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QCQMB'21 workshop, May 19, 2021

Quantum contextualitieS

Contextuality has a variety of interpretations associated with their inventors, say Bohr, Bell, Kochen and Specker, Cabello, and recently Dzhafarov.

In fact, Bohr was the first who pointed to contextuality of quantum measurements as a part of formulation of his principle of complementarity. (Instead of "contextuality", he considered dependence on "experimental conditions.") But contextuality counterpart of the complementarity principle was overshadowed by the issue of incompatibility of observables.

Interest for contextuality of quantum measurements rose again only in connection with the Bell inequality. The original Bohr's contextuality, as contextuality of each quantum measurement, was practically forgotten.

Bohr's viewpoint on contextuality of quantum measurements was highlighted in my works, with applications both to physics and cognition.

Khrennikov, A. *Contextual Approach to Quantum Formalism*; Springer: Berlin, Germany; New York, NY, USA, 2009.

Khrennikov, A. (2010). *Ubiquitous quantum structure: from psychology to finances*; Springer: Berlin-Heidelberg-New York.

In this talk, the theory of open quantum systems is applied to formalization of Bohr's contextuality within the the scheme of indirect measurements. This scheme is widely used in quantum information theory and it leads to the theory of quantum instruments (Davis-Lewis-Ozawa). In this scheme, Bohr's viewpoint on contextuality of quantum measurements is represented in the formal mathematical framework.

Bohr's contextuality has the clear physical meaning

"Strictly speaking, the mathematical formalism of quantum mechanics and electrodynamics merely offers rules of calculation for the deduction of expectations pertaining to observations obtained **under well-defined experimental conditions** *specified by classical physical concepts."*

Instead of "contextuality", he considered dependence on "experimental conditions." (We remark that neither Bell nor Kochen and Specker operated with the notion of contextuality. It was invented later by Beltrametti and Cassinelli.)

Bell, Kochen-Specker and their followers considered a variety of "derivatives" of Bohr's contextuality related to the very special case of joint measurement of pairs of compatible observables.

Joint-measurement contextuality: **If A, B, C are three quantum observables, such that A is compatible with B and C, a measurement of A might give different result depending upon whether A is measured with B or with C.**

This formulation is based on counterfactual argument and from my viewpoint it cannot be tested experimentally, so it has no relation to physics. (But, Svozil and Griffiths claimed that they elaborated experimental tests for joint-measurement contextuality.)

This formulation due to Bell who tried in this way to find an alternative to nonlocality in explaining violation of his inequality. Cabello widely advertized identification of contextuality with violation of the Bell type ineqialities, so called noncontextual inequalities.

Bohr-contextuality is experimentally tested through incompatibility and theoretically it is formulated in terms of commutators. The basic test is based on the Heisenberg uncertainty relation in its general form of the Schrödinger-Robertson inequality.

Cabello-contextuality can be tested experimentally in experiments by demonstration of violation of various noncontextual inequalities.

Contextuality component of Bohr's principle of complementarity

We follow my previous works devoted to Bohr's principle of complementarity and its contextual component. We start with the well known citate of Bohr ([?], v. 2, p. 40-41):

"This crucial point ... implies *the impossibility of any sharp separation between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.* In fact, the individuality of the typical quantum effects finds its proper expression in the circumstance that any attempt of subdividing the phenomena will demand a change in the experimental arrangement introducing new possibilities of interaction between objects and measuring instruments which in principle cannot be controlled. Consequently, evidence obtained under different experimental conditions cannot be comprehended within a single picture ... ."

The contextual component of this statement can be formulated as

Principle 1 (Contextuality) **The output of any quantum observable is indivisibly composed of the contributions of the system and the measurement apparatus.**

- There is no reason to expect that all experimental contexts can be combined and all observables can be measured jointly.
- Hence, incompatible observables (complementary experimental contexts) may exist.
- Moreover, they should exist, otherwise the contextuality principle would have the empty content.
- Really, if all experimental contexts can be combined into single context C and all observables can be jointly measured in this context, then the outputs of such joint measurements can be assigned directly to a system.
- To be more careful, we have to say: "assigned to a system and context $-C$". But, the latter can be omitted, since this is the same context for all observables.

The above reasoning implies:

Principle 2 (Incompatibility) There exist incompatible observables (complementary experimental contexts).

Since both principles, contextuality and ncompatibility, are so closely interrelated, it is natural to unify them into the single principle, **Contextuality-Incompatibility principle.**

This is my understanding of the Bohr's Complementarity principle.

Indirect measurement scheme: apparatus with meter interacting with a system

This scheme represents Bohr's framework - the outcomes of measurements are created in the process of the interaction of a system S with a measurement apparatus M. The latter is combined of a complex physical device interacting with S and a pointer showing the outcomes of measurements; for example, it can be the "spin up or spin down" arrow.

The system S by itself is not approachable by the observer who can see only the pointer of M. Then the observer associates pointer's outputs with the values of measured observable A for the system S.

The indirect measurement scheme can be represented as the block of following interrelated components:

- the states of the systems S and the apparatus M; they are represented in complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively;
- the unitary operator U representing the interaction-dynamics for the compound system $S + M$;
- the meter observable M_A giving outputs of the pointer of the apparatus M.

It is assumed that the compound system $S + M$ is isolated. The dynamics of pure states of the compound system is described by the Schrödinger equation:

$$i\frac{d}{dt}|\Psi\rangle(t) = H|\Psi\rangle(t), \ |\Psi\rangle(0) = |\Psi\rangle_0, \tag{1}$$

where H is it Hamiltonian (generator of evolution) of $S + M$. The state $|\Psi\rangle(t)$ evolves as

$$|\Psi\rangle(t) = U(t)|\Psi\rangle_0,$$

where $U(t)$ is the unitary operator represented as

$$U(t) = e^{-itH}.$$

Hamiltonian (evolution-generator) describing information interactions has the form

$$H = H_S \otimes I + I \otimes H_M + H_{S,M},$$

The Schrödinger equation implies that evolution of the density operator $\boldsymbol{R(t)}$ of the system $\boldsymbol{S+M}$ is described by the von Neumann equation:

$$\frac{d\boldsymbol{R}}{dt}(t) = -i[\boldsymbol{H}, \boldsymbol{R}(t)], \;\; \boldsymbol{R}(0) = \boldsymbol{R}_0. \tag{2}$$

However, the state $\boldsymbol{R(t)}$ is too complex to be handled consistently: the apparatus includes many degrees of freedom.

Suppose an observable on the system S which is represented by Hemitian operator A, acting in system's state space $\mathcal{H}$. The *indirect measurement model* for measurement of the A-observable was introduced by Ozawa in [?] as a "(general) measuring process"; this is a quadruple

$$(\mathcal{K}, \sigma, U, M_A)$$

consisting of a Hilbert space $\mathcal{K}$, a density operator $\sigma \in \mathrm{S}(\mathcal{K})$, a unitary operator U on the tensor product of the state spaces of S and M, $U : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$, and a Hermitian operator M_A on $\mathcal{K}$.

Here $\mathcal{K}$ represents the states of the apparatus M, U describes the time-evolution of system $S + M$, σ describes the initial state of the apparatus M before the start of measurement, and the Hermitian operator M_A is the meter observable of the apparatus M (say the pointer of M). This operator represents indirectly outcomes of an observable A for the system S.

The probability distribution $\mathbf{Pr}\{A = x \| \rho\}$ in the system state $\rho \in \mathrm{S}(\mathcal{H})$ is given by

$$\mathbf{Pr}\{A = x \| \rho\} = \mathbf{Tr}[(I \otimes E^{M_A}(x))U(\rho \otimes \sigma)U^{\star}], \tag{3}$$

where $E^{M_A}(x)$ is the spectral projection of M_A for the eigenvalue x. We reall that operator M_A is Hermitian. In the finite dimensional case, it can be represented in the form:

$$M_A = \sum_k x_k E^{M_A}(x_k), \tag{4}$$

where (x_k) is the set of its eigenvalues and $E^{M_A}(x_k)$ is the projector on the subspace of eigenvectors corresponding to eigenvalue x_k.

The change of the state $\boldsymbol{\rho}$ of the system $\boldsymbol{S}$ caused by the measurement for the outcome $\boldsymbol{A} = \boldsymbol{x}$ is represented with the aid of the map $\boldsymbol{\mathcal{I}_A(x)}$ in the space of density operators defined as

$$\text{(5)} \qquad \boldsymbol{\mathcal{I}_A(x)\rho = \mathrm{Tr}_{\mathcal{K}}[(I \otimes E^{M_A}(x))U(\rho \otimes \sigma)U^{\star}]},$$

where $\mathbf{Tr}_{\boldsymbol{\mathcal{K}}}$ is the partial trace over $\boldsymbol{\mathcal{K}}$. The map $\boldsymbol{x \mapsto \mathcal{I}_A(x)}$ is a quantum instrument. We remark that conversely any quantum instrument can be represented via the indirect measurement model (see Ozawa).

Bohr's contextuality from the indirect measurement scheme

We take the basic part of the aforementioned citate of Bohr: ".. the impossibility of any sharp separation between the behaviour of atomic objects and the interaction with the measuring instruments" and establish correspondence with the indirect measurement scheme.

- "atomic object" - state $\boldsymbol{\rho}$;
- "measuring instrument" - state $\boldsymbol{\sigma}$;
- "interaction" - unitary operator $\boldsymbol{U}$.

The triple $\boldsymbol{C} = (\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{U})$ represents the complex of the "experimental conditions", the context of measurement.

The interrelation between contextuality (in Bohr's sense) and incompatibility is completely clear: incompatibility is so to say the "derivative" of contextuality.

There is no reason to expect that any pair of contexts, $\boldsymbol{C_1} = (\boldsymbol{\rho_1}, \boldsymbol{\sigma_1}, \boldsymbol{U_1})$ and $\boldsymbol{C_2} = (\boldsymbol{\rho_2}, \boldsymbol{\sigma_2}, \boldsymbol{U_2})$ can be unified in the joint measurement scheme, even if $\boldsymbol{\rho_1} = \boldsymbol{\rho_2} = \boldsymbol{\rho}$.

Thus, for the fixed system's state measurement context is given by the pair $\boldsymbol{C'} = (\boldsymbol{\sigma}, \boldsymbol{U})$. For a macroscopic apparatus, we can assume that its state is also fixed. So, the main characteristic of a context is the system-apparatus interaction $\boldsymbol{U}$.

Since for the same observable $\boldsymbol{A}$ interactions can be different, quantum mechanics is contextual in the Bohr's sense.

Can here be given any meaning to Cabello-contextuality without incompatibility?

Bohr's contextuality is the genuine measurement contextuality.

Complementarity is its consequence.

We claim that "under natural conditions" Cabello-contextuality (= violation of noncontextual inequalities) is the same as complementarity – for quantum observables!

A. Khrennikov, Can here be given any meaning to contextuality without incompatibility? Int. J. Theor. Phys.; https://doi.org/10.1007/s10773-020-04666-z

In this sense, there is no meaning to check these inequalities at all! Since we know from Heisenberg that incompatible quantum observables exist.

CHSH in the noncontextual framework:

$$|\langle \boldsymbol{X}_1\boldsymbol{X}_2\rangle + \langle \boldsymbol{X}_2\boldsymbol{X}_3\rangle + \langle \boldsymbol{X}_3\boldsymbol{X}_4\rangle - \langle \boldsymbol{X}_4\boldsymbol{X}_1\rangle| \leq \mathbf{2}. \tag{6}$$

Since we work with quantum observables, we proceed under the compatibility assumption

$$[\hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_2] = \mathbf{0}, [\hat{\boldsymbol{X}}_3, \hat{\boldsymbol{X}}_2] = \mathbf{0}, [\hat{\boldsymbol{X}}_3, \hat{\boldsymbol{X}}_4] = \mathbf{0}, [\hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_4] = \mathbf{0}. \tag{7}$$

Now set

$$\hat{\boldsymbol{M}}_{\mathbf{13}} = \boldsymbol{i}[\hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_3] \text{ and } \hat{\boldsymbol{M}}_{\mathbf{34}} = \boldsymbol{i}[\hat{\boldsymbol{X}}_2, \hat{\boldsymbol{X}}_4]. \tag{8}$$

These are Hermitian operators, so they represent some quantum observables $\boldsymbol{M}_{\mathbf{13}}$ and $\boldsymbol{M}_{\mathbf{34}}$. We remark that these observables are compatible:

$$[\hat{\boldsymbol{M}}_{\mathbf{13}}, \hat{\boldsymbol{M}}_{\mathbf{34}}] = \mathbf{0}. \tag{9}$$

Theorem 1. *Condition*

$$\hat{\boldsymbol{M}}_{13} \circ \hat{\boldsymbol{M}}_{34} \neq \mathbf{0}. \tag{10}$$

is necessary and sufficient for violation of the noncontextuality inequality (6) for some quantum state.

Proof's scheme. Consider the operator

$$\hat{\boldsymbol{\Gamma}} = \hat{\boldsymbol{X}}_1\hat{\boldsymbol{X}}_2 + \hat{\boldsymbol{X}}_2\hat{\boldsymbol{X}}_3 + \hat{\boldsymbol{X}}_3\hat{\boldsymbol{X}}_4 - \hat{\boldsymbol{X}}_4\hat{\boldsymbol{X}}_1. \tag{11}$$

Then we have

$$\hat{\boldsymbol{\Gamma}}^2 = 4 + [\hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_3][\hat{\boldsymbol{X}}_2, \hat{\boldsymbol{X}}_4] = 4 + \hat{\boldsymbol{M}}_{13}\hat{\boldsymbol{M}}_{34}. \tag{12}$$

Then it is easy to show that $||\hat{\boldsymbol{\Gamma}}^2|| > 4$, if and only if condition (10) holds.

Finally, we note that

$$\sup_{\|\psi\|=1} |\langle\psi|\hat{\Gamma}|\psi\rangle = \|\hat{\Gamma}\| = \sqrt{\|\hat{\Gamma}^2\|}. \tag{13}$$

We remark that condition (10) is trivially satisfied for incompatible observables, if the state space and observables have the tensor product structure: $H = H_{13} \otimes H_{24}$ and

$$\hat{X}_i = \hat{\mathrm{X}}_i \otimes I, \hat{X}_j = I \otimes \hat{\mathrm{X}}_j, \tag{14}$$

where

$$\hat{\mathrm{X}}_i : H_{13} \to H_{13}, i = 1, 3, \hat{\mathrm{X}}_j : H_{24} \to H_{44}, j = 2, 4. \tag{15}$$

Here condition (10) is reduced to incompatibility condition:

$$[\hat{\mathrm{X}}_i, : \hat{\mathrm{X}}_j] \neq 0, i = 1, 3; j = 2, 4. \tag{16}$$

In particular, **for compound systems, contextuality ("non-locality") is exactly incompatibility.**

The same is valid for any tensor decomposition of the state space of a single quantum system with observables of the type (14). In the tensor product case, contextuality without incompatibility leads to the notion with the empty content.

But, it may happen that $\boldsymbol{X_i}$-observables, $\boldsymbol{i = 1, 3,}$ and $\boldsymbol{X_j}$-observables, $\boldsymbol{j = 2, 4,}$ are not connected via the tensor product structure. In this case, the interpretation of constraint (10) is nontrivial.

What is its physical meaning?

I have no idea.

A. Khrennikov, Get rid of nonlocality from quantum physics. Entropy, 21(8), 806 (2019).

The same reasoning leads to conclusion that nonlocality is apparent, it is just another word for incompatibility.

Bell, J.S. On the problem of hidden variables in quantum theory. *Rev. Mod. Phys.* **1966**, *38*, 450.