A hierarchical measure of contextuality

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Systems of random variables

A system $\mathcal{R}$ of random variables is a set of double-indexed random variables R_q^c, where

- $q \in Q$ denotes their content,
- $c \in C$ is their context.

A system can be presented as

$$\mathcal{R} = \{R_q^c : q \in Q, c \in C, q \prec c\},$$

$q \prec c$ indicates that content q is responded to in context c.

Connections

Any two variables in the subset

$$\mathcal{R}_q = \{R_q^c : c \in C, q \prec c\}$$

are *stochastically unrelated.*

The set $\mathcal{R}_q$ is called the *connection* for content q.

Bunches

The variables of the subset

$$R^c = \{R_q^c : q \in Q, q \prec c\}$$

are *jointly distributed*.

The set R^c is called the *bunch* corresponding to context c.

A system $\mathcal{R}$ with $n = |Q|$ and $m = |C|$ where all variables R_q^c take values 0/1 can be represented by three vectors:

$$\mathbf{l}^\star = (p_q^c)_{c \in C, q \in Q, q \prec c}$$

$$p_q^c = \Pr(R_q^c = 1)$$

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$$\mathbf{l}^\star = (p_q^c)_{c \in C, q \in Q, q \prec c}$$
$$\mathbf{c}^\star = (\min\{p_q^{c_1}, p_q^{c_2}\})_{c_1, c_2 \in C, q \in Q, q \prec c}$$

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$$
\begin{aligned}
\mathbf{l}^\star &= (p_q^c)_{c \in C, q \in Q, q \prec c} \\
\mathbf{c}^\star &= (\min\{p_q^{c_1}, p_q^{c_2}\})_{c_1, c_2 \in C, q \in Q, q \prec c} \\
\mathbf{b}^\star &= \begin{pmatrix} \mathbf{b}_2^\star \\ \vdots \\ \mathbf{b}_r^\star \end{pmatrix}
\end{aligned}
$$

$$p_q^c = \Pr(R_q^c = 1)$$

$$\mathbf{b}_s^\star = (p_{q_1,\ldots,q_s}^c)_{q_1,\ldots,q_s \in Q, c \in C, q_1,\ldots,q_s \prec c, q_1,\ldots,q_s} \text{ are distinct}$$

$$s = 2, \ldots, r; r \leqslant n; p_{q_1,\ldots,q_s}^c = \Pr(R_{q_1}^c = \cdots = R_{q_s}^c = 1)$$

Hierarchical structure of $b^\star$

Note that

$$\max(0, p^c_{q_1} + p^c_{q_2} - 1) \leqslant p^c_{q_1,q_2} \leqslant \min(p^c_{q_1}, p^c_{q_2})$$

Hierarchical structure of $\mathrm{b}^\star$

Note that

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$$\max\left(0, \frac{p^c_{q_1,q_2} + p^c_{q_1,q_3} + p^c_{q_2,q_3} - 1}{2}\right) \leqslant$$

$$p^c_{q_1,q_2,q_3} \leqslant \min(p^c_{q_1,q_2}, p^c_{q_1,q_3}, p^c_{q_2,q_3})$$

Hierarchical structure of $\mathbf{b}^\star$

and in complete generality,

$$\max\left(0, \frac{1}{s-1}\left(\sum_{k=1}^{s} p^{c}_{\{q_1,\ldots,q_s\}\setminus\{q_k\}} - 1\right)\right) \leqslant p^{c}_{q_1,\ldots,q_s} \leqslant \min(p^{c}_{\{q_1,\ldots,q_s\}\setminus\{q_k\}})_{k=1,\ldots,s},$$

for $s = 2, \ldots, r$.

That is, the elements of $\mathbf{b}^\star$ are hierarchically bounded, with $\mathbf{l}^\star$ providing the bounds for $\mathbf{b}^\star_2$ and $\mathbf{b}^\star_{s-1}$ determining the bounds for $\mathbf{b}^\star_s$ if $2 < s \leqslant r$.

A hierarchy of linear systems

A system represented by vectors $\mathbf{l}^\star$, $\mathbf{c}^\star$, and $\mathbf{b}^\star$ is noncontextual if and only if there is a vector $\mathbf{h} \geqslant 0$ (component-wise) such that

$$\mathbf{M}\mathbf{h} = \begin{pmatrix} 1 \\ \mathbf{l}^\star \\ \mathbf{c}^\star \\ \mathbf{b}^\star \end{pmatrix},$$

$\mathbf{h}$ are probabilities assigned to all combinations of values (1's and 0's) assigned to all random variables R_q^c

$\mathbf{M}$ is an incidence (Boolean) matrix

A hierarchy of linear systems

Denote the rows of $\mathbf{M}$ that correspond to $\mathbf{l}^\star$ and $\mathbf{c}^\star$ by $\mathbf{M}_\mathbf{l}$ and $\mathbf{M}_\mathbf{c}$.

Similarly, let $\mathbf{M}_s$ be the submatrix formed by the rows of $\mathbf{M}$ corresponding to the elements of $\mathbf{b}_s^\star$, for $2 \leqslant s \leqslant r$.

A hierarchy of linear systems

Consider the systems of equations

$$\begin{pmatrix} \mathbf{1}^\intercal \\ \mathbf{M}_\mathbf{l} \\ \mathbf{M}_\mathbf{c} \\ \mathbf{M}_2 \end{pmatrix} \mathbf{h}_2 = \begin{pmatrix} 1 \\ \mathbf{l}^\star \\ \mathbf{c}^\star \\ \mathbf{b}_2^\star \end{pmatrix}$$

$$\vdots$$

A hierarchy of linear systems

$$\vdots$$

$$\begin{pmatrix} \mathbf{1}^\intercal \\ \mathbf{M}_\mathbf{l} \\ \mathbf{M}_\mathbf{c} \\ \mathbf{M}_2 \\ \vdots \\ \mathbf{M}_s \end{pmatrix} \mathbf{h}_s = \begin{pmatrix} 1 \\ \mathbf{l}^\star \\ \mathbf{c}^\star \\ \mathbf{b}_2^\star \\ \vdots \\ \mathbf{b}_s^\star \end{pmatrix}$$

$$\vdots$$

A hierarchy of linear systems

$$\vdots$$

$$\mathbf{M}\mathbf{h} = \begin{pmatrix} \mathbf{1}^\intercal \\ \mathbf{M}_l \\ \mathbf{M}_c \\ \mathbf{M}_2 \\ \vdots \\ \mathbf{M}_r \end{pmatrix} \mathbf{h}_r = \begin{pmatrix} 1 \\ \mathbf{l}^\star \\ \mathbf{c}^\star \\ \mathbf{b}_2^\star \\ \vdots \\ \mathbf{b}_r^\star \end{pmatrix}$$

Contextuality and the hierarchy of linear systems

Noncontextual systems

When the system is noncontextual,

$$\mathbf{M}\mathbf{h} = \begin{pmatrix} 1 \\ \mathbf{l}^\star \\ \mathbf{c}^\star \\ \mathbf{b}^\star \end{pmatrix}$$

has a solution $\mathbf{h}^* \geqslant 0$.

This solution implies that all the systems have solutions $\mathbf{h}_s^* = \mathbf{h}^*$, $2 \leqslant s \leqslant r$.

Contextuality and the hierarchy of linear systems

Contextual systems

When the system is contextual, there is a value $2 \leqslant s^* \leqslant r$ such that there is no solution $\mathbf{h_s} \geqslant 0$ for any $s \geqslant s^*$ while there is a solution for each $s < s^*$.

We shall say the system is contextual at level s^*.

The degree of contextuality at level s^*: CNT$_2^{s^*}$

find	minimizing	subject to
$\mathbf{x}$	$\mathbf{1}^\intercal \mathbf{d}$	$-\mathbf{d} \leqslant \mathbf{b}^\star_{s^*} - \mathbf{M}_{s^*}\mathbf{x} \leqslant \mathbf{d}$
		$\mathbf{x}, \mathbf{d} \geqslant 0$
		$\mathbf{1}^\intercal \mathbf{x} = 1$
		$\mathbf{M}_\mathbf{l}\mathbf{x} = \mathbf{l}^\star$
		$\mathbf{M}_\mathbf{c}\mathbf{x} = \mathbf{c}^\star$
		$\mathbf{M}_2\mathbf{x} = \mathbf{b}^\star_2$
		$\vdots$
		$\mathbf{M}_{s^*-1}\mathbf{x} = \mathbf{b}^\star_{s^*-1}$

For any solution $\mathbf{x}^*$, compute

$$\text{CNT}_2^{s^*} = \left\| \mathbf{b}^\star_{s^*} - \mathbf{M}_{s^*}\mathbf{x}^* \right\|_1 .$$

The degree of noncontextuality at levels $s < s^*$: NCNT_2^s

$$\mathrm{NCNT}_2^s = \min_{i=1,\dots,K_s}\{\min(d_i^{*-}, d_i^{*+})\},$$

where K_s is the number of elements of $\mathbf{b}_s^\star$, and d_i^{*-}, d_i^{*+} are solutions to

find	maximizing	subject to	find	maximizing	subject to
$\mathbf{x}, d_i^-$	d_i^-	$\mathbf{b}_s^\star - d_i^- \mathbf{e}_i = \mathbf{M}_s\mathbf{x}$	$\mathbf{x}, d_i^+$	d_i^+	$\mathbf{b}_s^\star + d_i^+ \mathbf{e}_i = \mathbf{M}_s\mathbf{x}$
		$\mathbf{x}, d_i^- \geqslant 0$			$\mathbf{x}, d_i^+ \geqslant 0$
		$\mathbf{1}^\intercal\mathbf{x} = 1$			$\mathbf{1}^\intercal\mathbf{x} = 1$
		$\mathbf{M}_l\mathbf{x} = \mathbf{l}^\star$			$\mathbf{M}_l\mathbf{x} = \mathbf{l}^\star$
		$\mathbf{M}_c\mathbf{x} = \mathbf{c}^\star$			$\mathbf{M}_c\mathbf{x} = \mathbf{c}^\star$
		$\mathbf{M}_2\mathbf{x} = \mathbf{b}_2^\star$			$\mathbf{M}_2\mathbf{x} = \mathbf{b}_2^\star$
		$\vdots$			$\vdots$
		$\mathbf{M}_{s-1}\mathbf{x} = \mathbf{b}_{s-1}^\star$			$\mathbf{M}_{s-1}\mathbf{x} = \mathbf{b}_{s-1}^\star$

.

Example

R_1^1	R_2^1	R_2^1	c_1
R_1^2	R_2^2	R_2^2	c_2
q_1	q_2	q_3	$\mathcal{R}$

such that

- $\Pr(R_{q_i}^{c_j} = 1) = 1/2$ for all variables in $\mathcal{R}$.
- Variables in R^{c_1} are independent.
- Variables in R^{c_2} are pairwise independent.

Example

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- Noncontextual at level 2, $\mathrm{NCNT}_2^2 = 0$.

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- Variables in R^{c_1} are independent.
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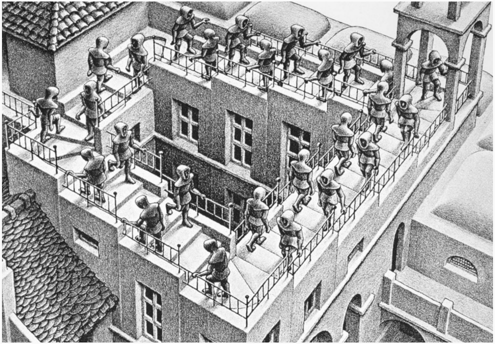
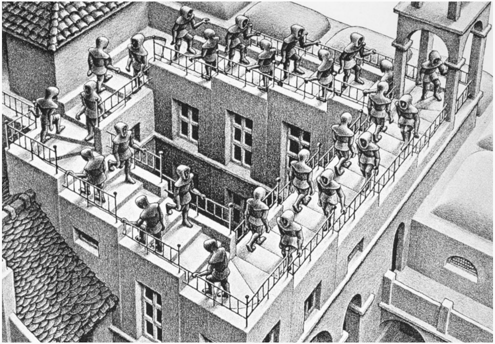
- Noncontextual at level 2, $\mathrm{NCNT}_2^2 = 0$.
- Contextual at level 3 whenever $p_{q_1,q_2,q_3}^{c_2} \neq 1/8$ with $\mathrm{CNT}_2^3 = |p_{q_1,q_2,q_3}^{c_2} - 1/8|$.



Thank you!

Ascending and Descending



Ascending and Descending

We approach the Ascending and Descending lithograph by M. C. Escher by considering the four stair flights as four contents, q_1, q_2, q_3, q_4.

Ascending and Descending

In one context we describe how the four stair flights appear together: either all staircases are described as ascending ($R^1_{q_i} = 1,$ for $i = 1, \dots, 4$) or all of them are descending ($R^1_{q_i} = 0,$ for $i = 1, \dots, 4$).

Ascending and Descending

In a second context, we describe the physically realizable patterns. Four stair flights forming a closed loop cannot ascend or descend indefinitely: the number of ascending stair flights should be precisely two to counterbalance the descending ones.

0	0	1	1	c_2
q_1	q_2	q_3	q_4	

1	1	0	0	c_2
q_1	q_2	q_3	q_4	

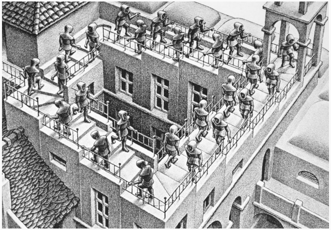
0	1	0	1	c_2
q_1	q_2	q_3	q_4	

1	0	1	0	c_2
q_1	q_2	q_3	q_4	

1	0	0	1	c_2
q_1	q_2	q_3	q_4	

0	1	1	0	c_2
q_1	q_2	q_3	q_4	

Ascending and Descending



The resulting system

R_1^1	R_2^1	R_3^1	R_4^1	c_1
R_1^2	R_2^2	R_3^2	R_4^2	c_2
q_1	q_2	q_3	q_4	$\mathcal{E}$

is contextual at the level $s^* = 2$ with $\text{CNT}_2^2 = 2$.

Cyclic systems

A system is cyclic (containing one or more disconnected cycles) if every bunch and every connection contains precisely two random variables.

It has been shown that removing a bunch/connection with a single variable preserves the (non)contextuality status of the system.

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Hence, a system without cycles is noncontextual.

A system without cycles is noncontextual

If the system $\mathcal{R}$ does not have cycles, then there exists some bunch $R^{c'}$ or some connection $\mathcal{R}_{q'}$ that has a single variable.

If the system has a single variable, take $\mathbf{h}^* = (p_1^1, 1 - p_1^1)^\intercal$

If the system has more than one variable, remove $R^{c'}$ or $\mathcal{R}_{q'}$ from the system to obtain a system $\mathcal{R}'$ with the same contextuality status as $\mathcal{R}$.

Example. A system can be contextual without contextual cyclic subsystems

Consider the system

R_1^1	R_2^1	R_2^1	c_1
R_1^2	R_2^2	R_2^2	c_2
q_1	q_2	q_3	$\mathcal{R}$

such that

- $\Pr(R_{q_i}^{c_j} = 1) = 1/2$ for all variables in $\mathcal{R}$.
- Variables in R^{c_1} are independent.
- Variables in R^{c_2} are pairwise independent.