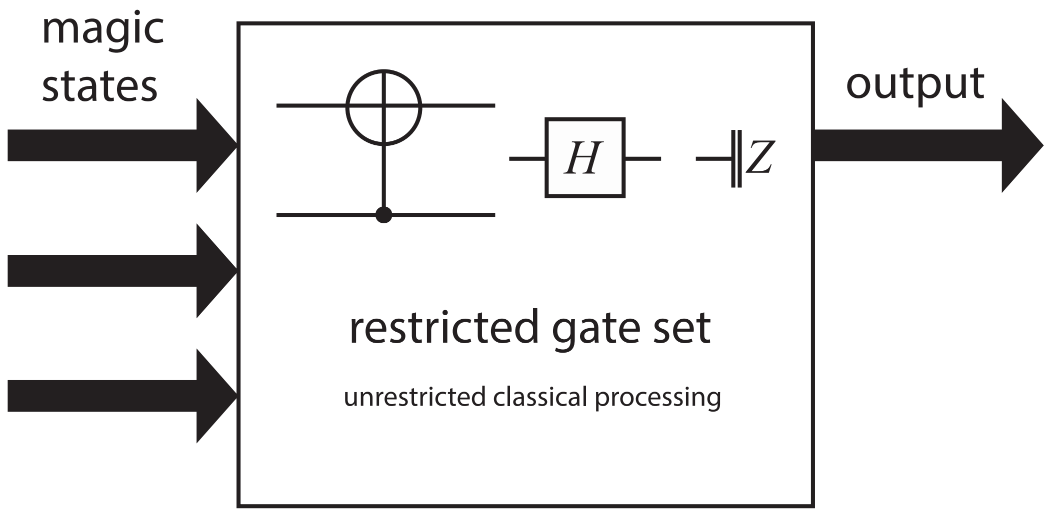


- Non-universal restricted gate set: *e.g. Clifford gates*.
- Universality reached through injection of *magic states*.

\+ *As of now, leading scheme for fault-tolerant QC.*

Computational power is shifted from gates to states

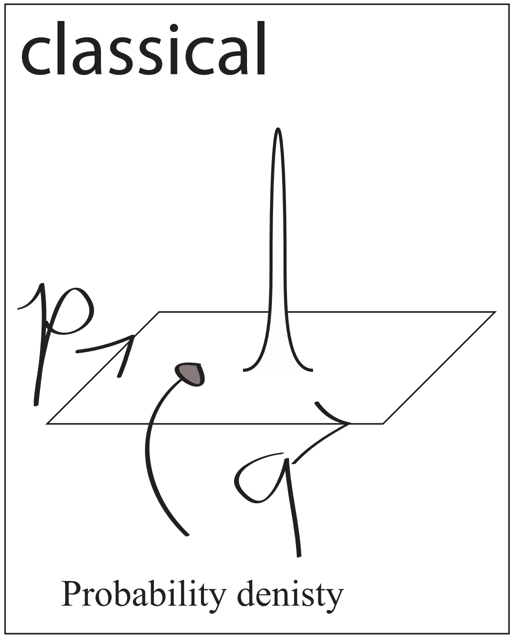
A question

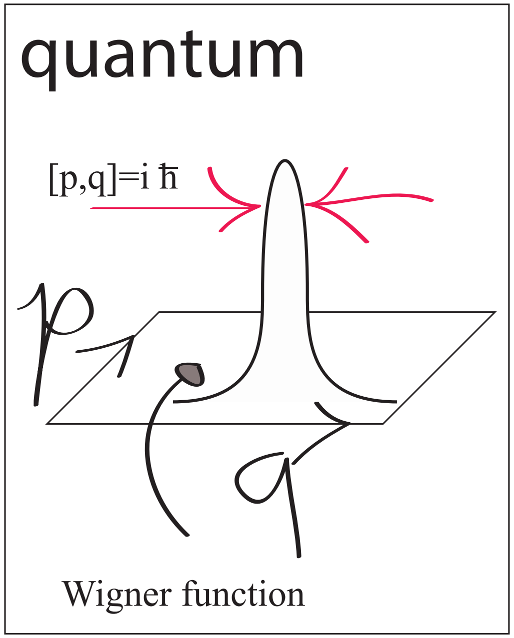


Which properties must the magic states have to enable a speedup?

A: Wigner function negativity

[quantum] mechanics in phase space





- The Wigner function is a quasi-probability distribution that can represent each quantum state. It can take negative values.
- It is the closest quantum counterpart to the classical probability distribution over phase space.

Results

Theorem[1]–[3]: *Quantum computation with magic states can have a quantum speedup only if the Wigner function of the initial magic states is negative.*

Negativity in the Wigner function
is necessary
for quantum computation[1] - [3]

[1] Qudits in odd d: V. Veitch *et al.*, New J. Phys. 14, 113011 (2012).

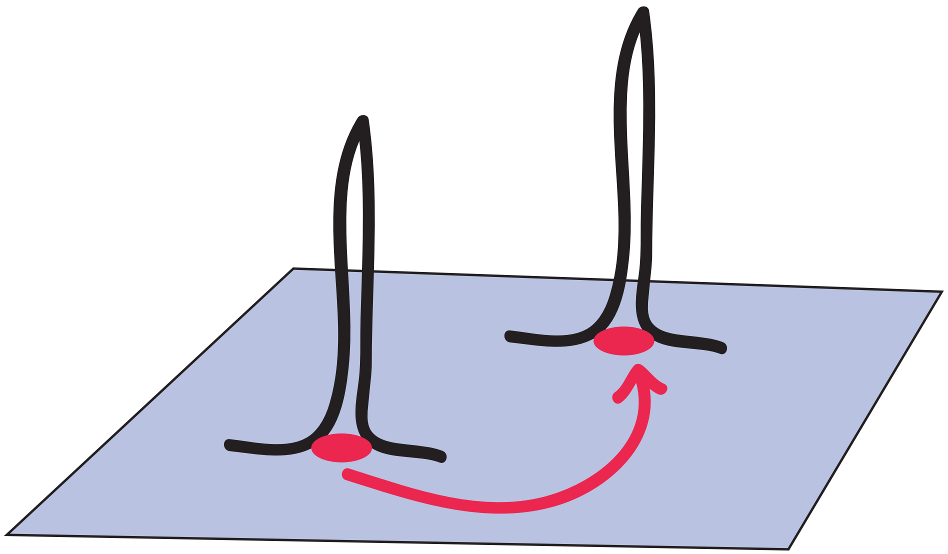
[2] Rebits: N. Delfosse *et al.*, Phys. Rev. X 5, 021003 (2015).

[3] Qubits: R. Raussendorf *et al.*, Phys. Rev. A 101, 012350 (2020).

Covariance of the qudit Wigner function

If and only if the local Hilbert space dimension is odd, then:

- The n-qudit Wigner function transforms *covariantly* under all Clifford unitaries, $W_{U\rho U^\dagger}(\mathbf{v}) = W_\rho(L_U \mathbf{v}), \ \forall \rho$.

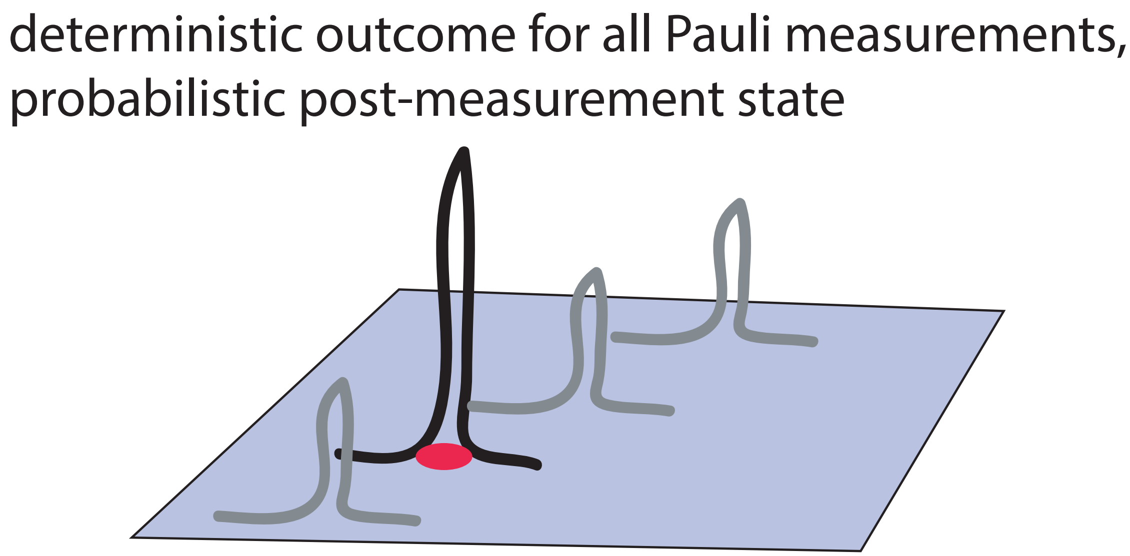


D. Gross, PhD thesis, Imperial College London (2005).

Positivity preservation under Pauli measurement

If and only if the local Hilbert space dimension is odd, then:

- The n-qudit Wigner function *preserves* positivity under all Pauli measurements.

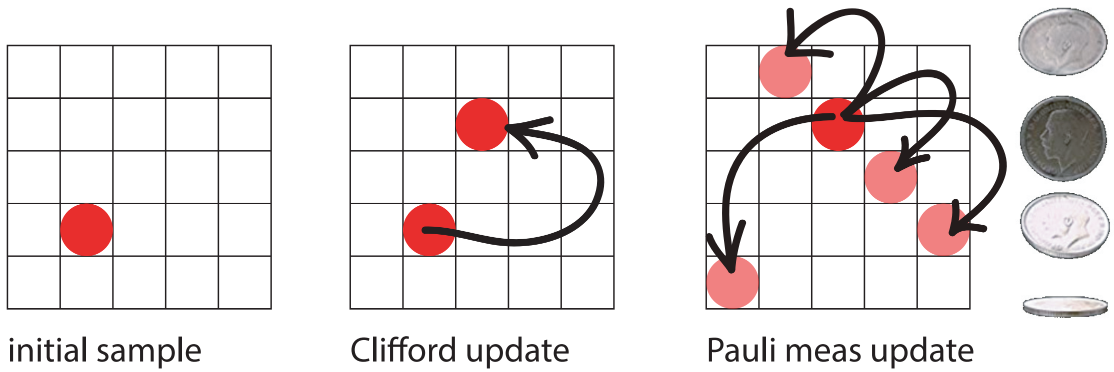


Pauli measurement on a peaked Wigner function produces a ridge.

Theorem – proof idea

We will show that:

$\boxed{\text{If } W_{\rho_{\text{magic}}} \geq 0 \Rightarrow \text{efficiently classical simulation} \Rightarrow \text{no speedup.}}$



Simulation algorithm:

1. $W_{\rho_{\text{magic}}} \geq 0$ is a probability distribution. $\longrightarrow$ Sample from it! Each sample is a point in phase space.

2. Update the phase space points under Clifford gates and Pauli measurement.

Classical simulation in the presence of negativity

.. is hard

How hard?

- **Theorem [1].** The cost O of classically simulating magic state quantum computation scales as

$$\boxed{O \propto \frac{\left(|W_\rho|_1\right)^2}{\epsilon^2}}$$

where ϵ is the desired accuracy.

- Numerically [2] (as well as analytically upper-bounded),

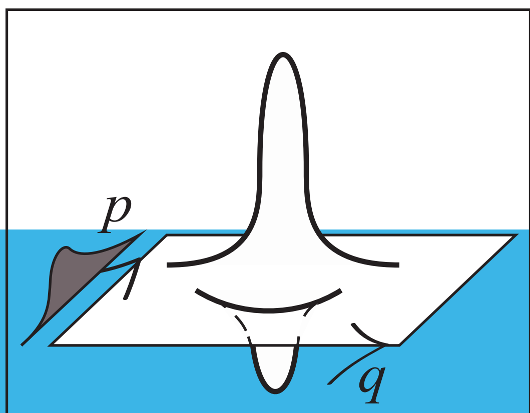
$$\left|W_{\rho^{\otimes n}}\right|_1 \propto e^{\kappa n}.$$

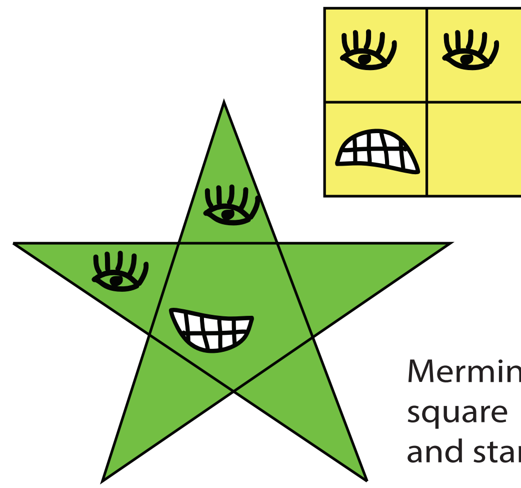
[1] H. Pashayan, J.J. Wallman, S.D. Bartlett, PRL 115, 070501 (2015).

[2] M. Heinrich and D. Gross, Quantum 3, 132 (2019).

The trouble with qubits

quantum optics:
Hilbert space dimension infinite





Mermin's square and star

Odd: all nice & safe

quantum computation:
Hilbert space dimension finite

Even: monsters lurking

The trouble with qubits

Consider a Wigner function W such that for all states ρ

$$\rho = \sum_{\mathbf{v}} W_\rho(\mathbf{v}) A_{\mathbf{v}}.$$

Phase point operators $\{A_{\mathbf{v}}\}$ span the space of density matrices.

- **Trouble with covariance:**

  Theorem [*] If $\{A_{\mathbf{v}}\}$ is an operator basis then W cannot be Clifford covariant.

- **Trouble with contextuality:**

  Theorem [**] A memory of $O(n^2)$ bits is required for simulating contextuality on n-qubit systems.

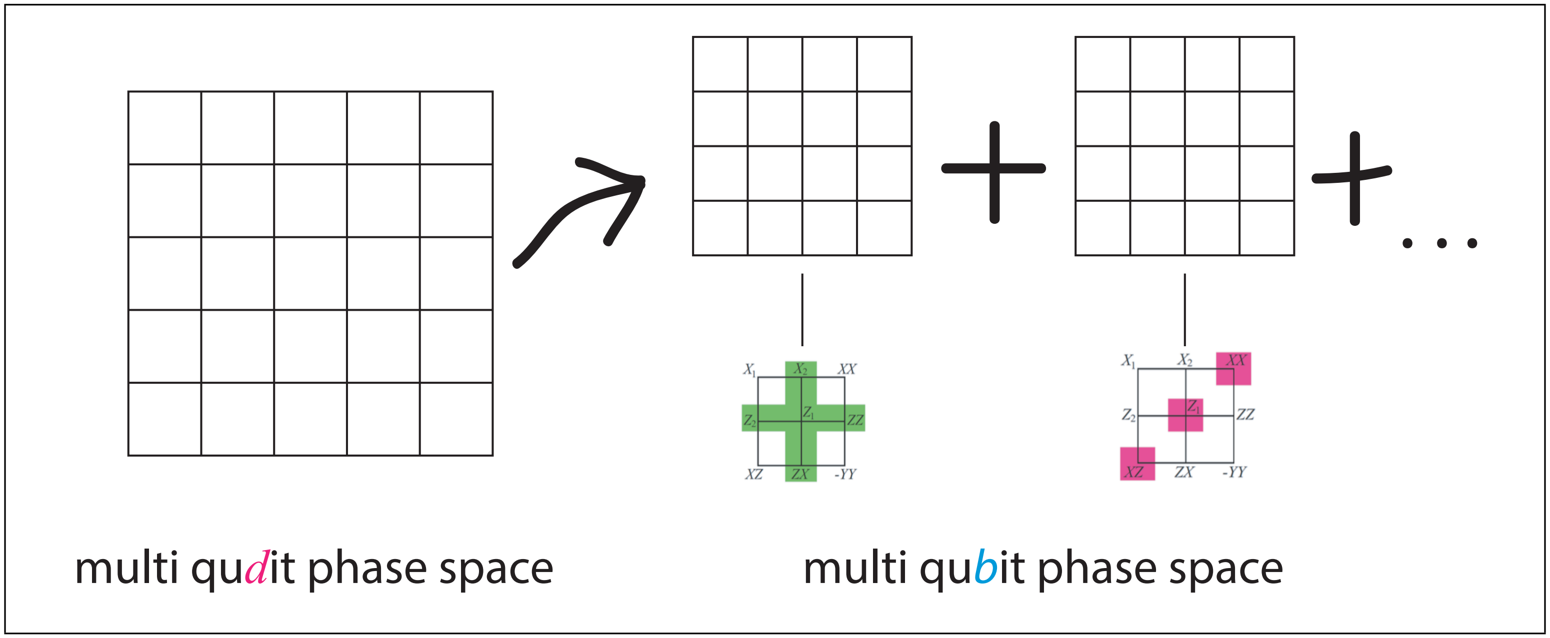
Lesson: Base quasiprobability functions on over-complete sets $\{A_{\mathbf{v}}\}$.

[*] H. Zhu, Phys. Rev. Lett. **116**, 040501 (2016).

[**] A. Karanjai *et al.*, arXiv:1802.07744.

Resolution for qubits

Lesson: Base Wigner functions on over-complete sets $\{A_{\mathbf{v}}\}$.



R. Raussendorf *et al.*, Phys. Rev. A 101, 012350 (2020),

also see: W.M. Kirby and P.J. Love, Phys. Rev. Lett. 123, 200501 (2019).

Resolution for qubits

Theorem: Quantum computation with magic states can have a quantum speedup only if the quasiprobability distribution $\tilde{W}_\rho$ of the initial magic state ρ is negative.*

Price to pay:

- Quasiprobability function $\tilde{W}$ is not unique.
- Phase space way more complicated than for qudits and rebits.

*: Btw, that is not the result of this talk.

R. Raussendorf *et al.*, Phys. Rev. A 101, 012350 (2020).

Earlier results ... and a counterpoint

Theorem[1]–[3]: *Quantum computation with magic states can have a quantum speedup only if the Wigner function of the initial magic states is negative.*

Negativity in the Wigner function
is necessary
for quantum computation[1] - [3]

[1] Qudits in odd d: V. Veitch *et al.*, New J. Phys. 14, 113011 (2012).
[2] Rebits: N. Delfosse *et al.*, Phys. Rev. X 5, 021003 (2015).
[3] Qubits: R. Raussendorf *et al.*, Phys. Rev. A 101, 012350 (2020).

Counterpoint: The above theorem hinges on the precise definition of the Wigner function. *Quantum states and their update under measurement can be represented positively.*

[4] M. Zurel, C. Okay and T. Raussendorf, PRL 125, 260404 (2020).

A hidden variable model for multi-qubit states, Clifford gates and Pauli measurements

Model and result

•

Compatibility with the PBR and KS theorems

•

What does this mean for classical simulation?

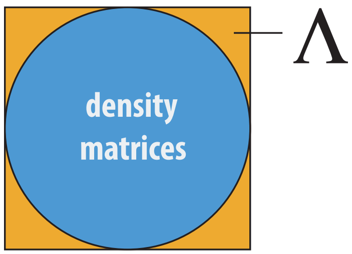
The HVM state space

Question: *For any number n of qubits, what is the largest state space Λ_n that is closed under Pauli measurement?*

Tinkering with this question brings up the following construct:

Definition: For any $n \in \mathbb{N}$, denote by $\mathcal{S}_n$ the set of pure n-qubit stabilizer states. Then, the polytope Λ_n is the set of operators $O \in \text{Op}(\mathbb{C}^{2^n})$ with the properties

1. O is Hermitian.
2. $\text{Tr}(O) = 1$.
3. $\text{Tr}(O\,|\sigma\rangle\langle\sigma|) \geq 0$, for all $|\sigma\rangle \in \mathcal{S}_n$.



- We can describe the state polytope Λ_n by its extremal vertices $\{A_\alpha,\ \alpha \in \mathcal{V}_n\}$. $\mathcal{V}_n$ is the generalized HVM phase space.
- $\mathcal{V}_n$ is finite for all n.

Everything is positive

Theorem 1. $\mathcal{V}_n$ has the following properties.

(i) *Positive representation.* All quantum states ρ are represented by a probability function $p: \mathcal{V}_n \to \mathbb{R}_{\geq 0}$,

$$\rho = \sum_{\alpha \in \mathcal{V}_n} p_\rho(\alpha) A_\alpha.$$

(ii) *Positivity preservation.* Denote by $\Pi_{a,s}$ the projection corresponding to the measurement of the Pauli observable T_a with outcome s. Then,

$$\Pi_{a,s} A_\alpha \Pi_{a,s} = \sum_{\beta \in \mathcal{V}_n} q_{\alpha,a}(\beta, s) A_\beta,$$

where the $q_{a,\alpha}$ are probability functions.

(iii) *Born rule.* Denote by $P_{\rho,a}(s)$ the probability of obtaining outcome s in the measurement of the Pauli observable T_a on the state ρ. Then,

$$P_{\rho,a}(s) = \mathrm{Tr}(\Pi_{a,s}\rho) = \sum_{\alpha \in \mathcal{V}_n} p_\rho(\alpha) Q_a(s|\alpha),$$

where all Q_a are conditional probability functions.

Proof of Theorem 1 – Part (ii)

Statement:

(ii) *Positivity preservation.* Denote by $\Pi_{a,s}$ the projection corresponding to the measurement of the Pauli observable T_a with outcome s. Then,

$$\Pi_{a,s} A_\alpha \Pi_{a,s} = \sum_{\beta \in \mathcal{V}_n} q_{\alpha,a}(\beta, s) A_\beta, \tag{1}$$

where the $q_{a,\alpha}$ are probability functions.

Proof: Recall from the definition of Λ_n that $\mathrm{Tr}\left(A_\alpha \left|\sigma\right\rangle\left\langle\sigma\right|\right) \geq 0$, $\forall |\sigma\rangle \in \mathcal{S}_n$. This property is inherited under Pauli projection.

$$\begin{aligned} \mathrm{Tr}\left(\Pi_{a,s} A_\alpha \Pi_{a,s} \left|\sigma\right\rangle\left\langle\sigma\right|\right) &= \mathrm{Tr}\left(A_\alpha \Pi_{a,s} \left|\sigma\right\rangle\left\langle\sigma\right| \Pi_{a,s}\right) \\ &= \mathrm{Tr}\left(A_\alpha \left(\Pi_{a,s} \left|\sigma\right\rangle\left\langle\sigma\right| \Pi_{a,s}\right)\right) \\ &= (c \geq 0) \cdot \mathrm{Tr}\left(A_\alpha |\sigma'\rangle\langle\sigma'|\right), \\ &= \geq 0. \end{aligned}$$

Main case: $\mathrm{Tr}(\Pi_{a,s} A_\alpha) > 0$. $\frac{\Pi_{a,s} A_\alpha \Pi_{a,s}}{\mathrm{Tr}(\Pi_{a,s} A_\alpha)} \in \Lambda_n$ & Eq. (1) holds.

Other case: $\mathrm{Tr}(\Pi_{a,s} A_\alpha) = 0$. Then, $\mathrm{Tr}\left(\Pi_{a,s} A_\alpha \Pi_{a,s} \left|\sigma\right\rangle\left\langle\sigma\right|\right) = 0$ for all $|\sigma\rangle \in \mathcal{S}_n$, hence $\Pi_{a,s} A_\alpha \Pi_{a,s} = 0$, and Eq. (1) holds with $q_{\alpha,a} \equiv 0$. $\square$

Clifford gates are also positively represented

Lemma. If $X \in \Lambda_n$ then $UXU^\dagger \in \Lambda_n$, for all Clifford unitaries U.

Proof. For any $X \in \Lambda_n$, U Clifford and all stabilizer states $|\sigma\rangle$ it holds that

$$\begin{aligned} \mathrm{Tr}(UXU^\dagger|\sigma\rangle\langle\sigma|) &= \mathrm{Tr}(X\,U^\dagger|\sigma\rangle\langle\sigma|U) \\ &= \mathrm{Tr}(X\,|\sigma'\rangle\langle\sigma'|) \\ &\geq 0, \end{aligned}$$

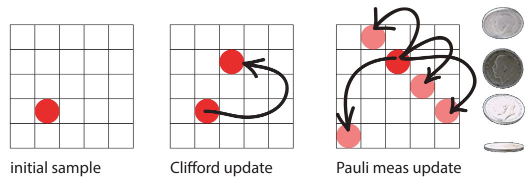
Hence, $UXU^\dagger \in \Lambda_n$. $\square$

Consequence: $UA_\alpha U^\dagger = \sum_{\beta\in\mathcal{V}_n} q_{\alpha,U}(\beta) A_\beta$, with all $q_{\alpha,U}(\beta) \geq 0$.

[In fact, we have Clifford covariance: $UA_\alpha U^\dagger = A_{U\alpha}$.]

Application to QC with magic states

Theorem 2. Universal quantum computation by Clifford unitaries and Pauli measurements on magic states can be described by iterated sampling from probability functions.



This is about universal QC, hence all quantum mechanics in finite Hilbert space dimension.

Both the states and the operations are positively represented.

We do not claim to efficiently simulate universal quantum computation.

If not negativity of quasiprobability functions, then what makes classical simulation by sampling hard?

Consistency

and consequences

Consistency with Kochen-Specker

Consider arbitrary sequences of Pauli measurements on the Mermin square ...

Our HVM simulation will never run into a contradiction.

The value assignments in our HVM are probabilistic. Hence, the Kochen-Specker theorem does not apply.

Consistency with PBR

The Pusey-Barrett-Rudolph theorem states that

Theorem. Any model in which a quantum state represents mere information about an underlying physical state of the system [HVM], *and in which systems that are prepared independently have independent physical states*, must make predictions which contradict those of quantum theory.

The condition in italics is called "preparation independence". Our HVM does not satisfy it,

$$A_\alpha \otimes A_\beta \neq A_\gamma.$$

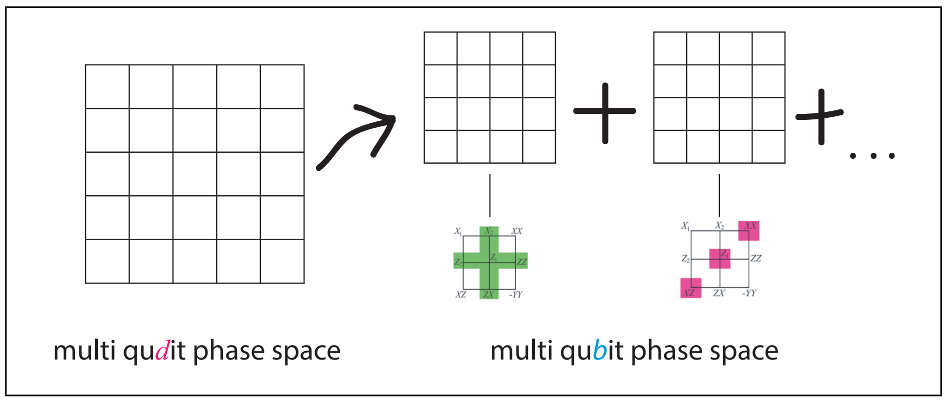
Thus no contradiction with PBR.

What can be efficiently classically simulated?

For a start ..

Everything that can be efficiently simulated with the stabilizer formalism can be efficiently simulated with the present method.

Why is this so?—This already holds for the smaller model



and this model is contained as a special case in the present one.

And so the real question is: How much *more* can we simulate efficiently?

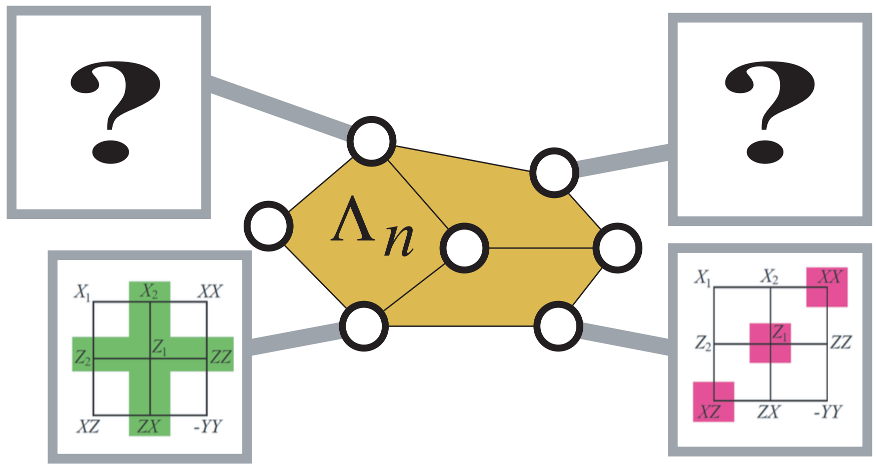
Conclusion

- We have described a hidden variable model for universal quantum computation where all states and the necessary operations are represented by classical probabilities.
- No negativity is required anywhere.
- The classical simulation algorithm is not necessarily efficient.

PRL 125, 260404 (2020).

Outlook

From the perspective of quantum computation, the interesting objects are the extremal vertices A_α of the state polytope Λ_n.



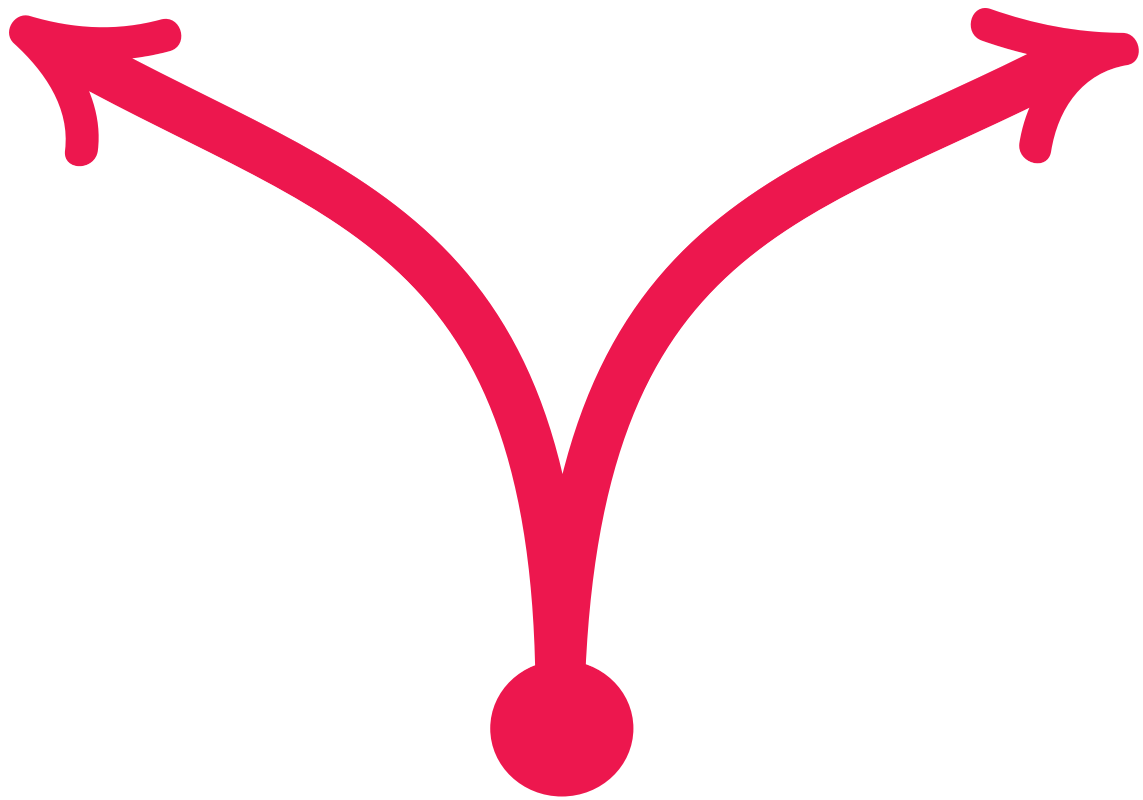
Can those vertices be fully classified?

Where and how is quantumness hiding in them?

PRL 125, 260404 (2020).

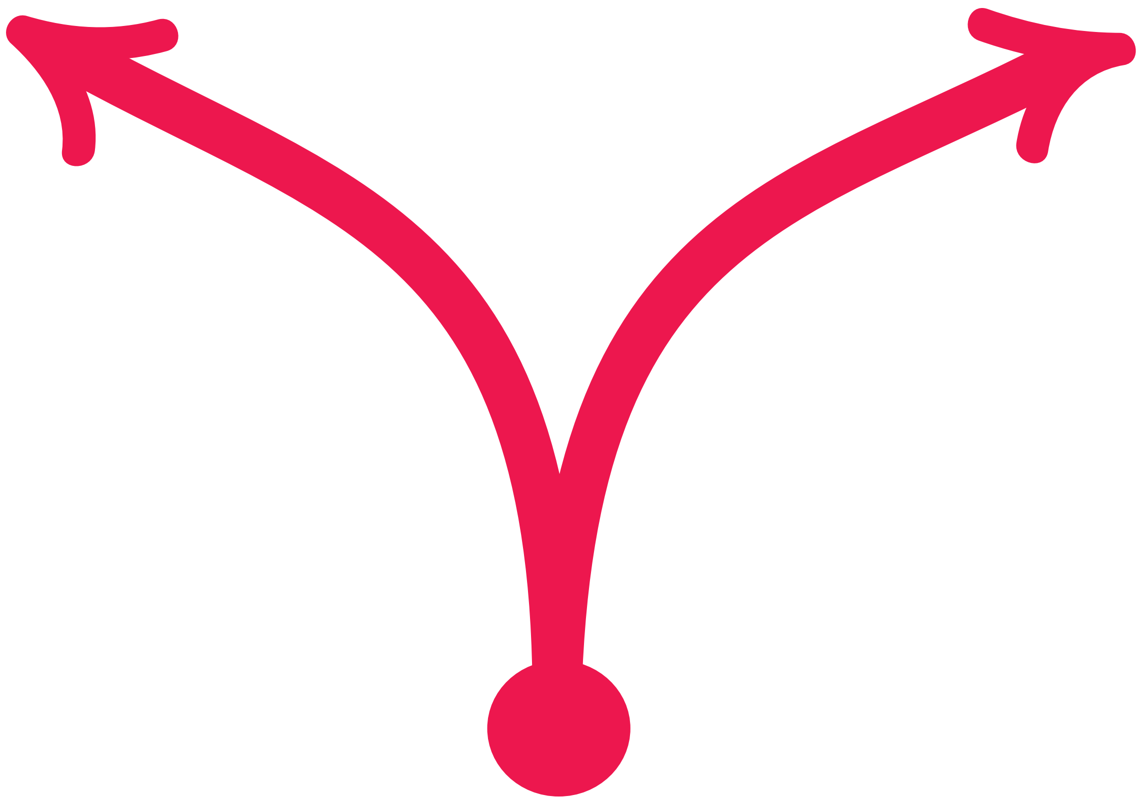
Whiteboard

Approaching junctions



Are non-unique quasiprobability functions OK?

Approaching junctions



Should preparation independence be required?