A generalized cohomology theory for quantum contextuality

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Linear constraint systems

A *linear constraint system* (LCS) is specified by a linear equation $Mx = b$ over the ring $\mathbb{Z}_d$ of integers mod d.

An *operator solution* consists of unitary matrices $A_i \in U_m$ that satisfy three conditions:

(1) d-torsion, (2) commutativity and (3) linear constraint.

Today's talk

Our goal is to introduce a generalized cohomology theory that can be used to classify operator solutions of LCSs.

1. Constructing a space of contexts
2. Interpreting operator solutions as maps
3. Stabilization and generalized cohomology
4. Mermin class

Satisfiability gap: For d odd are there LCSs which admit an operator solution over U_m for $m \geq 2$ but not over U_1?

Contexts in unitary groups

For us a context is a set

$$C = \{A_1, A_2, \cdots, A_n |\, A_i \in U_m\}$$

such that (1) d-torsion: $A_i^d = \mathbb{I}$ and (2) commuting: $A_iA_j = A_jA_i$.

We will think of contexts as group homomorphisms

$$C : \mathbb{Z}_d^n \to U_m.$$

Hypergraph formulation

A LCS $Mx = b$ can be described by a pair $(\mathfrak{H}, \tau)$ consisting of

1. a set of vertices V encoding the variables x,
2. a set of edges E encoding the non-zero rows of M,
3. an incidence weight $\epsilon_e(v)$ encoding the matrix M,
4. a function $\tau : E \to \mathbb{Z}_d$ encoding the column b.

$$\begin{bmatrix} \vdots & & & & & & \vdots \\ M_{k1} & \cdots & M_{ki} & \cdots & M_{kj} & \cdots & M_{kc} \\ \vdots & & & & & & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_c \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_c \end{bmatrix}$$

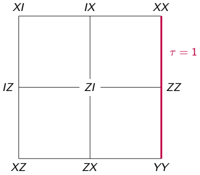
Operator solutions and contextuality

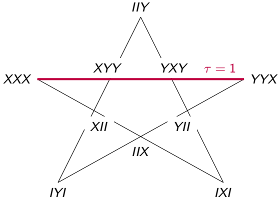
An operator solution is a function $T : V \to U_m$ such that for each $e \in E$ the set $\{T(v) | \, v \in e\}$ is a context and

(3) linear constraint: $\prod_{v \in e} T(v)^{\epsilon_e(v)} = \omega^{\tau(e)} \mathbb{I} \qquad (\omega = e^{2\pi i/d})$

A LCS is *contextual* if it admits an operator solution over U_m for $m \geq 2$ but does not admit an operator (*scalar*) solution over U_1.

Ex. Mermin square and star LCSs are contextual for $d = 2$.

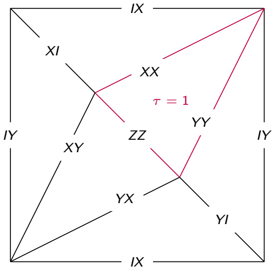


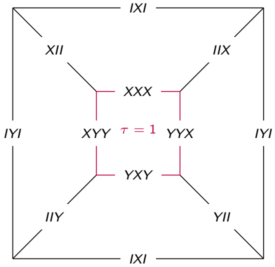


Topological realization

A *topological realization* for $\mathfrak{H}$ is a connected 2-dimensional cell complex $X_{\mathfrak{H}}$ such that

1. 1-cells are labeled by V,
2. 2-cells are labeled by E,
3. attaching maps encode linear constraints (up to a scalar).





Homotopy and contextuality

The function $\tau : E \to \mathbb{Z}_d$ corresponds to $[\tau] \in H^2(X_{\mathfrak{H}}, \mathbb{Z}_d)$.

1. $(\mathfrak{H}, \tau)$ admits a scalar solution if and only if $[\tau] = 0$ for any topological realization $X_{\mathfrak{H}}$.
2. If $(\mathfrak{H}, \tau)$ admits an operator solution but no scalar solutions then any topological realization $X_{\mathfrak{H}}$ has a non-trivial fundamental group[1].

   Generalizes Arkhipov's graph-theoretic characterization.

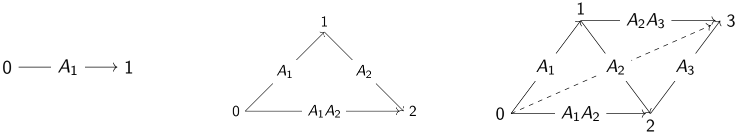
Can we interpret operator solutions in a topological way?

[1] O. and Raussendorf. Quantum 4 (2020)

Space of contexts

Let $B(d, m)$ denote the cell complex whose set of n-cells is given by n-tuples of contexts

$$\{C : \mathbb{Z}_d^n \to U_m |\ \text{group homomorphism}\}$$



Ex. $B(d, 1) = B\mu_d$ where $\mu_d = \{\omega^k |\ 0 \le k \le d-1\}$.

[2] Adem and Gómez. Algebr. Geom. Topol. 15 (2015)

Classifying space for contextuality

Let $\bar{B}(d,m)$ denote the quotient space of $B(d,m)$ obtained by

$$(A_1, \cdots, A_n) \sim (\alpha_1 A_1, \cdots, \alpha_n A_n) \quad \alpha_i \in \mu_d.$$

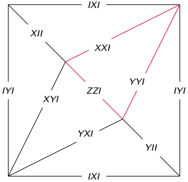
An operator solution $T : V \to U_m$ can be turned into a map

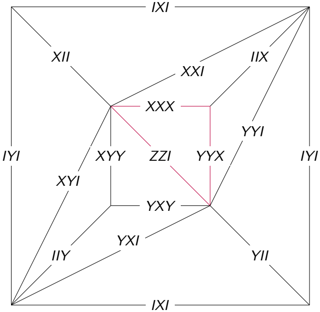
$$f_T : X_{\mathfrak{H}} \to \bar{B}(d,m).$$

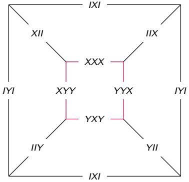
Homotopy classes of maps induce an equivalence relation

$$T \sim T' \iff f_T \simeq f_{T'}.$$

Example - homotopic operator solutions







f_T and $f_{T'}$ are homotopic as maps $S^1 \times S^1 \to \bar{B}(2, 2^3)$.

Stabilization

It is hard to study homotopy classes of maps $X_{\mathfrak{H}} \to \bar{B}(d,m)$ but there is a (two step) procedure to resolve this difficulty:

Step 1

$$U_1 \to U_2 \to \cdots \to U_m \xrightarrow{A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}} U_{m+1} \to \cdots \to U_\infty$$

On the level of spaces we have

$$B(d,1) \to B(d,2) \to \cdots \to B(d,m) \to B(d,m{+}1) \to \cdots \to B(d,\infty)$$

To study $B(d,\infty)$ we need to pass to projectors.

Measurements associated to contexts

$C = (A_1, A_2, \cdots, A_n)$ specifies a projective measurement

$$\Pi : \mathbb{Z}_d^n \to \mathrm{Proj}(\mathbb{C}^m)$$

where $\Pi(s_1 \cdots s_n)$ is the projector onto $V^{(1)}_{\omega^{s_1}} \cap V^{(2)}_{\omega^{s_2}} \cap \cdots \cap V^{(n)}_{\omega^{s_n}}$
where $V^{(i)}_{\omega^s}$ is the eigenspace of A_i corresponding to ω^s.

Stabilization gives a projective measurement

$$\Pi : \mathbb{Z}_d^n \to \mathrm{Proj}(\mathbb{C}^\infty)$$

where each $\Pi(s_1 \cdots s_n)$ projects onto a finite dimensional subspace.

Projections form a Γ-space

After stabilization, passing to projectors and imposing partial sum

$$\Pi_1 \perp \Pi_2 \quad \Rightarrow \quad \Pi_1 \oplus \Pi_2$$

we obtain a Γ-space

$$\begin{gathered} B(d,\infty) = \{C : \mathbb{Z}_d^n \to U_\infty\}_{\geq n} \\ \rightsquigarrow \\ k\mu_d = \{\Pi : \mathbb{Z}_d^n \to \mathrm{Proj}(\mathbb{C}^\infty)\}_{n\geq 0} \end{gathered}$$

Every Γ-space[3] gives a generalized cohomology theory.

[3] Segal. Topology 13 (1974)

Stabilization of $\bar{B}(d,m)$

The generalized cohomology theory we will use to label operator solutions of LCSs is obtained from $k\mu_d$.

Step 2

$$
\begin{tikzcd}
B(d,m) \arrow[r, rightsquigarrow] \arrow[d] & k\mu_d \arrow[d] \\
\bar{B}(d,m) \arrow[r, rightsquigarrow] & C(d,m)
\end{tikzcd}
$$

$C(d,m)$ is a quotient of $k\mu_d$ in a suitable category of generalized cohomology theories.

$C(d, m)$-cohomology

([4]) For a connected 2-dimensional cell complex

$$C(d,m)(X) \cong H^1(X, \mathbb{Z}_{(d,m)}) \oplus H^2(X, \mathbb{Z}_{(d,m)})$$

where (d, m) denotes the greatest common divisor.

For LCSs the second cohomology class coincides with $[\tau]$:

$$(\mathfrak{H}, \tau, T) \rightsquigarrow f_T : X_{\mathfrak{H}} \to \bar{B}(d, m) \rightsquigarrow (\varphi_1; \varphi_2) \in C(d, m)(X_{\mathfrak{H}})$$

[4] O. arXiv:2006.07542 (2020)

Mermin class

If d and m are coprime then every $(\mathfrak{H}, \tau)$ that admits an operator solution over U_m admits a scalar solution:

$$(d, m) = 1 \Rightarrow C(d, m)(X_{\mathfrak{H}}) = 0 \Rightarrow [\tau] = 0$$

Operator solution for the Mermin square LCS defines a class M_n

$$\begin{aligned} C(2, 2^n)(S^1 \times S^1) &= H^1(S^1 \times S^1, \mathbb{Z}_2) \oplus H^2(S^1 \times S^1, \mathbb{Z}_2) \\ &= (\mathbb{Z}_2)^2 \oplus \mathbb{Z}_2. \end{aligned}$$

It turns out that the Mermin class $M_n = (0, 0; 1)$.

Interpretation of stable classes

For $m = d^n$ we have

$$C(d, d^n)(X) = H^1(X, \mathbb{Z}_d) \oplus H^2(X, \mathbb{Z}_d).$$

For d odd when does a stable class come from a LCS?

Real version ($\mathbb{C} \rightsquigarrow \mathbb{R}$) captures information about SPT phases

$$\underset{\langle \text{Gu–Wen} \rangle}{\mathbb{Z}_2} \hookrightarrow C_{\mathbb{R}}(2, 2^n)(S^2) \twoheadrightarrow \underset{\langle M_n \rangle}{\mathbb{Z}_2}$$

Thank you very much for your attention!