On Quantum Sets of the Coloured-Graph Approach to Contextuality

Marcelo Terra Cunha

Joint work with Lina Vandré

arXiv 2105.08561

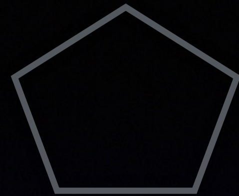
Universidade Estadual de Campinas - Unicamp

QCQMB21

19/05/21

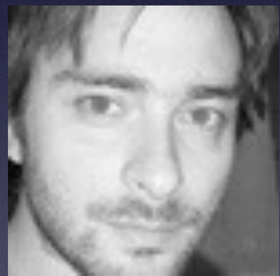
Barbara's birthday + Physicists Day

Graph-Approach



$$\sum_{i\in V(C_5)} p(v_i) \overset{NC}{\leq} 2 \overset{MQ}{\leq} \sqrt{5} \overset{E}{\leq} \frac{5}{2}$$

Adán Cabello



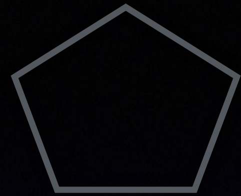
Simone Severini

Andreas Winter

arXiv: quant-ph/1010.2163

Phys. Rev. Lett. **112**, 040401

Graph-Approach



$$\sum_{i\in V(G)} p(v_i) \overset{NC}{\leq} \alpha(G) \overset{MQ}{\leq} \vartheta(G) \overset{E}{\leq} \alpha^*(G)$$

Independence Number

Lovász Number

Fractional Packing Number

Adán Cabello

Simone Severini

Andreas Winter

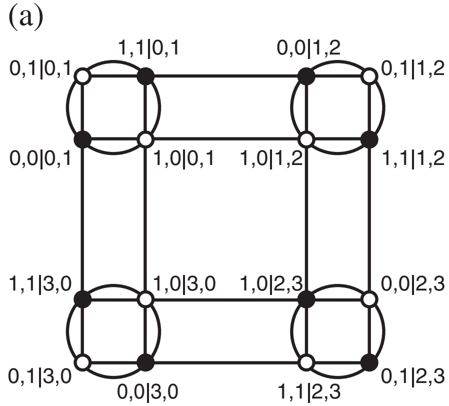
arXiv: quant-ph/1010.2163

Phys. Rev. Lett. **112**, 040401

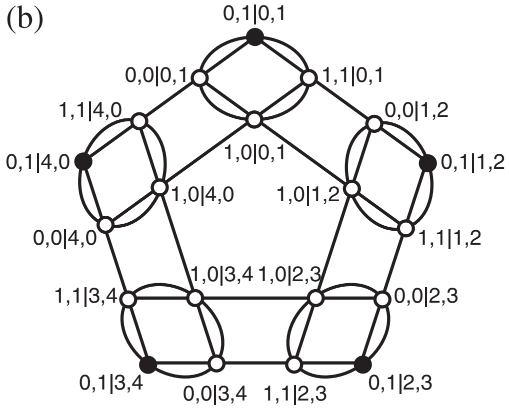
CSW Graph Approach

Exclusivity Graph of a Scenario

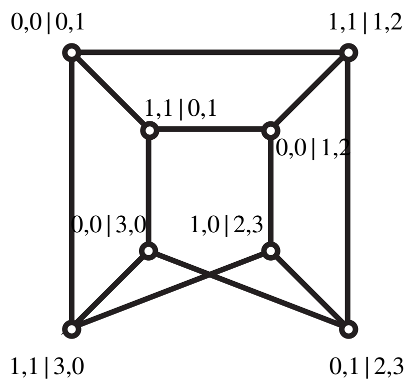
CHSH

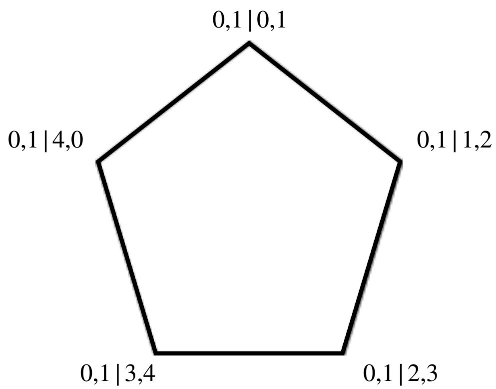


KCBS



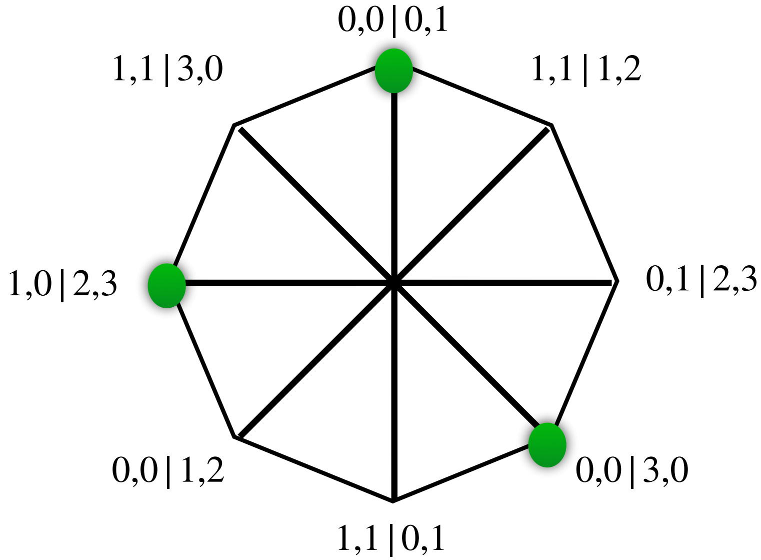
Exclusivity Graph of an Inequality





Cabello, Severini, Winter, *Phys. Rev. Lett.* **112**, 040401 Clauser, Horne, Shimony, Holt, *Phys. Rev. Lett.* **23**, 880 Klyachko, Can, Binicioglu, Shumovsky, *Phys. Rev. Lett.* **101**, 020403

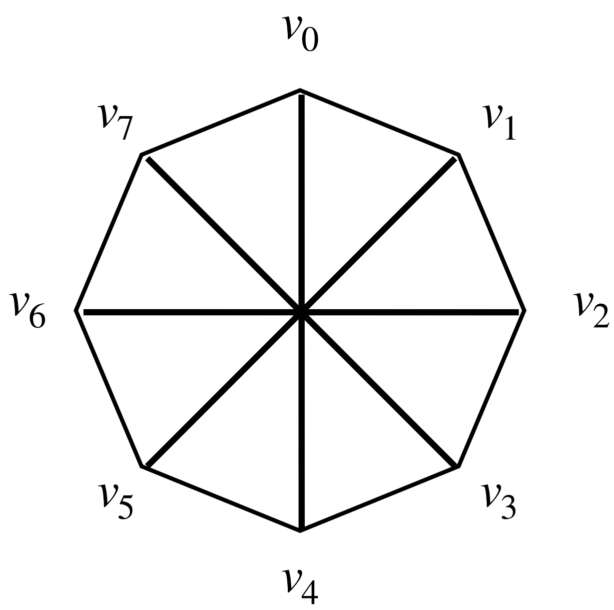
CHSH Inequality Graph



$$S(G_{CHSH}) = p(0,0|0,1) + p(1,1|1,2) + p(0,1|2,3) + p(0,0|3,0) + \\ + p(1,1|0,1) + p(0,0|1,2) + p(1,0|2,3) + p(1,1|3,0) \leq 3$$

Independence Number
Non-contextual Bound
Classical Reasoning
Classical Probability Theory

CHSH Inequality Graph



$$S(G_{CHSH}) = p(v_0) + p(v_1) + p(v_2) + p(v_3) + p(v_4) + p(v_5) + p(v_6) + p(v_7) \leq 3$$

Independence Number
Non-contextual Bound
Classical Reasoning
Classical Probability Theory

CHSH Inequality Graph

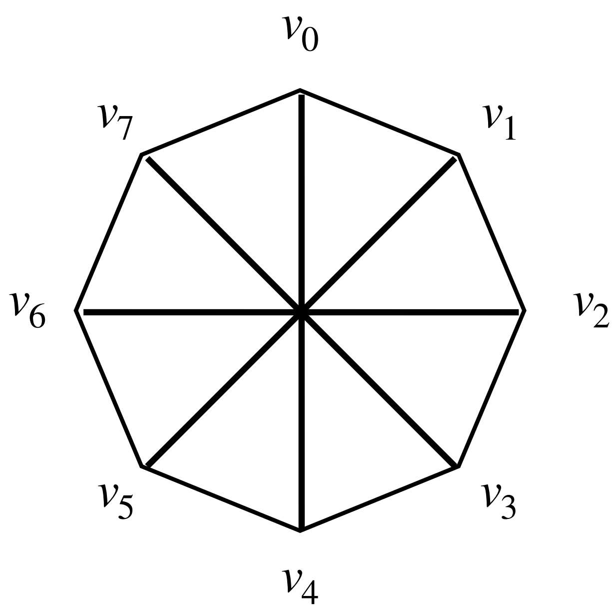
Orthogonal Representation

$$v_i \mapsto \mathbf{v}_i$$

$$v_i \sim v_j \Rightarrow \mathbf{v}_i \cdot \mathbf{v}_j = 0$$

Handle ψ

$$p(v_i) = |\mathbf{v}_i \cdot \psi|^2$$



$$S(G_{CHSH}) = p(v_0) + p(v_1) + p(v_2) + p(v_3) + \\ + p(v_4) + p(v_5) + p(v_6) + p(v_7) \leq \vartheta(G_{CHSH}) = 2 + \sqrt{2}$$

Lovász theta Number

Quantum Bound (for the graph)

Quantum Reasoning

Born's Rule

CHSH Inequality Graph

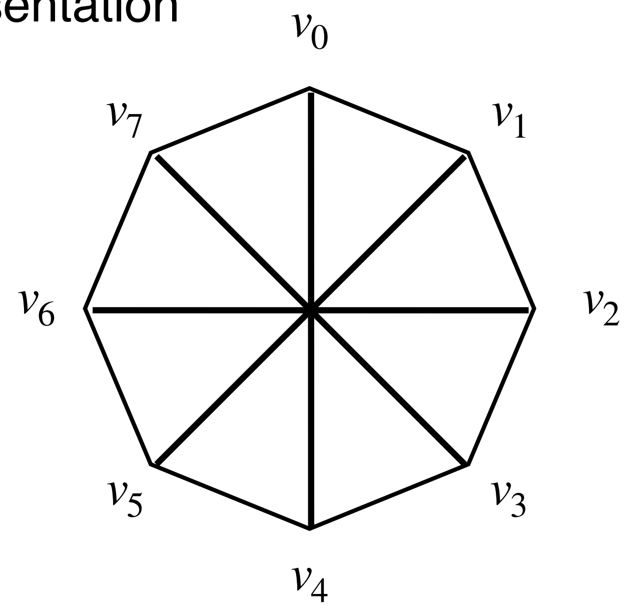
Orthogonal Projective Representation

$$v_i \mapsto \mathbf{\Pi}_i$$

$$v_i \sim v_j \Rightarrow \mathbf{\Pi}_i \mathbf{\Pi}_j = 0$$

Handle ψ

$$p(v_i) = \left\| \mathbf{\Pi}_i \psi \right\|^2$$



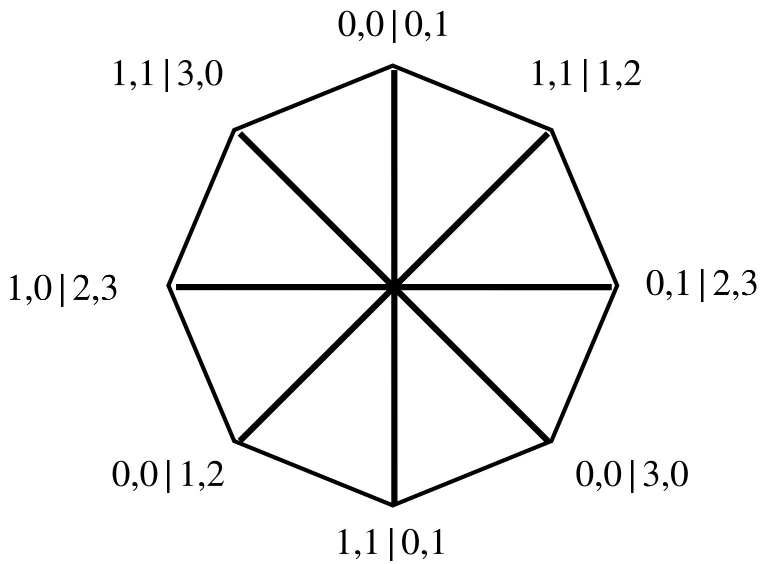
$$S(G_{CHSH}) = p(v_0) + p(v_1) + p(v_2) + p(v_3) + \\ + p(v_4) + p(v_5) + p(v_6) + p(v_7) \leq \vartheta(G_{CHSH}) = 2 + \sqrt{2}$$

Lovász theta Number
Quantum Bound (for the graph)
Quantum Reasoning
Born's Rule

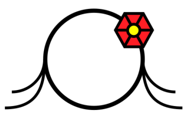
Rabelo, Duarte, López-Tarrida, Terra Cunha, Cabello, *J. Phys. A: Math. Theor.* **47**, 424021

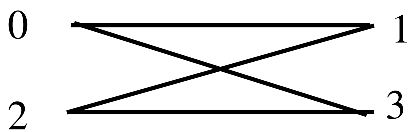
CHSH Inequality Graph

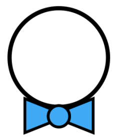
Where are Alice and Bob?



The Compatibility Graph



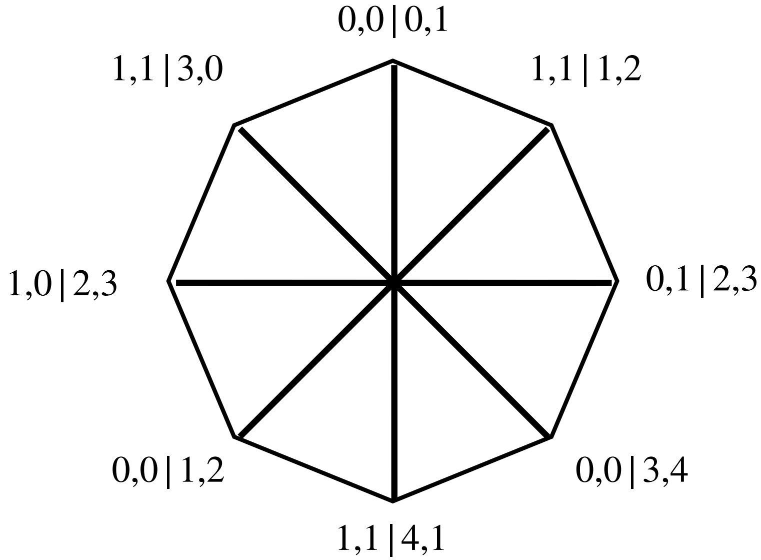




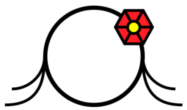
It allows for them!

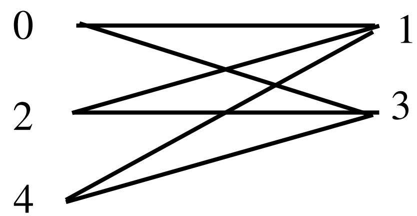
CHSH Inequality Graph

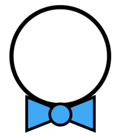
Another Bell labelling



The Compatibility Graph







It allows for them!

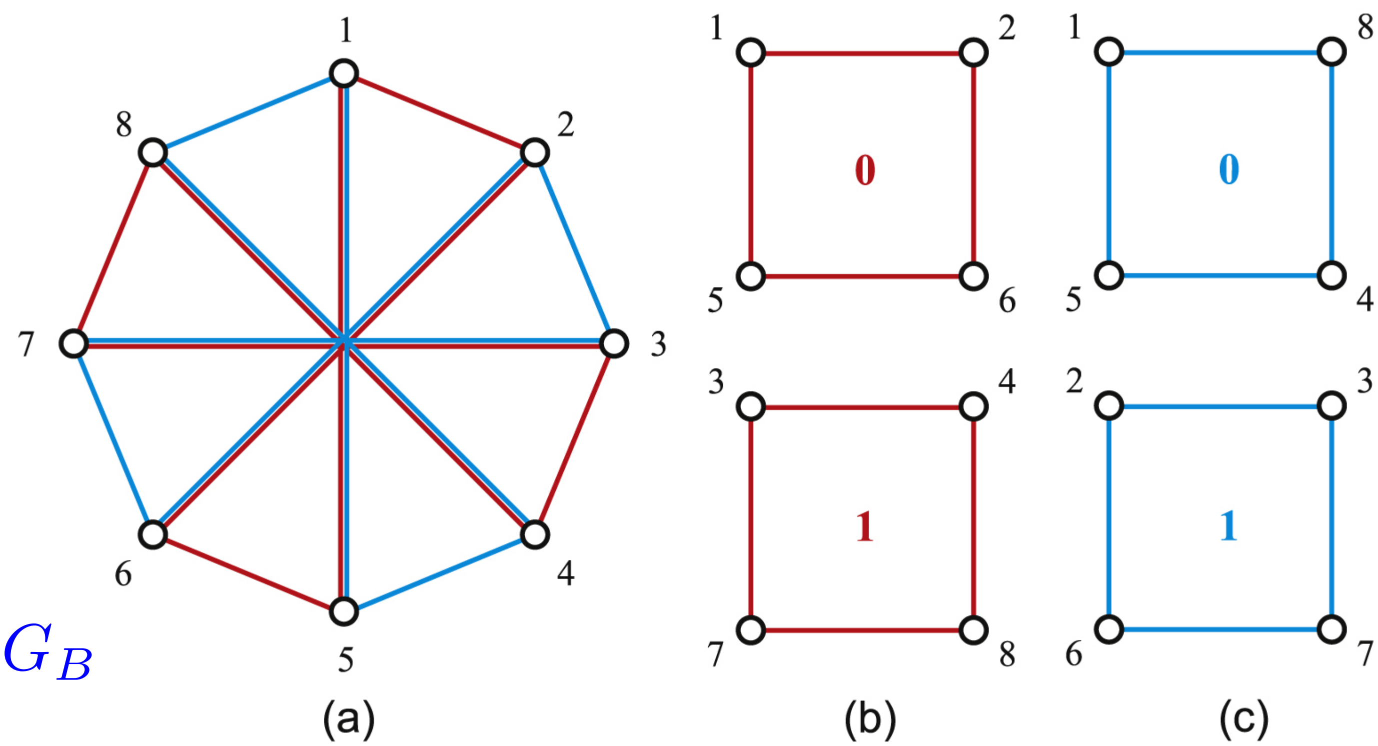
How can we demand their presence?


Rafael Rabelo
Cristhiano Duarte
Antonio López Tarrida
(a)

CHSH Coloured Graph

Figure 1 from Rafael Rabelo et al 2014 J. Phys. A: Math. Theor. 47 424021

1	00 | 00
2	11 | 01
3	10 | 11
4	00 | 10
5	11 | 00
6	00 | 01
7	01 | 11
8	11 | 10



$$G = G_A \sqcup G_B$$

$$S(G) = p(00|00) + p(00|01) + p(00|10) + p(01|11) + \\ + p(11|00) + p(11|01) + p(11|10) + p(10|11) \leq 3$$

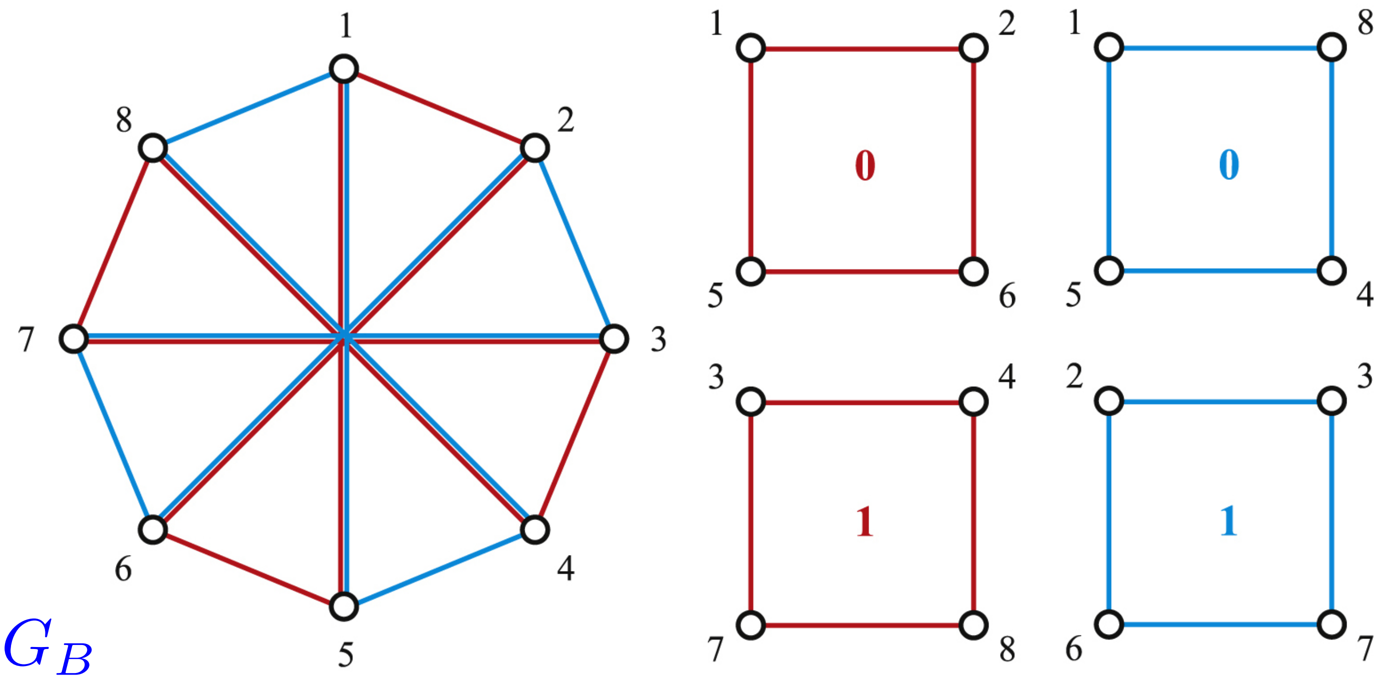
CHSH inequality is essentially encoded on the coloured graph!

Rabelo, Duarte, López-Tarrida, Terra Cunha, Cabello, *J. Phys. A: Math. Theor.* **47**, 424021

CHSH Coloured Graph

1	00 I 00
2	11 I 01
3	10 I 11
4	00 I 10
5	11 I 00
6	00 I 01
7	01 I 11
8	11 I 10

Figure 1 from Rafael Rabelo et al 2014 J. Phys. A: Math. Theor. 47 424021



(a) (b) (c)

$$\mathrm{G} = {\color{red}G_A} \sqcup {\color{blue}G_B}$$

Orthogonal Projective Representation

$$v \mapsto \mathbf{\Pi}_v^A \otimes \mathbf{\Pi}_v^B$$

$$v \sim_X u \Rightarrow \mathbf{\Pi}_v^X \mathbf{\Pi}_u^X = 0$$

Handle ψ

$$p(v) = \left\| \mathbf{\Pi}_v^A \otimes \mathbf{\Pi}_v^B \psi \right\|^2$$

Rabelo, Duarte, López-Tarrida, Terra Cunha, Cabello, *J. Phys. A: Math. Theor.* **47**, 424021

2-Coloured Lovász Number

$\mathrm{G} = {\color{red}G_A} \sqcup {\color{blue}G_B}$

$$\theta(\mathrm{G}) = \sup_{\mathrm{OPR},\psi} \sum_{v \in V(\mathrm{G})} p(v)$$

Thanks to Tensor Product…

Not a Semi Definite Program (SDP)

A Hierarchy of SDPs, like in NPA

Orthogonal Projective Representation

$$v \mapsto \mathbf{\Pi}_v^A \otimes \mathbf{\Pi}_v^B$$

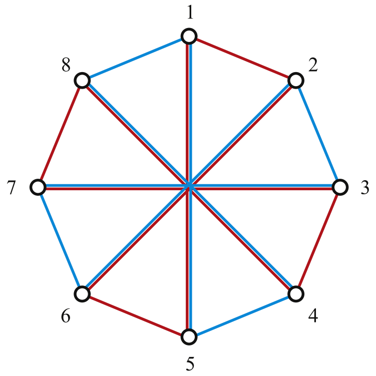
$$v \sim_X u \Rightarrow \mathbf{\Pi}_v^X \mathbf{\Pi}_u^X = 0$$

Handle ψ

$$p(v) = \left\| \mathbf{\Pi}_v^A \otimes \mathbf{\Pi}_v^B \psi \right\|^2$$

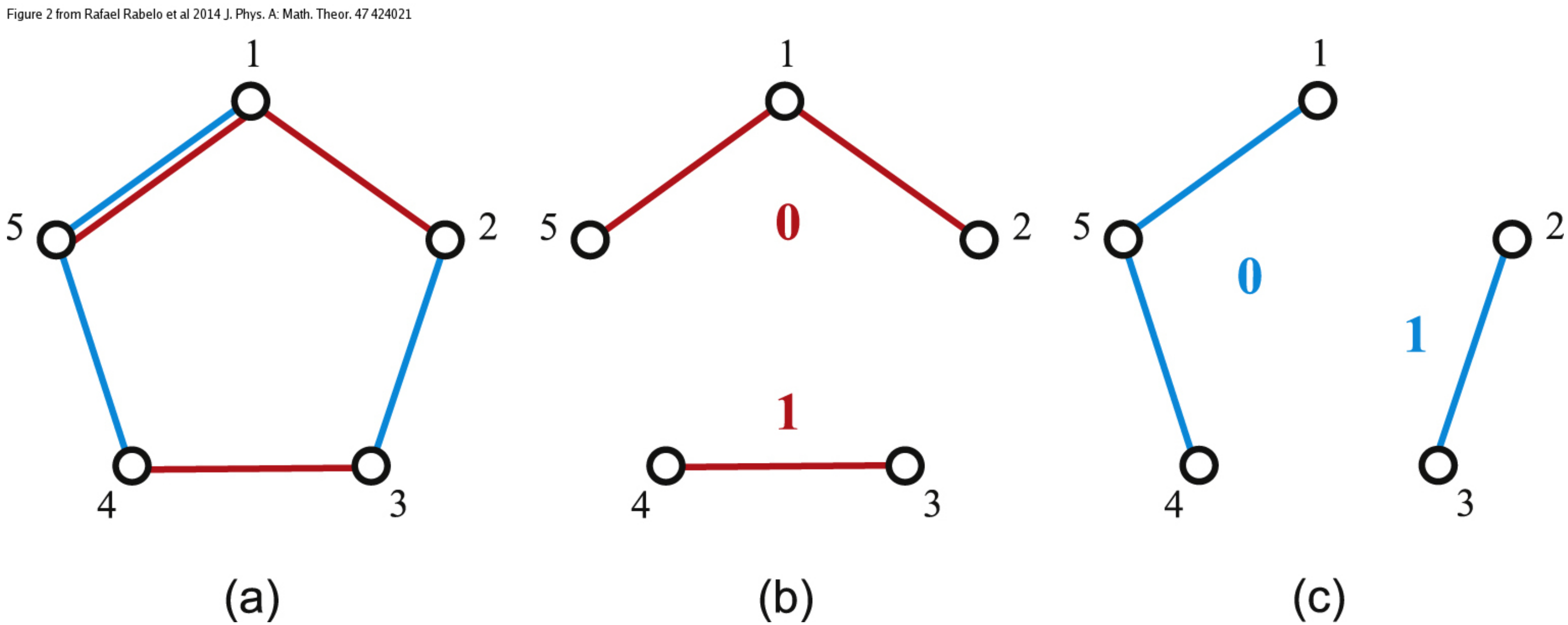
Navascués, Pironi, Acín, *Phys. Rev. Lett.* **98**, 010401; *New J. Phys.* **10**, 073013

Rabelo, Duarte, López-Tarrida, Terra Cunha, Cabello, *J. Phys. A: Math. Theor.* **47**, 424021



Pentagonal Bell Inequalities

$\theta \approx 2.178$



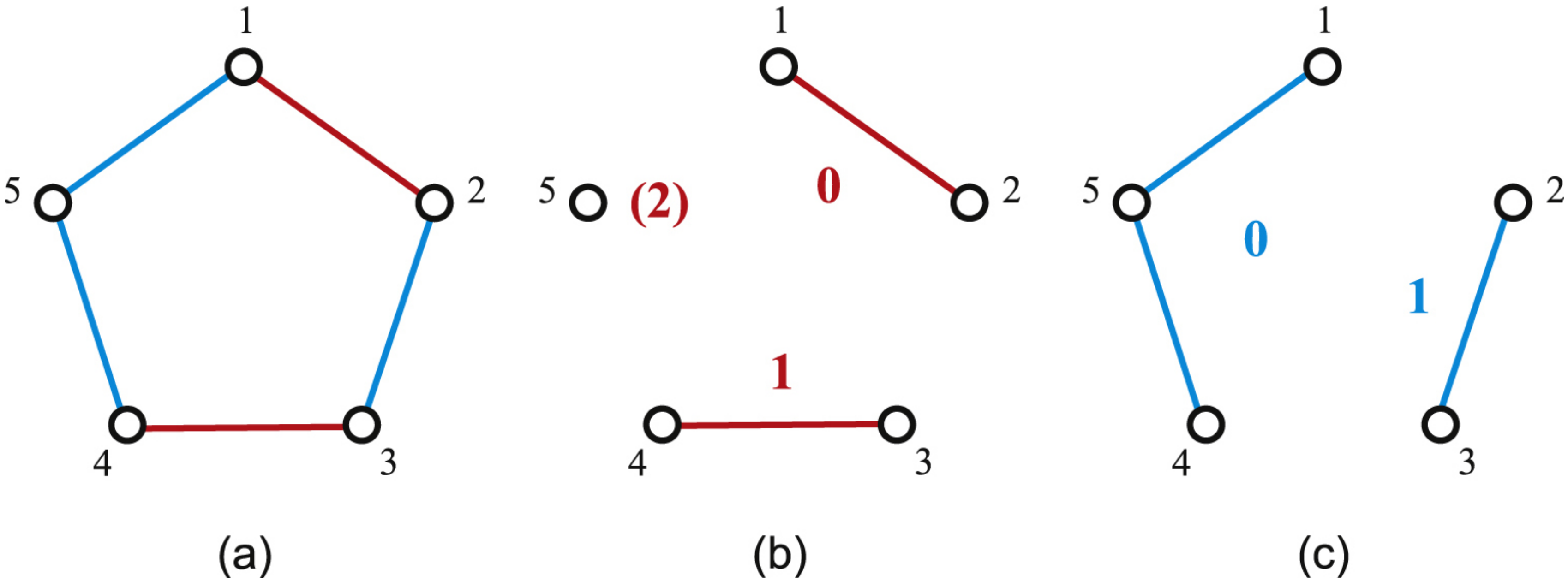
$$I_1^{\mathrm{P}} = P(00|00) + P(11|01) + P(10|11) + P(00|10) + P(11|00) \overset{\mathrm{LHV}}{\leqslant} 2,$$

Rabelo, Duarte, López-Tarrida, Terra Cunha, Cabello, *J. Phys. A: Math. Theor.* **47**, 424021

Pentagonal Bell Inequalities

$\theta \approx 2.207$

Figure 3 from Rafael Rabelo et al 2014 J. Phys. A: Math. Theor. 47 424021



$$I_2^{\mathrm{P}} = P(00|00) + P(11|01) + P(10|11) + P(00|10) + P(\_1|\_0) \overset{\mathrm{LHV}}{\leqslant} 2,$$

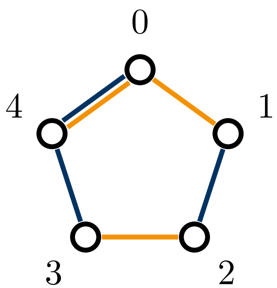
$$I_3^{\mathrm{P}} = P(00|00) + P(11|01) + P(10|11) + P(00|10) + P(11|20) \overset{\mathrm{LHV}}{\leqslant} 2,$$

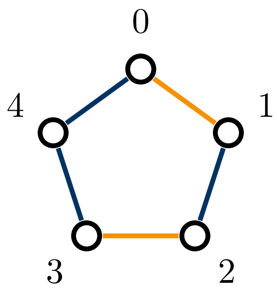
Rabelo, Duarte, López-Tarrida, Terra Cunha, Cabello, *J. Phys. A: Math. Theor.* **47**, 424021

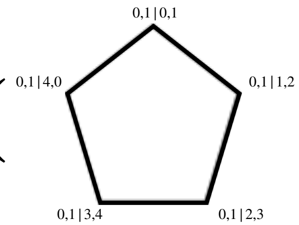
Refreshing

- CSW only sees exclusivities
- Coloured graphs characterise which part sees each exclusivity
- More restrictions, lower upper bounds

For Pentagons

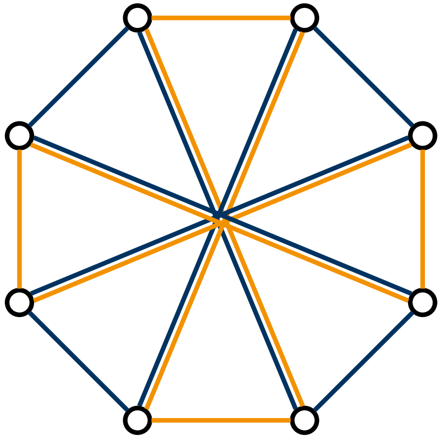
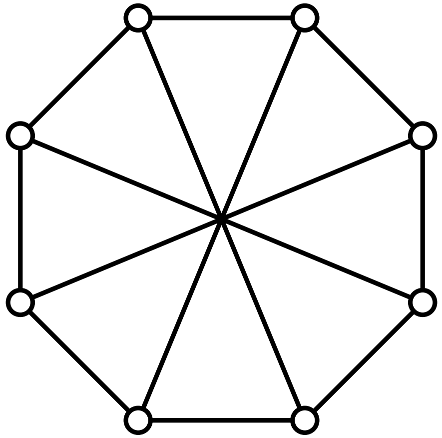
$\theta($ 

$) < \theta($ 

$) < \vartheta($ 

$)$

For CHSH

$\theta($  $) = \vartheta($  $)$

Beyond Numbers

Given G

Orthogonal Projective Representation

$$v \mapsto \mathbf{\Pi}_v$$

$$v \sim u \Rightarrow \mathbf{\Pi}_v \mathbf{\Pi}_u = 0$$

Handle ψ

$$p(v) = \left\| \mathbf{\Pi}_v \psi \right\|^2$$

Given $\mathscr{G}$

Orthogonal Projective Representation

$$v \mapsto \mathbf{\Pi}_v^A \otimes \mathbf{\Pi}_v^B$$

$$v \sim_X u \Rightarrow \mathbf{\Pi}_v^X \mathbf{\Pi}_u^X = 0$$

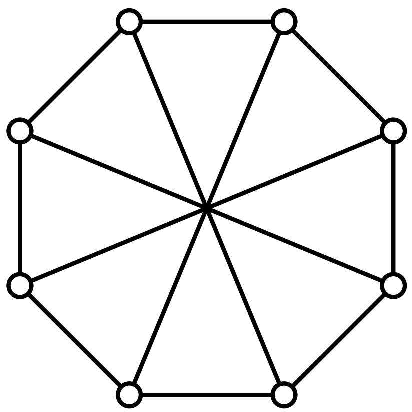
Handle ψ

$$p(v_i) = \left\| \mathbf{\Pi}_i^A \otimes \mathbf{\Pi}_i^B \psi \right\|^2$$

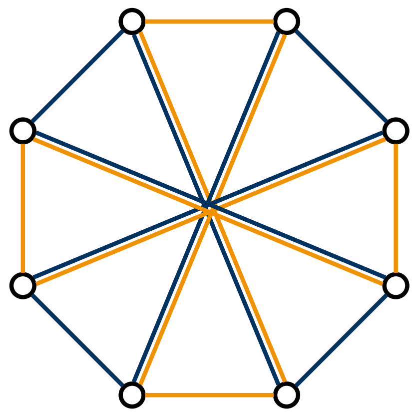
$$THETA\,(G) = \left\{ \big(p\,(v)\big), v \in V\,(G) \right\} \quad \supseteq \quad cTHETA\,\left(\mathscr{G}\right) = \left\{ \big(p\,(v)\big), v \in V\left(\mathscr{G}\right) \right\}$$

Quantum Sets

In the case of CHSH



CHSH NC-Inequality



CHSH Bell-Inequality

$$THETA(G) = \left\{ \left(p(v)\right), v \in V(G) \right\} \quad \supseteq \quad cTHETA\left(\mathscr{G}\right) = \left\{ \left(p(v)\right), v \in V\left(\mathscr{G}\right) \right\}$$

Can they be equal??


Lina Vandré

More than one inequality…

Weighted graphs: (G, ω) or $(\mathscr{G}, \omega)$

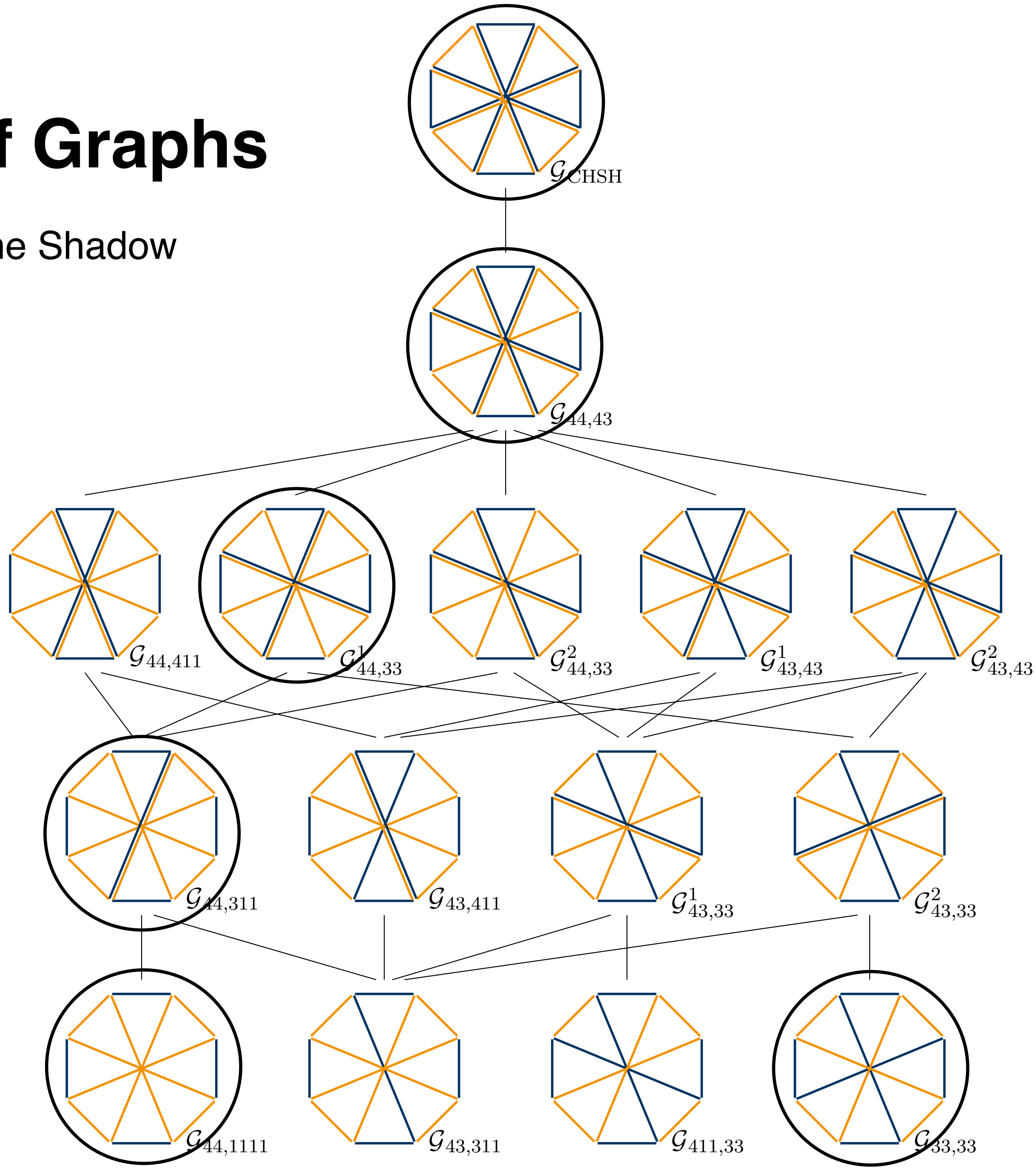
$$\omega : v \mapsto \omega_v \geq 0$$

Each weight vector, a new inequality $\omega \cdot p = \sum_{v \in V(G)} \omega_v p_v \leq \alpha(G, \omega)$

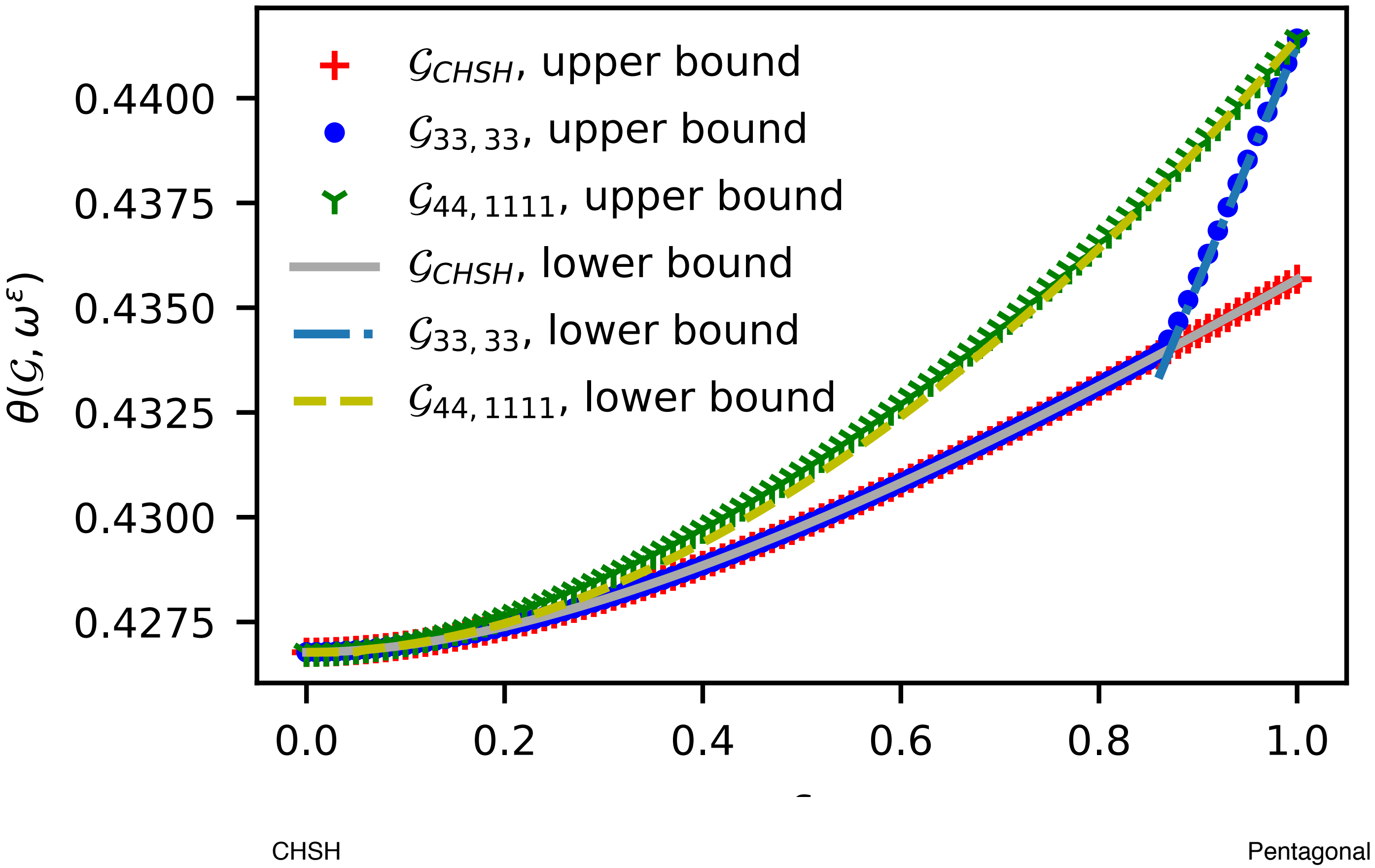
Each weight vector, a different direction in $THETA(G)$ or $cTHETA(\mathscr{G})$

A Family of Graphs

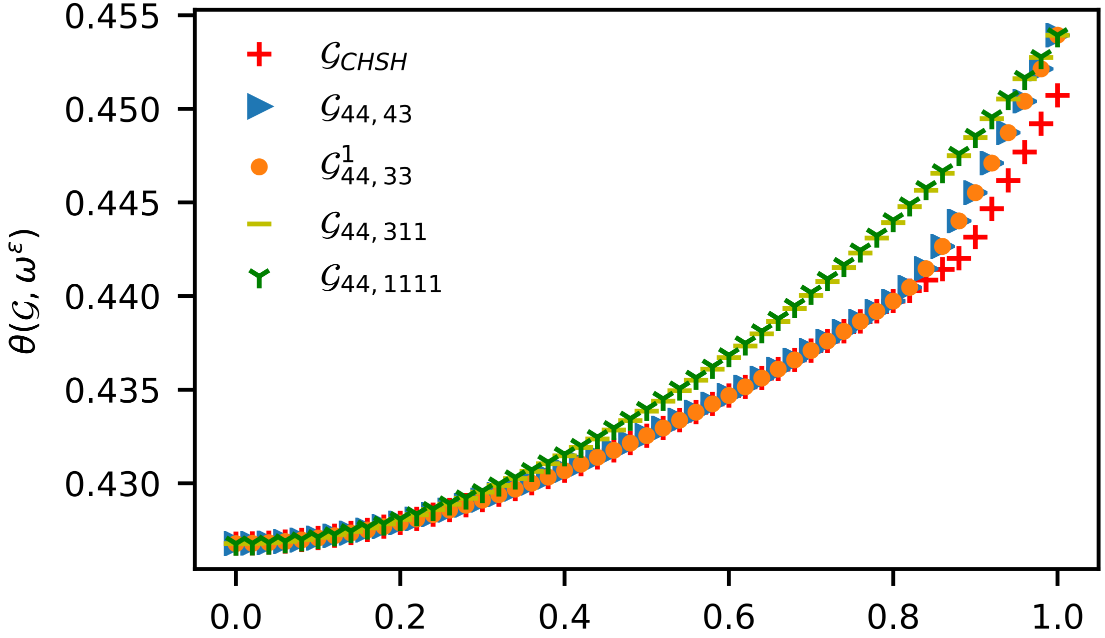
With the same Shadow



Clearly not equal



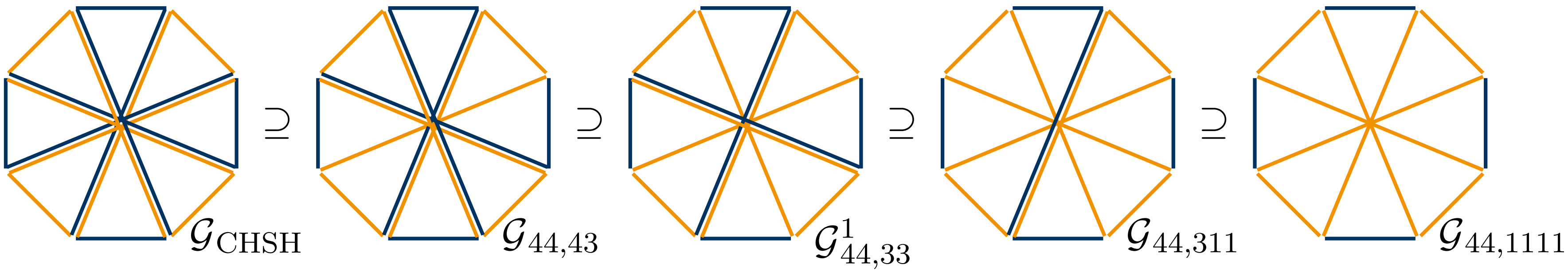
More than this



Some edge removing may not affect (these directions) coloured theta…

CHSH

Randomised Pentagonal



Wrap up

- We show that the Quantum Set of the Shadow of CHSH is strictly larger than the Quantum Set of original CHSH
- There are many coloured graphs in between CHSH and its shadow
  - Are there as many different quantum sets?
  - Or quantum sets of different coloured graphs can coincide?

Open Questions

- Is there a graph characterisation of (possibly) removable edges?
- Is it true that $\mathcal{Q}_{33,33} \subset \mathcal{Q}_{44,1111}$?
  - Why??
- To complete the "quasi-empirical"* version of here shown results

* in Svozil terms, earlier

Thank you, organisers
and participants!