All About Cyclic Systems

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(joint work with Janne V. Kujala and Víctor H. Cervantes)

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SYSTEM OF (BINARY) RANDOM VARIABLES

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

SYSTEM OF (BINARY) RANDOM VARIABLES

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

q_1, q_2, q_3, q_4, q_5 — contents (Yes/No questions)

SYSTEM OF (BINARY) RANDOM VARIABLES

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

q_1, q_2, q_3, q_4, q_5 — contents (Yes/No questions)

c^1, c^2, c^3, c^4— contexts, or conditions

BUNCHES

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

q_1, q_2, q_3, q_4, q_5 — contents (Yes/No questions)

c^1, c^2, c^3, c^4— contexts, or conditions

BUNCHES

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

q_1, q_2, q_3, q_4, q_5 — contents (Yes/No questions)

c^1, c^2, c^3, c^4— contexts, or conditions

STOCHASTICALLY UNRELATED BUNCHES

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

q_1, q_2, q_3, q_4, q_5 — contents (Yes/No questions)

c^1, c^2, c^3, c^4— contexts, or conditions

CONNECTIONS

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

q_1, q_2, q_3, q_4, q_5 — contents (Yes/No questions)

c^1, c^2, c^3, c^4— contexts, or conditions

CONNECTIONS

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

q_1, q_2, q_3, q_4, q_5 — contents (Yes/No questions)

c^1, c^2, c^3, c^4— contexts, or conditions

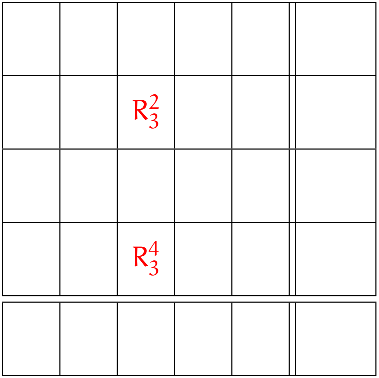
SAME-CONTENT RANDOM VARIABLES: (IN)CONSISTENT CONNECTEDNESS

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

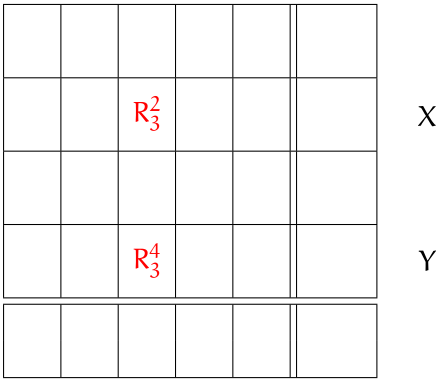
q_1, q_2, q_3, q_4, q_5 — contents (Yes/No questions)

c^1, c^2, c^3, c^4— contexts, or conditions

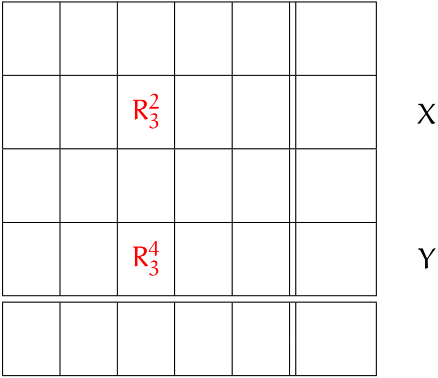
MAXIMAL COUPLINGS FOR PAIRS OF SAME-CONTENT RANDOM VARIABLES



MAXIMAL COUPLINGS FOR PAIRS OF SAME-CONTENT RANDOM VARIABLES

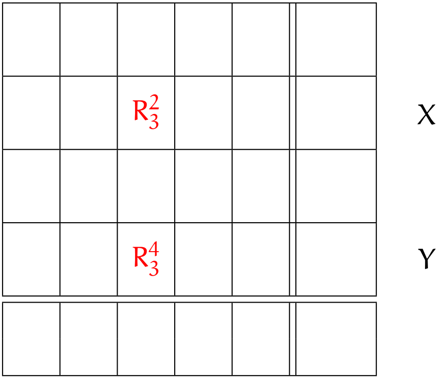


MAXIMAL COUPLINGS FOR PAIRS OF SAME-CONTENT RANDOM VARIABLES



$X \sim R_3^2, \quad Y \sim R_3^4, \quad JD(X, Y)$

MAXIMAL COUPLINGS FOR PAIRS OF SAME-CONTENT RANDOM VARIABLES



$X \sim R_3^2, \qquad Y \sim R_3^4, \qquad \mathrm{JD}\,(X, Y)$

$\max \Pr[X = Y]$ = measure of similarity of R_3^2 and R_3^4 in isolation

OVERALL COUPLING

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

OVERALL COUPLING

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

S_1^1	S_2^1		S_4^1		c^1
S_1^2		S_3^2			c^2
	S_2^3	S_3^3	S_4^3	S_5^3	c^3
		S_3^4		S_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{S}$

$\mathrm{JD}(\mathcal{S})$

OVERALL COUPLING

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

S_1^1	S_2^1		S_4^1		c^1
S_1^2		S_3^2			c^2
	S_2^3	S_3^3	S_4^3	S_5^3	c^3
		S_3^4		S_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{S}$

JD $(\mathcal{S})$

OVERALL COUPLING

R^1_1	R^1_2		R^1_4		c^1
R^2_1		R^2_3			c^2
	R^3_2	R^3_3	R^3_4	R^3_5	c^3
		R^4_3		R^4_5	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

S^1_1	S^1_2		S^1_4		c^1
S^2_1		S^2_3			c^2
	S^3_2	S^3_3	S^3_4	S^3_5	c^3
		S^4_3		S^4_5	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{S}$

JD $(\mathcal{S})$

OVERAL COUPLING

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

S_1^1	S_2^1		S_4^1		c^1
S_1^2		S_3^2			c^2
	S_2^3	S_3^3	S_4^3	S_5^3	c^3
		S_3^4		S_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{S}$

$\text{JD}(\mathcal{S})$

OVERAL COUPLING

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

S_1^1	S_2^1		S_4^1		c^1
S_1^2		S_3^2			c^2
	S_2^3	S_3^3	S_4^3	S_5^3	c^3
		S_3^4		S_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{S}$

JD $(\mathcal{S})$

(NON)CONTEXTUALITY

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

S_1^1	S_2^1		S_4^1		c^1
S_1^2		S_3^2			c^2
	S_2^3	S_3^3	S_4^3	S_5^3	c^3
		S_3^4		S_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{S}$

Is there an overall coupling in which any two same-content variables are as similar as they are in isolation?

(NON)CONTEXTUALITY

R_1^1	R_2^1		R_4^1		c^1
R_1^2		R_3^2			c^2
	R_2^3	R_3^3	R_4^3	R_5^3	c^3
		R_3^4		R_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}$

S_1^1	S_2^1		S_4^1		c^1
S_1^2		S_3^2			c^2
	S_2^3	S_3^3	S_4^3	S_5^3	c^3
		S_3^4		S_5^4	c^4
q_1	q_2	q_3	q_4	q_5	$\mathcal{S}$

Is there an overall coupling in which any two same-content variables are as similar as they are in isolation?

E.g., is $\Pr\left[S_3^2 = S_3^4\right]$ in the coupling maximal possible?

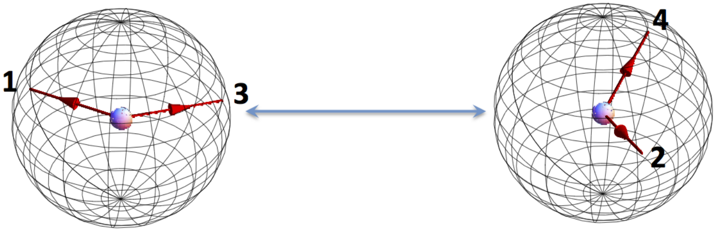
DEFINITION OF (NON)CONTEXTUALITY

Definition

A system is noncontextual if it has a coupling in which any two random variables sharing a content (answering the same question) coincide with the maximal possible probability.

If no such coupling exists, the system is contextual: the contexts are "forcing" the variables to be more dissimilar than they are when taken in isolation.

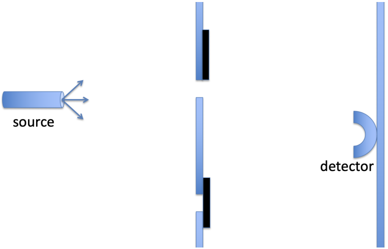
CYCLIC SYSTEMS: CONSISTENTLY CONNECTED



R_1^1	R_2^1			c^1
	R_2^2	R_3^2		c^2
		R_3^3	R_4^3	c^3
R_1^4			R_4^4	c^4
q_1	q_2	q_3	q_4	$\mathcal{R}_4$

$q_i =$ is the spin along axis i “up”?

CYCLIC SYSTEMS: INCONSISTENTLY CONNECTED



$R^{\circ\circ}_{\circ\cdot}$	$R^{\circ\circ}_{\cdot\circ}$			$c_{\circ\circ}$
	$R^{\times\circ}_{\cdot\circ}$	$R^{\times\circ}_{\times\cdot}$		$c_{\times\circ}$
		$R^{\times\times}_{\times\cdot}$	$R^{\times\times}_{\cdot\times}$	$c_{\times\times}$
$R^{\circ\times}_{\circ\cdot}$			$R^{\circ\times}_{\cdot\times}$	$c_{\circ\times}$
$q_{\circ\cdot}$	$q_{\cdot\circ}$	$q_{\times\cdot}$	$q_{\cdot\times}$	$\mathcal{R}_4$

$q_{\circ\cdot}$: hit through left open slit?

$q_{\cdot\circ}$: hit through right open slit?

$q_{\times\cdot}$: hit through left closed slit?

$q_{\cdot\times}$: hit through right closed slit?

CYCLIC SYSTEMS: EPISTEMIC VARIABLES

Nico: Zora's claim is true.

Zora: Max's claim is true.

Max: Alex's claim is true.

Alex: Nico's claim is false.

R^1_1	R^1_2			c^1
	R^2_2	R^2_3		c^2
		R^3_3	R^3_4	c^3
R^4_1			R^4_4	c^4
q_1	q_2	q_3	q_4	$\mathcal{R}_4$

q_1 : q_2 is true? q_2 : q_3 is true?

q_3 : q_4 is true? q_4 : q_1 is false?

CYCLIC SYSTEMS

R_1^1	R_2^1				c^1
	R_2^2	R_3^2			c^2
		R_3^3	R_4^3		c^3
			R_4^4	R_5^4	c^4
R_1^5				R_5^5	c^5
q_1	q_2	q_3	q_4	q_5	$\mathcal{R}_5$

R_1^1	R_2^1			c^1
	R_2^2	R_3^2		c^2
		R_3^3	R_4^3	c^3
R_1^4			R_4^4	c^4
q_1	q_2	q_3	q_4	$\mathcal{R}_4$

R_1^1	R_2^1		c^1
	R_2^2	R_3^2	c^2
R_1^3		R_3^3	c^3
q_1	q_2	q_3	$\mathcal{R}_3$

R_1^1	R_2^1	c^1
R_1^2	R_2^2	c^2
q_1	q_2	$\mathcal{R}_2$

CYCLIC SYSTEMS: PROBABILITY/EXPECTATION VECTORS

R_1^1	R_2^1		c^1
	R_2^2	R_3^2	c^2
R_1^3		R_3^3	c^3
q_1	q_2	q_3	$\mathcal{R}_3$

$$\Longrightarrow \left(\begin{array}{c} \left\langle R_1^1 \right\rangle, \left\langle R_2^1 \right\rangle, \left\langle R_2^2 \right\rangle, \ldots, \left\langle R_1^3 \right\rangle \\ \left\langle R_1^1 R_2^1 \right\rangle, \left\langle R_2^2 R_3^2 \right\rangle, \left\langle R_3^3 R_1^3 \right\rangle \end{array} \right)$$

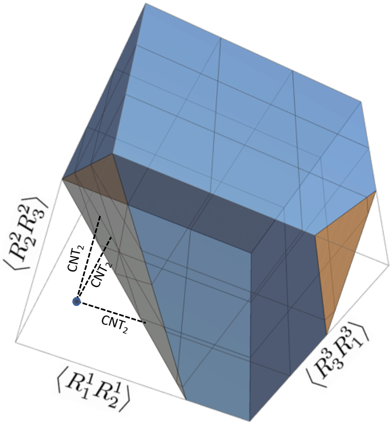
CYCLIC SYSTEMS: PROBABILITY/EXPECTATION VECTORS

R_1^1	R_2^1		c^1
	R_2^2	R_3^2	c^2
R_1^3		R_3^3	c^3
q_1	q_2	q_3	$\mathcal{R}_3$

$$\Longrightarrow \left(\begin{array}{c} \overbrace{\left\langle R_1^1 \right\rangle, \left\langle R_2^1 \right\rangle, \left\langle R_2^2 \right\rangle, \ldots, \left\langle R_1^3 \right\rangle}^{\text{fixed at observed values}} \\ \underbrace{\left\langle R_1^1 R_2^1 \right\rangle, \left\langle R_2^2 R_3^2 \right\rangle, \left\langle R_3^3 R_1^3 \right\rangle}_{\text{allowed to vary}} \end{array} \right)$$

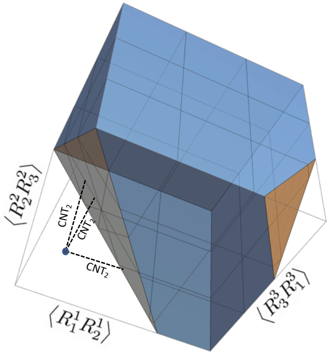
CNT_2 MEASURE OF CONTEXTUALITY FOR

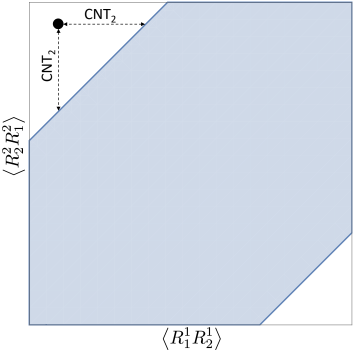
R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3



$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

DEGREE OF CONTEXTUALITY, CNT_2





Theorem (Dzh-Kujala-Cervantes 2020, Phys. Rev. A 101:042119)

$$\mathrm{CNT}_2 = \frac{1}{4}\left[\max \sum_{\text{odd } \#-} \pm \left\langle R_i^i R_{i\oplus 1}^i \right\rangle - \left(n - 2 + \sum_{i=1}^{n} \left| \left\langle R_i^i \right\rangle - \left\langle R_i^{i\ominus 1} \right\rangle \right| \right)\right]$$

DEGREE OF CONTEXTUALITY, CNT_2

$$\mathsf{CNT}_2 = \frac{1}{4}\left[\max \sum_{\text{odd } \# -} \pm \left\langle R_i^i R_{i\oplus 1}^i \right\rangle - \left(n - 2 + \sum_{i=1}^{n} \left| \left\langle R_i^i \right\rangle - \left\langle R_i^{i\ominus 1} \right\rangle \right| \right) \right]$$

DEGREE OF CONTEXTUALITY, CNT_2

$$\text{CNT}_2 = \frac{1}{4}\left[\max \sum_{\text{odd }\#-} \pm \left\langle R_i^i R_{i\oplus 1}^i \right\rangle - \left(n - 2 + \sum_{i=1}^{n} \left| \left\langle R_i^i \right\rangle - \left\langle R_i^{i\ominus 1} \right\rangle \right| \right) \right]$$

$$\max \sum_{\text{odd }\#-} \pm \left\langle R_i^i R_{i\oplus 1}^i \right\rangle > (n-2) + \sum_{i=1}^{n} \left| \left\langle R_i^i \right\rangle - \left\langle R_i^{i\ominus 1} \right\rangle \right|$$

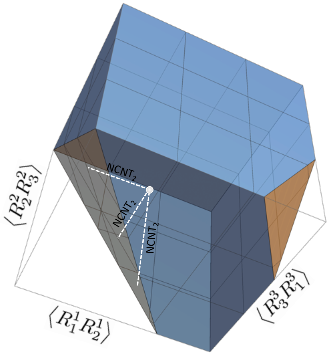
DEGREE OF CONTEXTUALITY, CNT_2

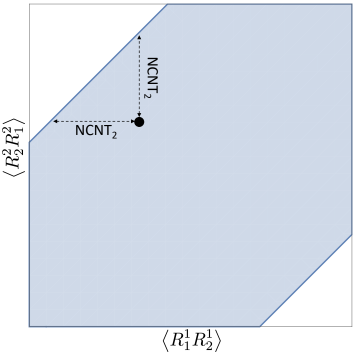
$$\mathrm{CNT}_2 = \frac{1}{4}\left[\max \sum_{\text{odd } \#-} \pm \left\langle R_i^i R_{i\oplus 1}^i \right\rangle - \left(n - 2 + \sum_{i=1}^{n} \left| \left\langle R_i^i \right\rangle - \left\langle R_i^{i\ominus 1} \right\rangle \right| \right)\right]$$

$$\max \sum_{\text{odd } \#-} \pm \left\langle R_i^i R_{i\oplus 1}^i \right\rangle > (n-2) + \sum_{i=1}^{n} \left| \left\langle R_i^i \right\rangle - \left\langle R_i^{i\ominus 1} \right\rangle \right|$$

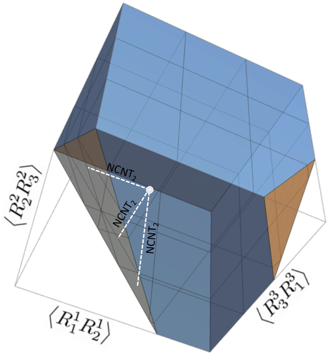
$$\max \sum_{\text{odd } \#-} \pm \left\langle R_i^i R_{i\oplus 1}^i \right\rangle > (n-2)$$

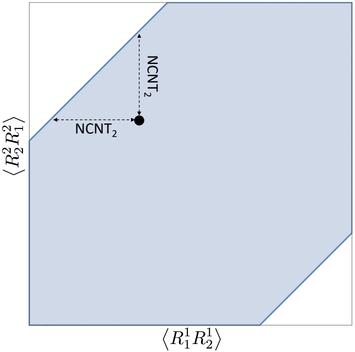
DEGREE OF NONCONTEXTUALITY, NCNT_2





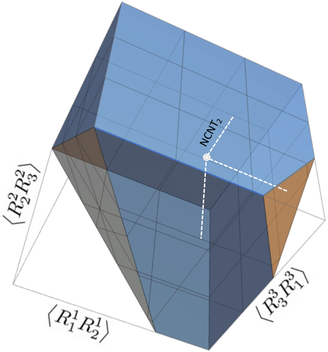
DEGREE OF NONCONTEXTUALITY, NCNT_2

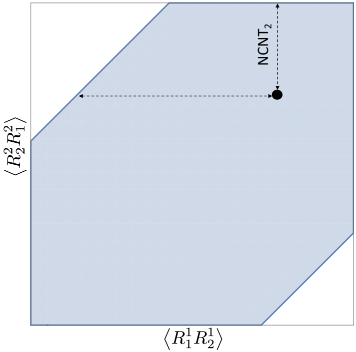




$$\frac{1}{4}\left[\left(n-2+\sum_{i=1}^{n}\left|\left\langle R_i^i\right\rangle-\left\langle R_i^{i\ominus 1}\right\rangle\right|\right)-\max\sum_{\text{odd }\#-}\pm\left\langle R_i^i R_{i\oplus 1}^i\right\rangle\right]$$

DEGREE OF NONCONTEXTUALITY, NCNT_2

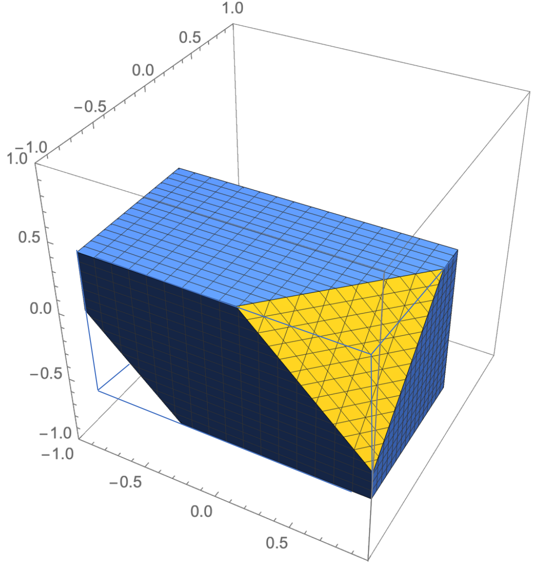




Theorem (Dzh-Kujala-Cervantes 2020, Phys. Rev. A 101:042119)

$$\mathsf{NCNT}_2 = \frac{1}{4}\min\left\{\begin{array}{c}\left(n-2+\sum_{i=1}^{n}\left|\left\langle R_i^i\right\rangle-\left\langle R_i^{i\ominus 1}\right\rangle\right|\right)-\max\sum_{\mathsf{odd}\ \#-}\pm\left\langle R_i^i R_{i\oplus 1}^i\right\rangle\\ m\left(\left\langle R_i^i R_{i\oplus 1}^i\right\rangle : i=1,\ldots,n\right)\end{array}\right.$$

R_1^1	R_2^1	
	R_2^2	R_3^2
R_1^3		R_3^3



EPISTEMIC ODDS OF CONTEXTUALITY

Theorem (Dzh-Kujala-Cervantes 2021, Eur. Phys. J. - Spec. Top.)

For any cyclic system of rank n,

$$\epsilon = 1 - \frac{\mathrm{vol}\mathbb{K}}{\mathrm{vol}\mathbb{B}} \leq \frac{2^{n-1}}{n!}.$$

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n	2	3	4	5	10	15	20	50	...
$\epsilon \leq$	1	6.67×10^{-1}	3.34×10^{-1}	1.34×10^{-1}	1.42×10^{-4}	1.26×10^{-8}	2.16×10^{-13}	1.86×10^{-50}	...

CYCLIC SYSTEMS ARE IMPORTANT, BUT NOT ALL IMPORTANT

Example (Dzh-Kujala-Cervantes 2020, Phys. Rev. A 101:042119)

A contextual system may have all its cyclic subsystems noncontextual.

R_1^1	R_2^1	R_3^1		c_1
	R_2^2	R_3^2	R_4^2	c_2
R_1^3		R_3^3	R_4^3	c_3
q_1	q_2	q_3	q_4	

R_1^1	R_2^1	R_3^1	
-1	-1	$+1$	$1/4$
-1	$+1$	-1	$1/4$
$+1$	-1	-1	$1/4$
$+1$	$+1$	$+1$	$1/4$

,

R_2^2	R_3^2	R_4^2	
-1	-1	$+1$	$1/4$
-1	$+1$	-1	$1/4$
$+1$	-1	-1	$1/4$
$+1$	$+1$	$+1$	$1/4$

,

R_1^3	R_3^3	R_4^3	
$+1$	$+1$	-1	$1/4$
$+1$	-1	$+1$	$1/4$
-1	$+1$	$+1$	$1/4$
-1	-1	-1	$1/4$

LITERATURE

- For a nontechnical introduction to the Contextuality-by-Default approach, see https://arxiv.org/abs/2103.07954 and https://arxiv.org/abs/2104.12495.
- The paper in Phys. Rev. A 101:042119, 2020 (corrected in accordance with the Erratum notes, in PRA 101:069902, 2020, and 103:059901, 2021) is in https://arxiv.org/abs/1907.03328.
- The paper on the epistemic odds of contextuality is in https://arxiv.org/abs/2002.07755.
- Some topics mentioned in the talks are discussed in https://arxiv.org/abs/1906.02718, https://arxiv.org/abs/1801.10593, and https://arxiv.org/abs/1703.06752.