Quantum Contextuality on Quantum Measurements and Beyond (QCQMB) 2021

Emergence of noncontextuality and classical limits

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With: Rafael Wagner, Cristhiano Duarte, Bárbara Amaral, Marcelo Terra Cunha

Arxiv 2104.05734

With: Marcelo Terra Cunha

Phys. Rev A 102, 052226(2020)/Arxiv 1811.00615

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The problem

Obs 1 Contextuality is a **nonclassical** feature

The problem

Obs 1 Contextuality is a **nonclassical** feature

Obs 2 Our everyday experience is **classical**

The problem

Obs 1 Contextuality is a **nonclassical** feature

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Obs 3 Quantum Theory is contextual

The problem

Contextuality is a **nonclassical** feature

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Classical limits must kill quantum contextuality

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Contextuality is a **nonclassical** feature

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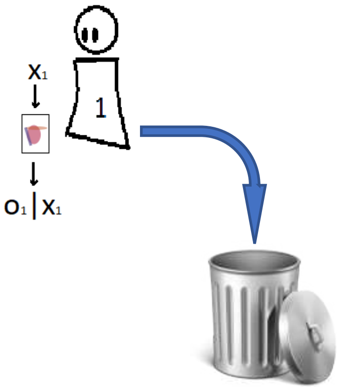
Classical limits must kill quantum contextuality

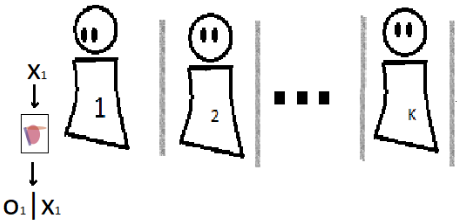
Does contextuality fade out under classical limits?

Outline

Part I

We analyze the best quantum realization for the odd N-cycle inequalities in **multiple** observers setup





Discuss the results as emergence of noncontextuality on open quantum systems

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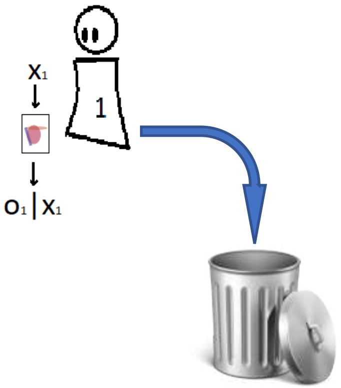
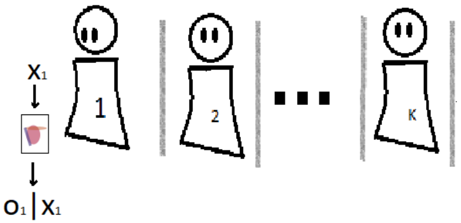
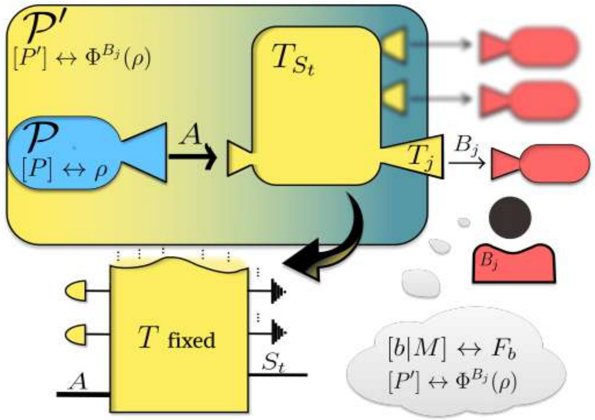




Discuss the results as emergence of noncontextuality on open quantum systems

Part II

Noncontextuality emerges under quantum Darwinism processes



Part I

Part I requests

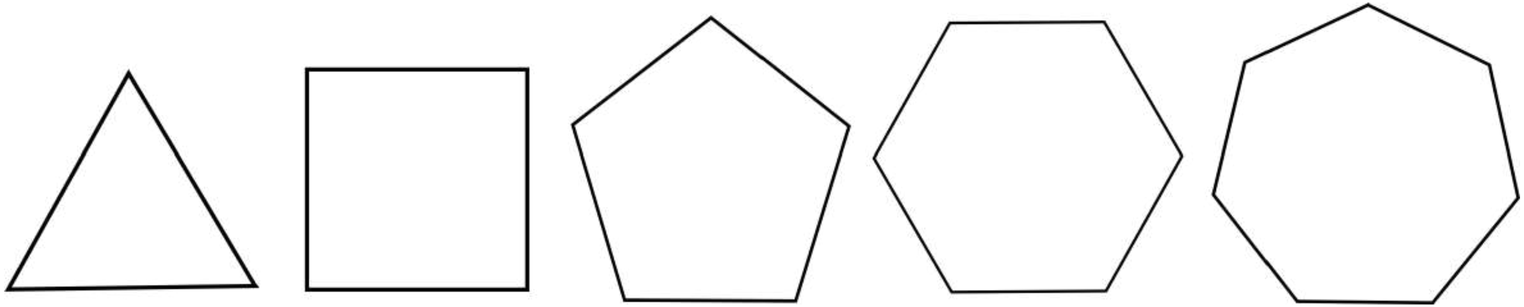
We need to set our ground:

N-cycle inequalities

Contextuality: N-cycle scenarios

- Simple compatibility scenarios.

$$\Gamma \equiv \{X, \mathcal{C}, O\}$$



$$\mathcal{C} = \{\{A_0, A_1\}, \{A_1, A_2\}, \ldots, \{A_{N-1}, A_0\}\}$$

M. Araújo et al, Phys. **Rev. A 88 022118 (2013)**

Contextuality: N-cycle scenarios

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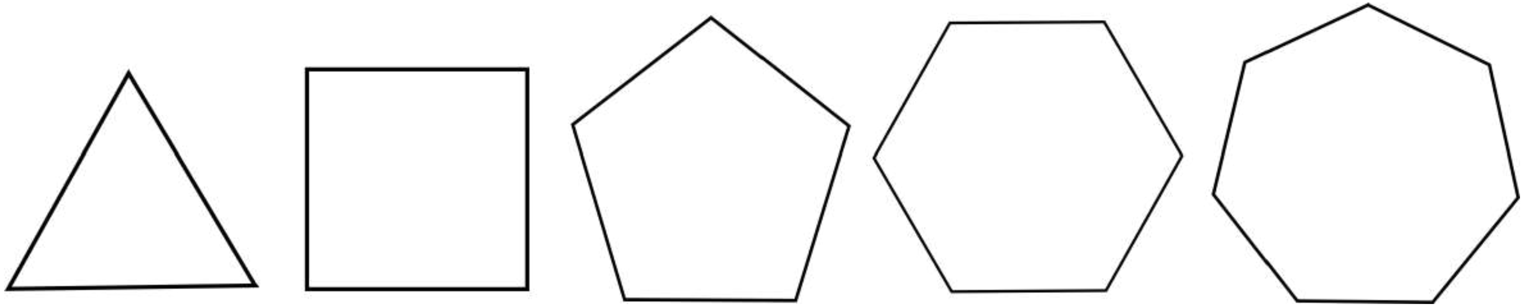
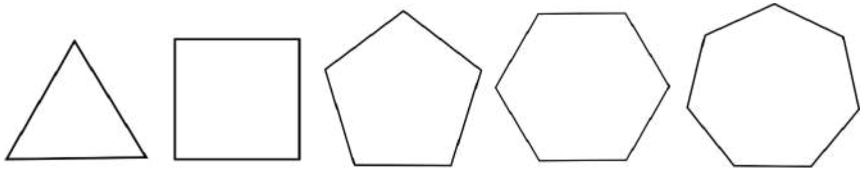


Scenarios with odd N cannot be split into 2 parties!

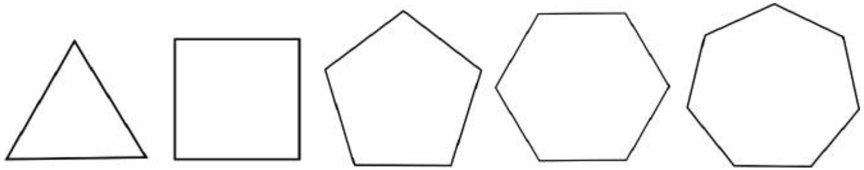
Contextuality: N-cycle scenarios



Odd N $o_i \in \{-1, 1\}$

$(+1, +1)$ $(+1, -1)$ $(-1, +1)$ $(-1, -1)$

Contextuality: N-cycle scenarios



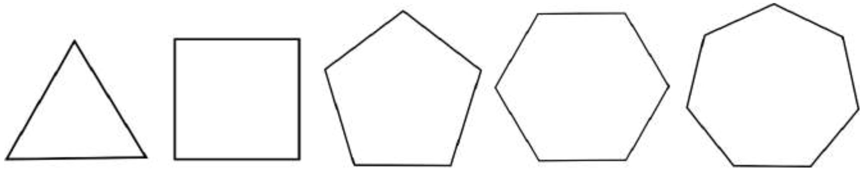
Odd N $o_i \in \{-1, 1\}$

$$(+1,+1)\ (+1,-1)\ (-1,+1)\ (-1,-1)$$

Inequality

$$\sum_{i=0}^{N-1} \langle A_i A_{i+1} \rangle \geq -N+2$$

Contextuality: N-cycle scenarios



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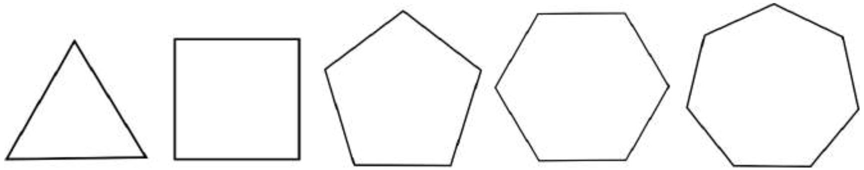
Extra assumption

Exclusiveness:

$$p(-1,-1|i, i \pm 1) = 0$$

A. Cabello, S. Severini, A. Winter, **arXiv:1010.2163**

Contextuality: N-cycle scenarios



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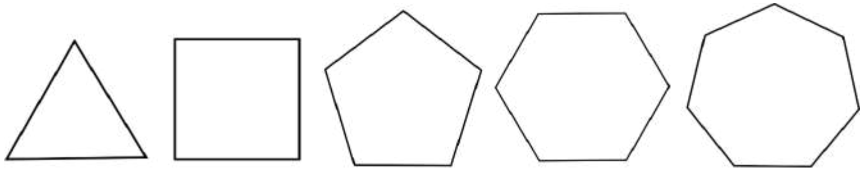
Exclusiveness

$$\sum_i p(1,1|i,i+1) \geq 1$$

$$\sum_i p(-1|i) \leq \frac{N-1}{2}$$

Contextuality: N-cycle scenarios



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Exclusiveness

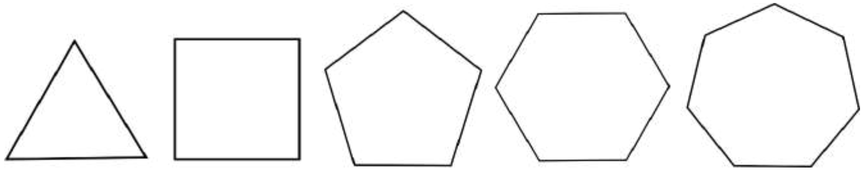
b_i

$$\sum_i p(1,1|i,i+1) \geq 1$$

$$\sum_i p(-1|i) \leq \frac{N-1}{2}$$

a_i

Contextuality: N-cycle scenarios



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$$\sum_i p(b_i) \geq 1$$

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Contextuality: N-cycle scenarios

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Measurement protocols

1. $\mathcal{M}_i = \{a_i, b_i, a_{i+1}\}_i$
2. $\mathcal{M}_i^a = \{a_i, \neg a_i\}_i$
3. $\mathcal{M}_i^b = \{b_i, \neg b_i\}_i,$

Contextuality: N-cycle scenarios

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Usually equivalent!

Quantum realizations

$$\sum_i p(b_i) \geq 1$$

$$\sum_i p(a_i) \leq \frac{N-1}{2}$$

- Hilbert space of dimension 3

$$A_i = \mathcal{I} - 2|a_i\rangle\langle a_i|$$

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- Hilbert space of dimension 3

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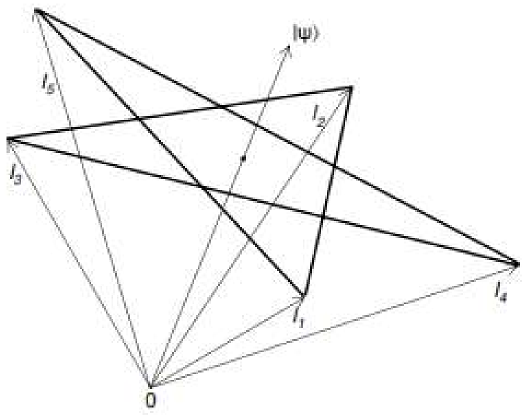
$$\langle a_i|a_{i+1}\rangle = 0$$

Exclusivity $\surd$



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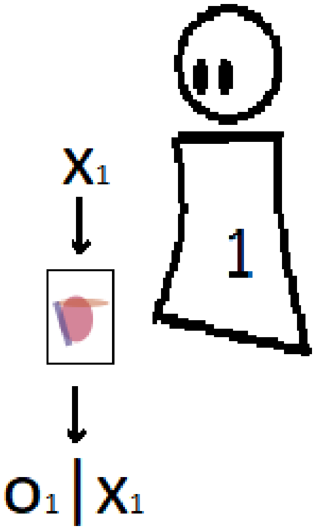
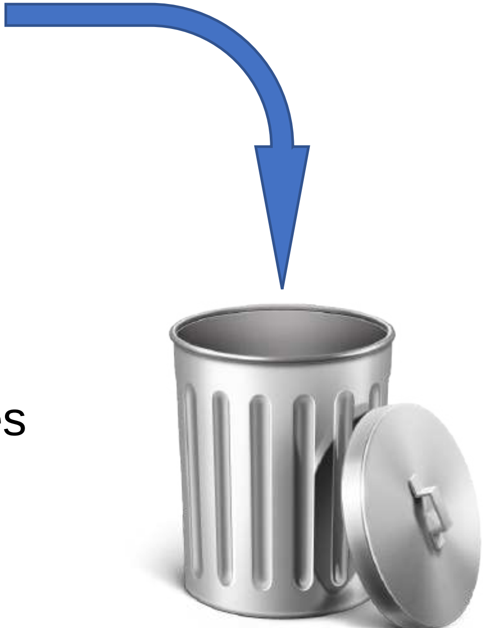


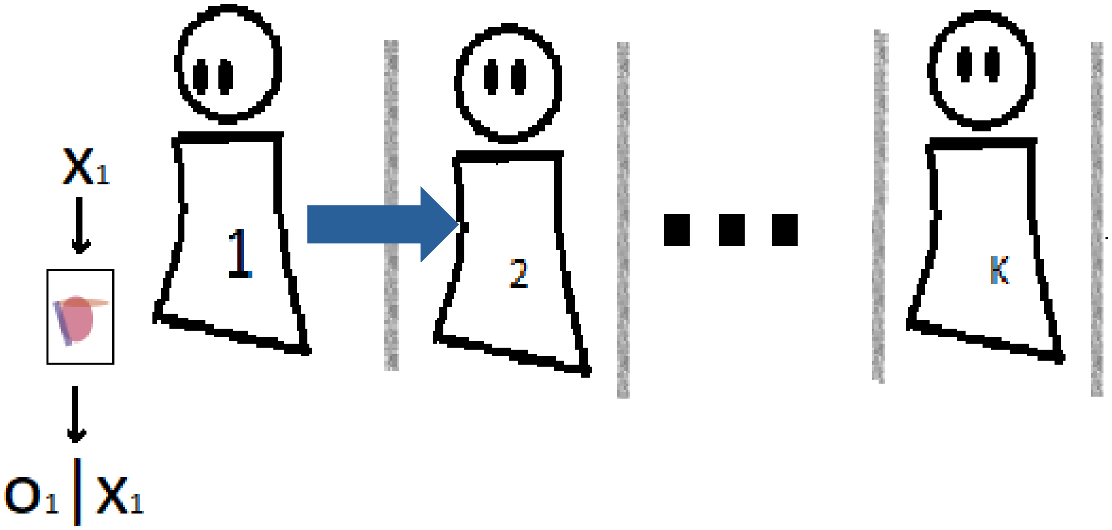


Usual way to test the inequalities

Sequential observers

$$\sum_i p(b_i) \geq 1$$

$$\sum_i p(a_i) \leq \frac{N-1}{2}$$



Our multiobserver setup proposal

Sequential observers

$$\sum_i p(b_i) \geq 1$$

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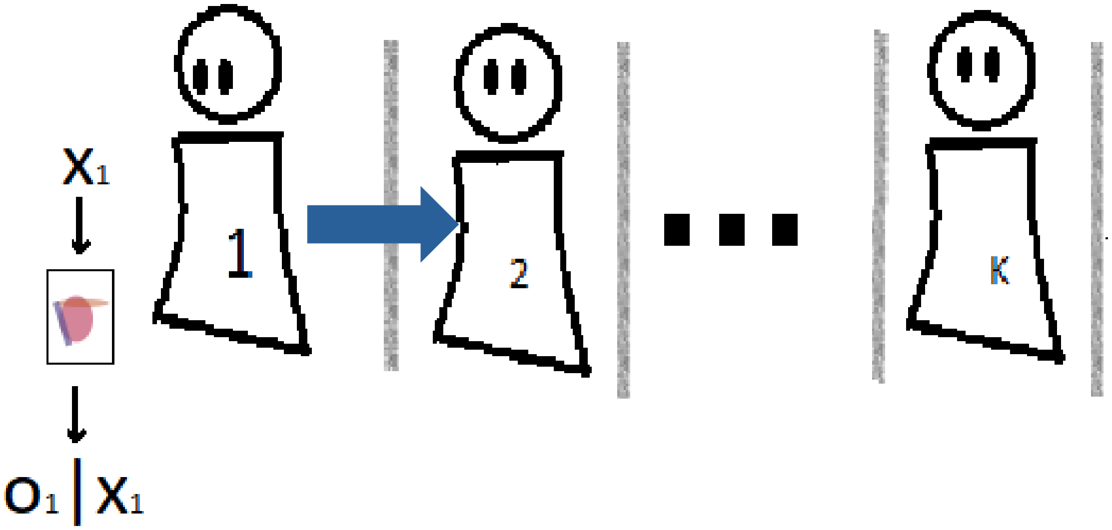
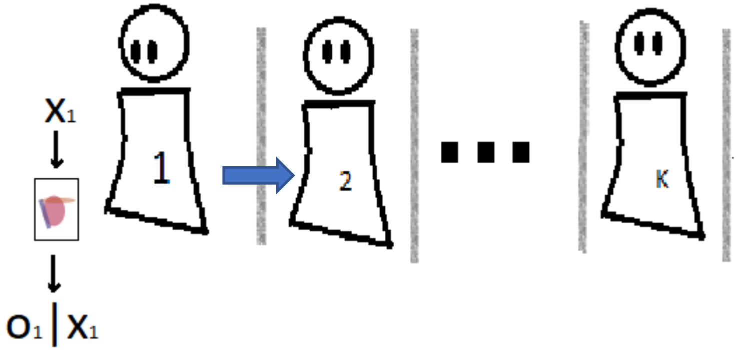


Best quantum realization

Sequential observers



RDB, Marcelo Terra Cunha,
Phys. Rev A 102, 052226(2020)/Arxiv 1811.00615

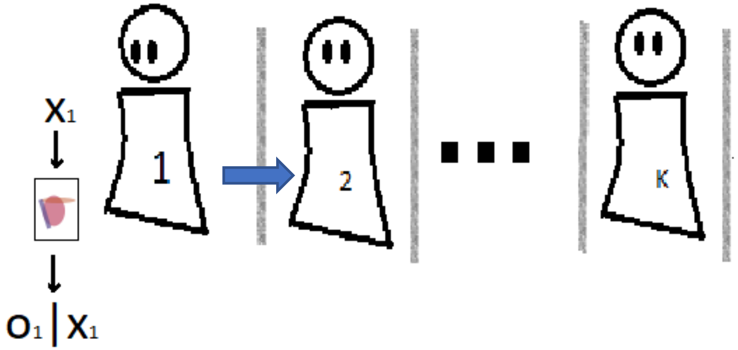
For the odd N-cycle

- Which observers still find violations?
  - -Does this depend on N?
  - -Does this depend on the measurement protocol?

- Already analized, in different setups, for non-locality and steering, but lacking for Contextuality.

S. Sasmal et al, Phys. Rev. A98, 012305 (2018)
R. Silva et al, Phys. Rev. Lett.114, 250401 (2015)
D. Das et al, Phys. Rev. A99, 022305 (2019).

Sequential observers



Here: we focus on the message we take regarding emergence of noncontexuality in open quantum systems

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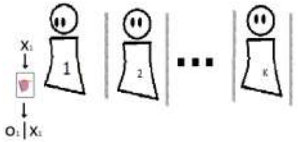
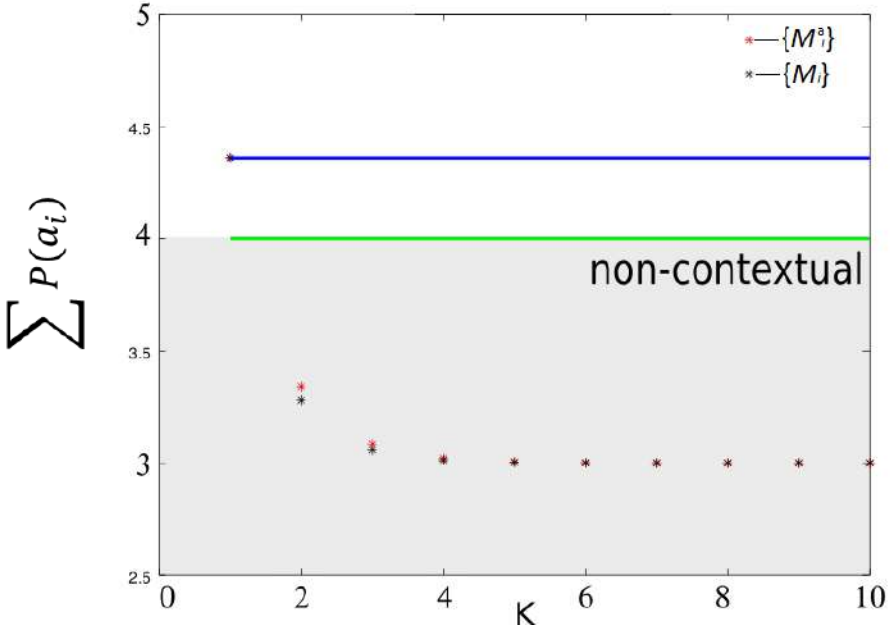
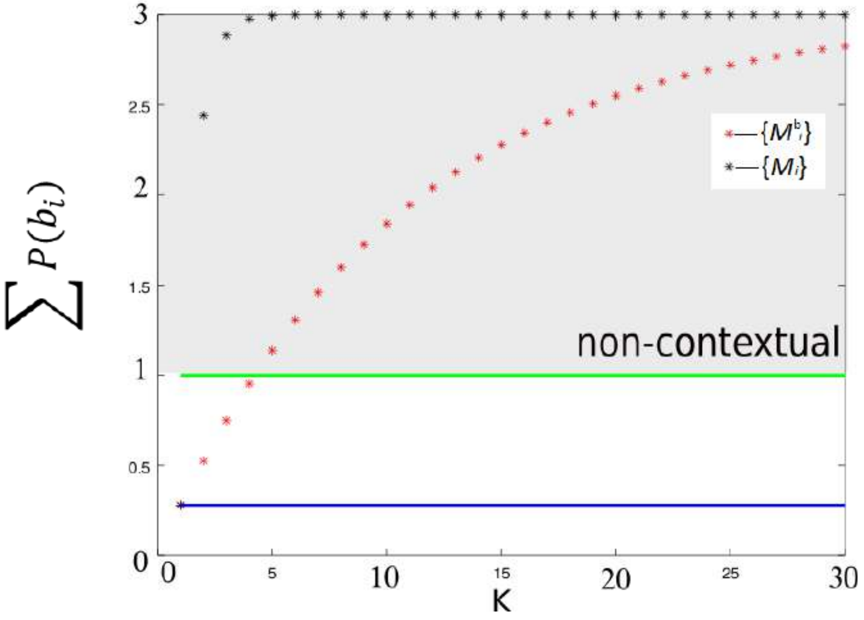
For the odd N-cycle

Results

The typical results (N=9)







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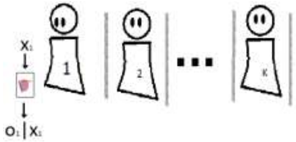

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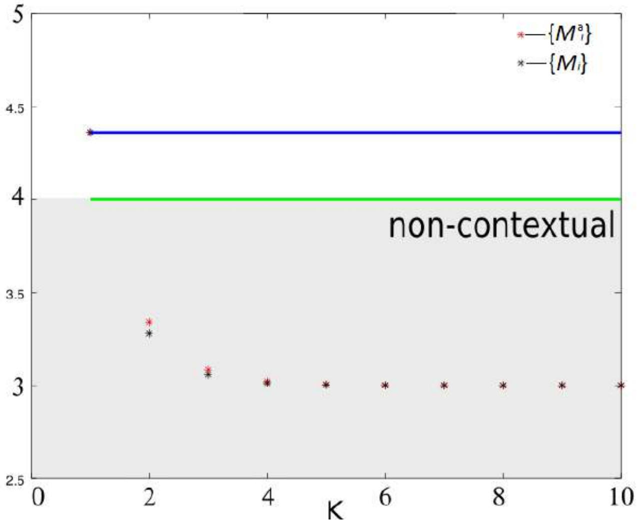

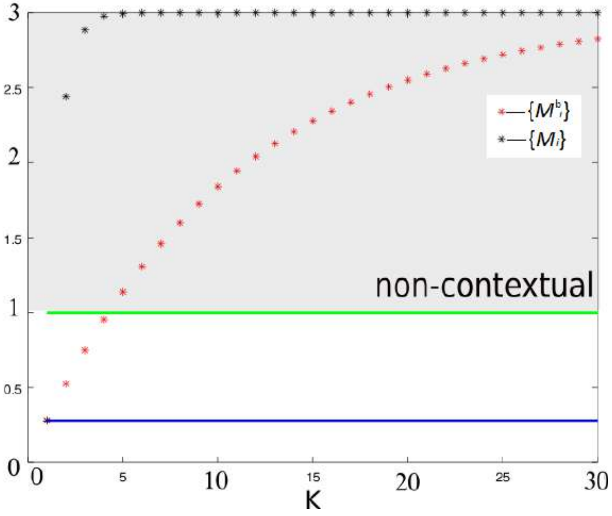

- Assyptotic limit without violation!
- Violation disappears quickly
- Protocols behave differently

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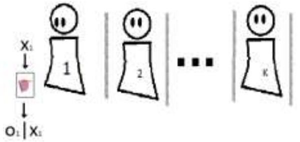

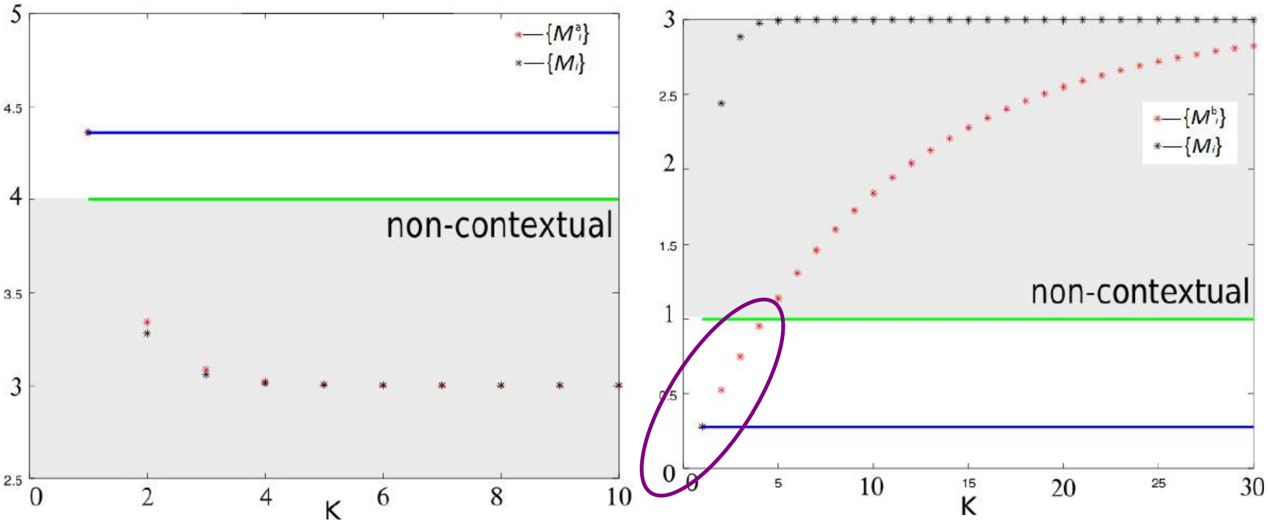

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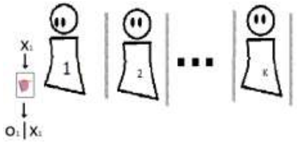

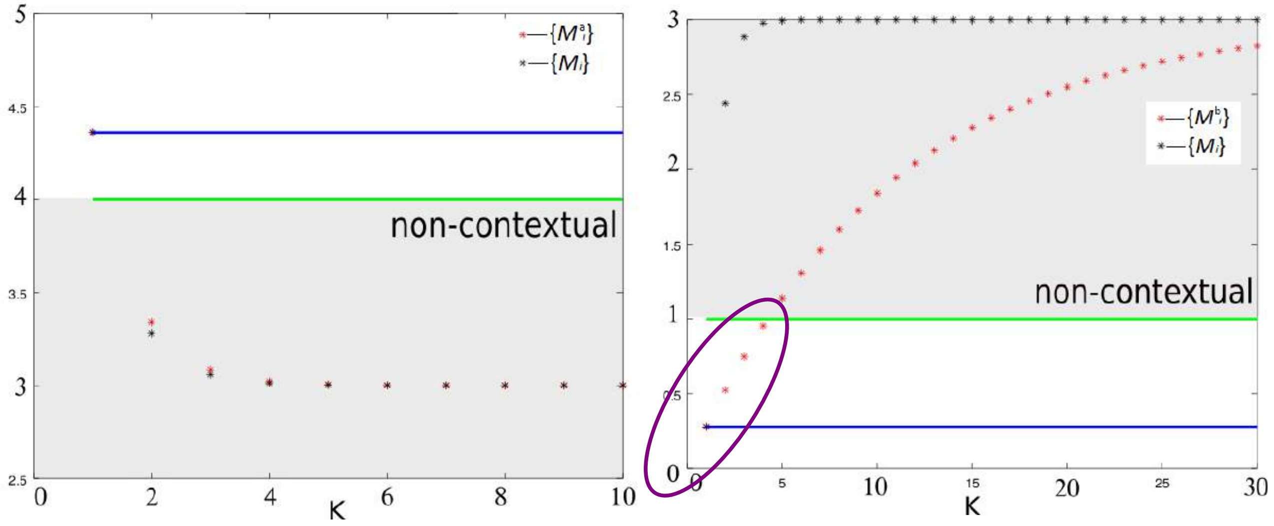

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Results

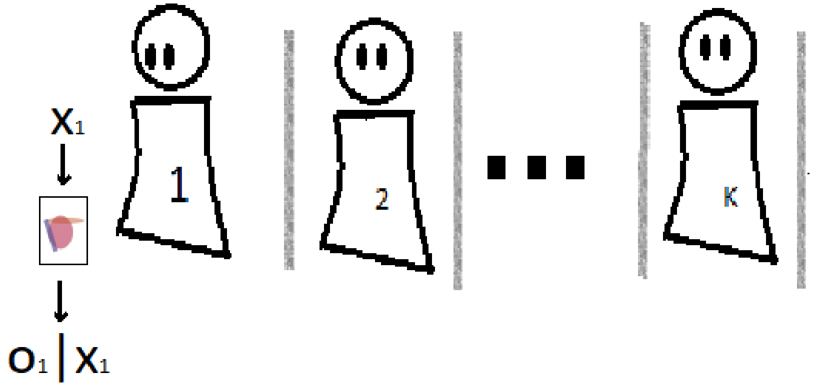
- The behaviour showed for N=9 is qualitatively the same for all N.

- No-violation is always atained for large enough K, tending to the assymptotic limit.

Interpreting as a classical limit

Interpreting as a classical limit

- Multiplayers setup as collisional models and multisystem environment



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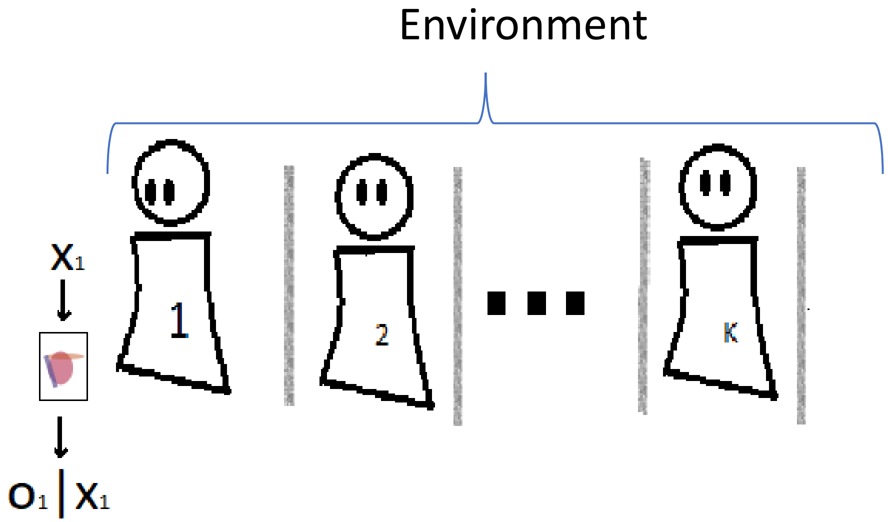

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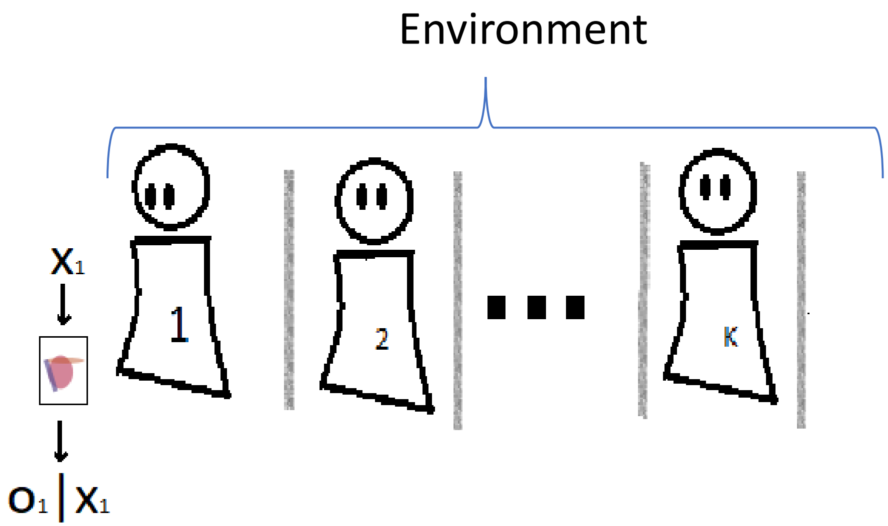

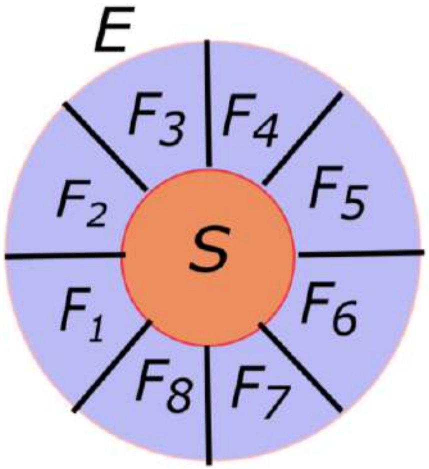

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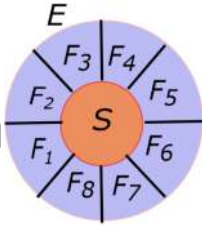

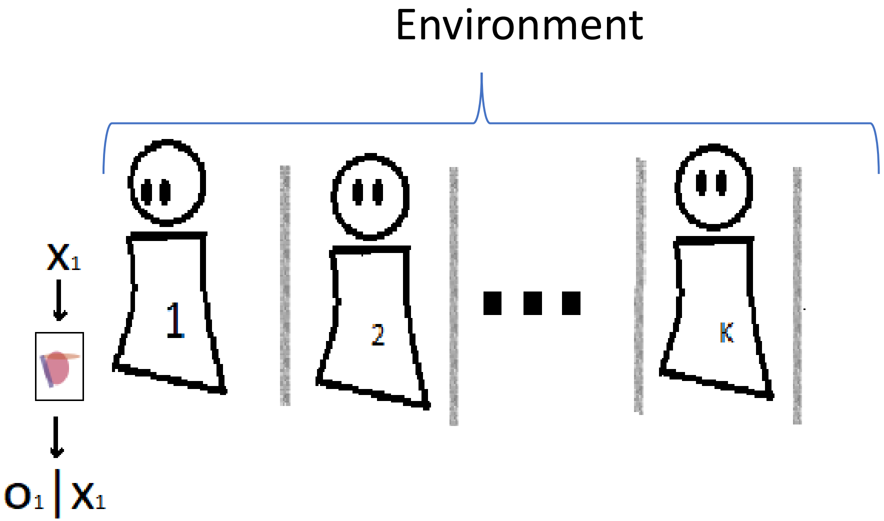

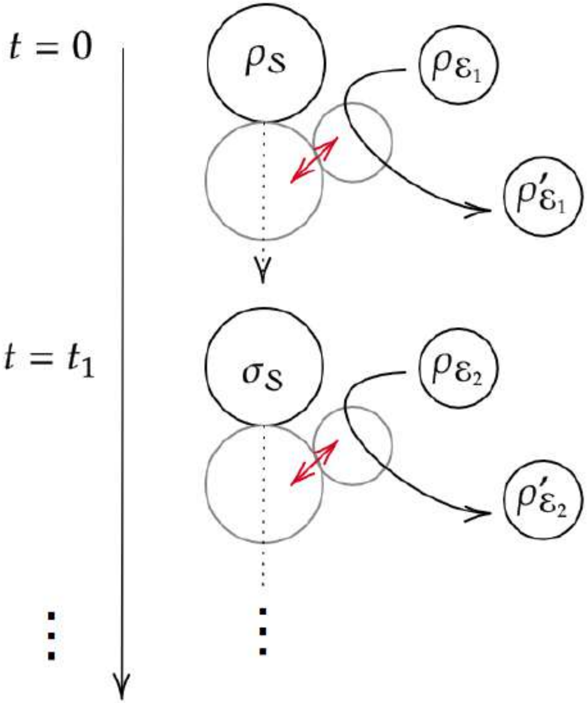

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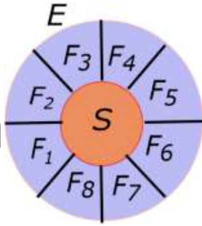

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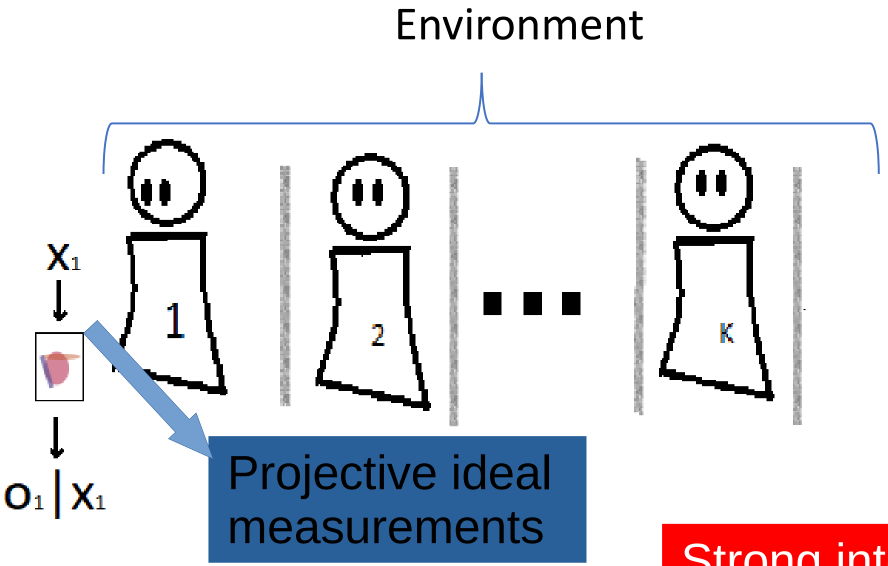

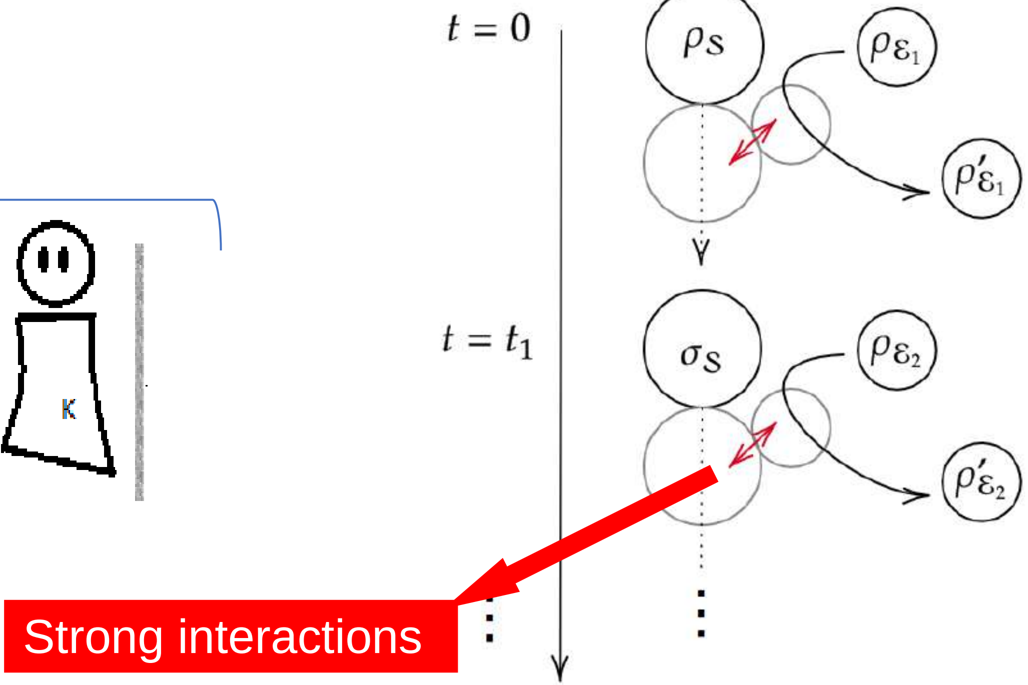

Interpreting as a classical limit

Our results tell us...

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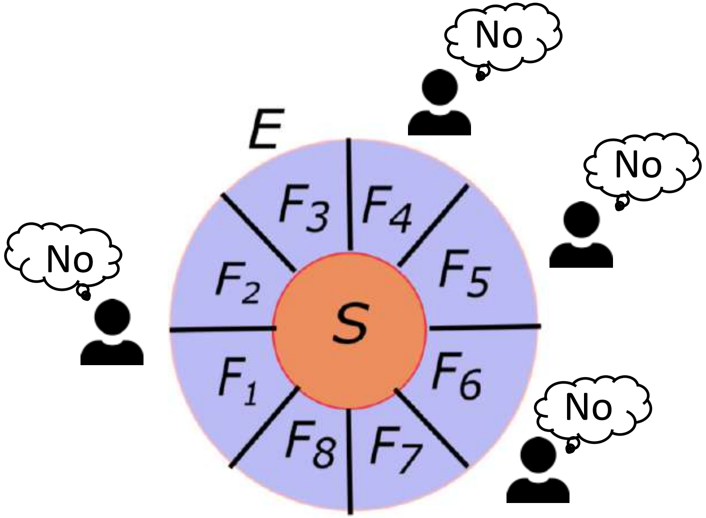
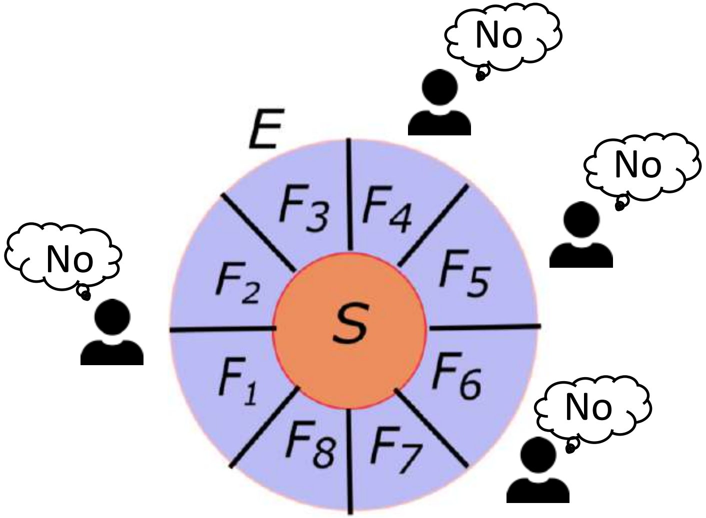


Interpreting as a classical limit

Our results tell us...

Can you witness contextuality?



Dynamics makes contextuality disappear, for all N and large enough ‘environment’.

Interpreting as a classical limit

Take home message:

In this very idealized and special environment, designed to test contextuality, noncontextuality emerges.

Interpreting as a classical limit

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In this very idealized and **special environment**, designed to test contextuality, noncontextuality emerges.

Does noncontextuality emerge in more generic classical limit processes?

Interpreting as a classical limit

Take home message:

In this very idealized and **special environment**, designed to test contextuality, noncontextuality emerges.

Suggestion:

Environment made of several (independent) subsystems

We look into information about a central system stored in these ‘subenvironments’

Interpreting as a classical limit

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In this very idealized and **special environment**, designed to test contextuality, noncontextuality emerges.

Suggestion:

Environment made of several (independent) subsystems

We look into information about a central system stored in these ‘subenvironments’

Quantum Darwinism!

Part II

Part II requests

We need to set our ground:

Quantum Darwinism (Brandão,Piani and Horodecki)

F. Brandão,M. Piani,P. Horodecki, Nat. Comm. 6 7908(2015)

Quantum Darwinism – BPH’s approach

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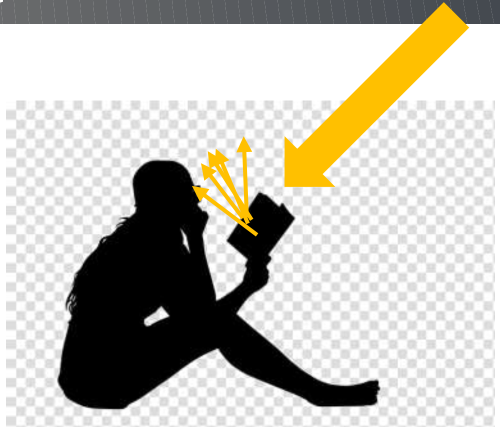
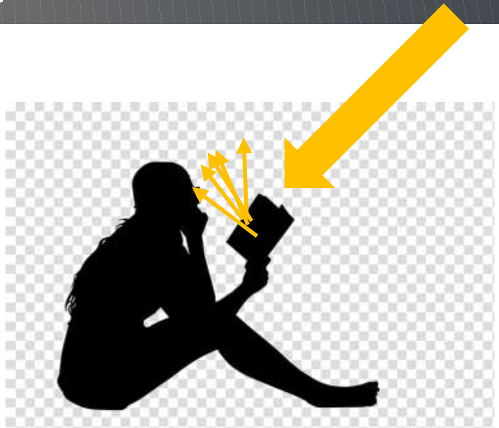
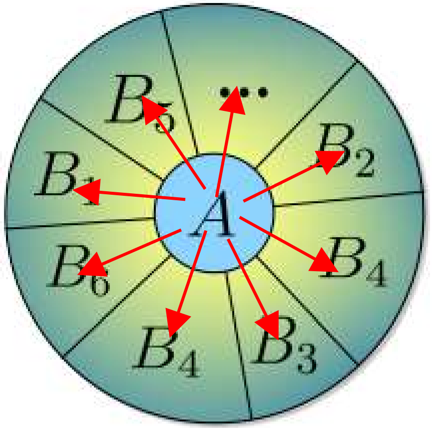


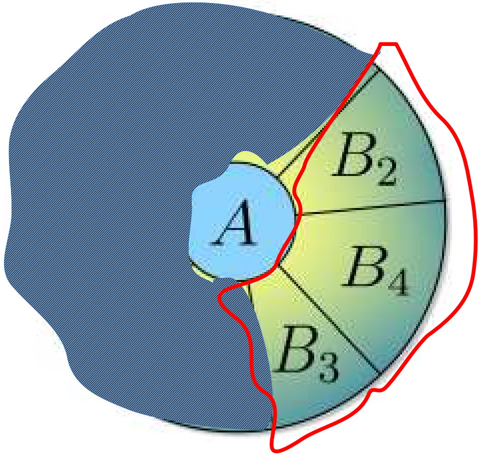
Quantum Darwinism – BPH's approach





Quantum Darwinism – BPH's approach

Environment as a witness dynamics



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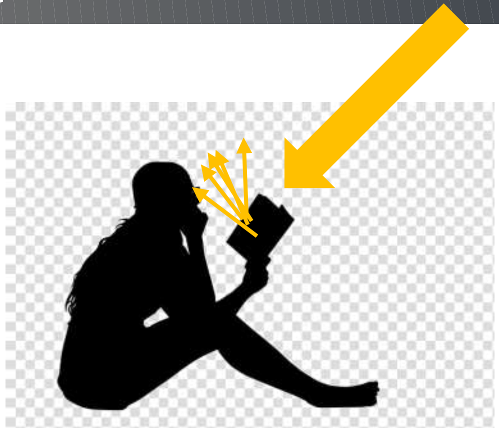

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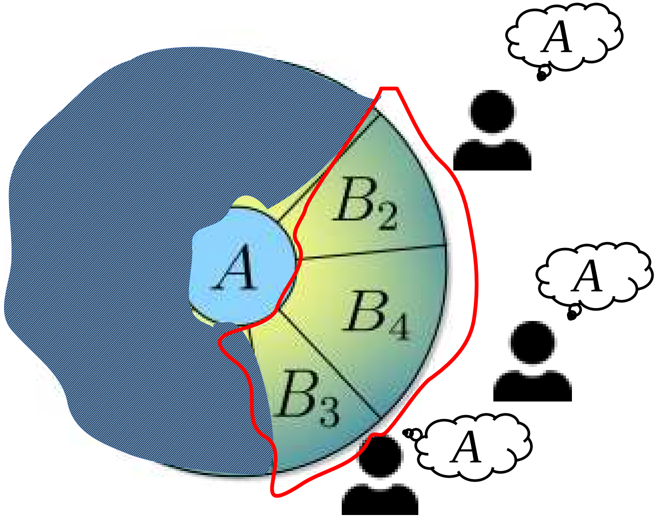


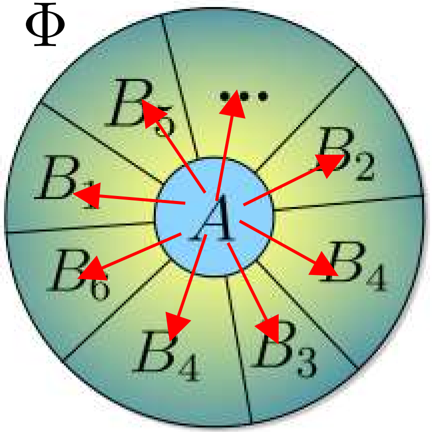
QD aims to explain why they see the same thing

Quantum Darwinism – BPH's approach

Let us formalize a bit...

Quantum Darwinism – BPH's approach

cptp Φ



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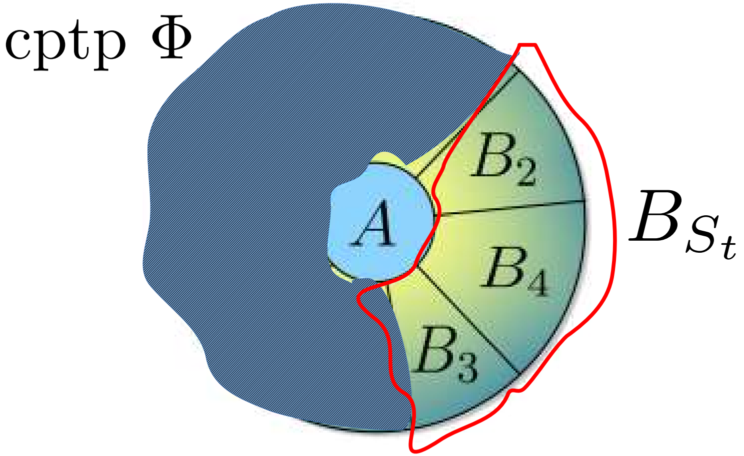

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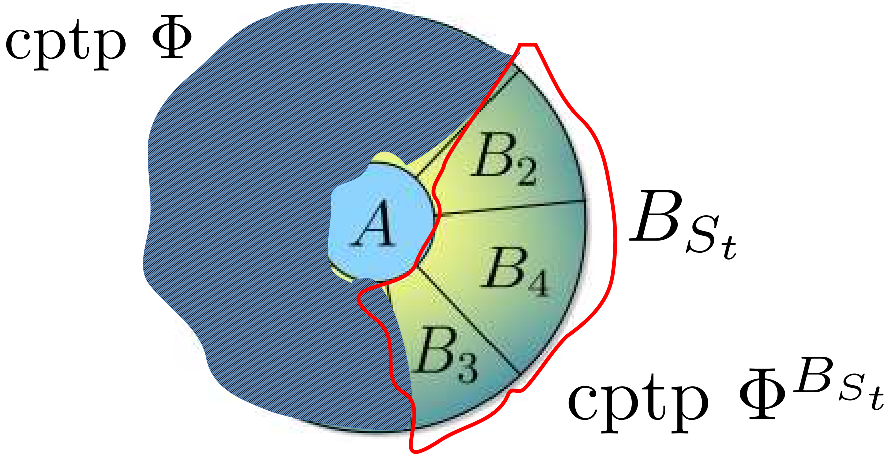

Def. **EWt-dynamics** for the subset $B_{S_t} = (B_j)_{j \in S_t}$: $\Phi^{B_{S_t}} := \mathrm{Tr}_{\mathcal{E} \setminus B_{S_t}} \circ \Phi$

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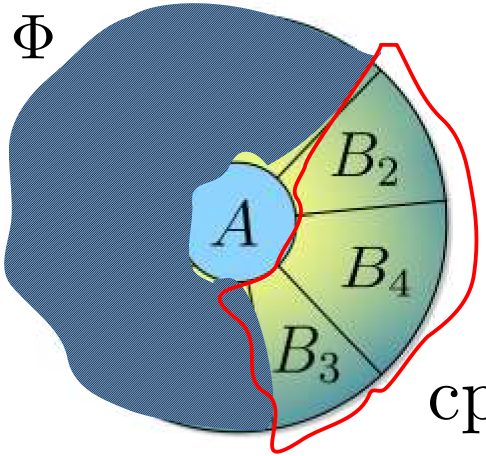
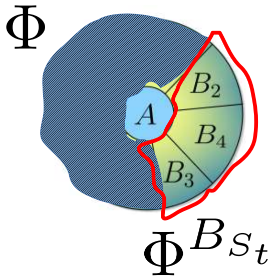


B_{S_t}

cptp $\Phi^{B_{S_t}}$

Quantum Darwinism – BPH's approach



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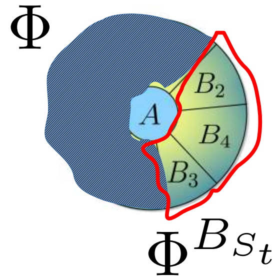
*F. Brandão,M. Piani,P. Horodecki, Nat. Comm. 6 7908(2015)



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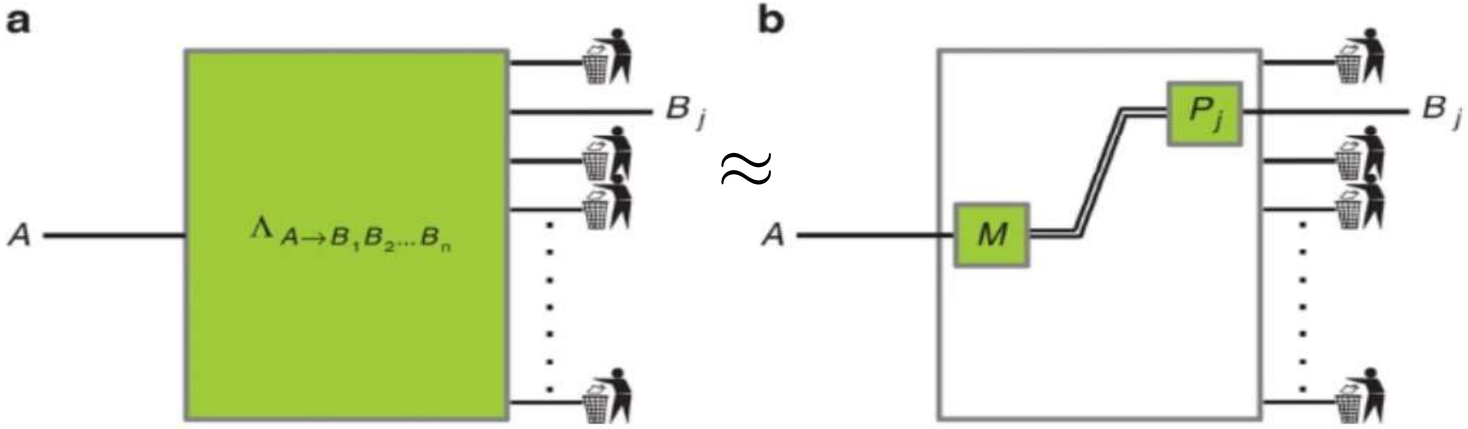

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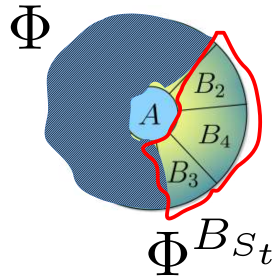

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Consider portion of size t. If $N \gg t$, then for most choices of B_{S_t} there exists $\{\tilde{E}_k\}$ and $\{\sigma_k^{B_{S_t}}\}s.t.$

$$\Phi^{B_{S_t}}(\rho^A) \approx \sum_k \mathrm{Tr}\{\tilde{E}_k \rho^A\}\sigma_k^{B_{S_t}}$$

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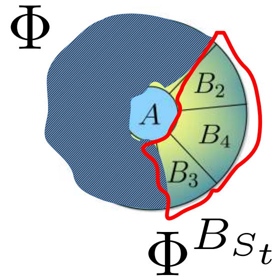
Quantum Darwinism – BPH's approach



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Quantum Darwinism – BPH's approach

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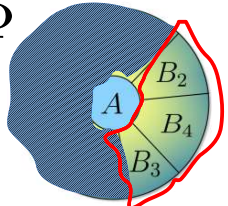

$\Phi^{B_{S_t}}$

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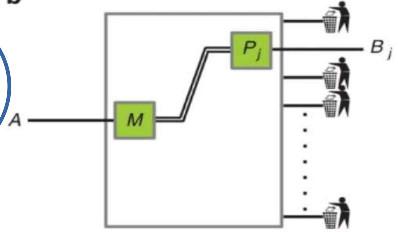

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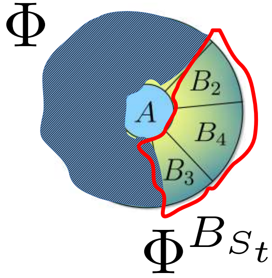
Quantum Darwinism – BPH's approach



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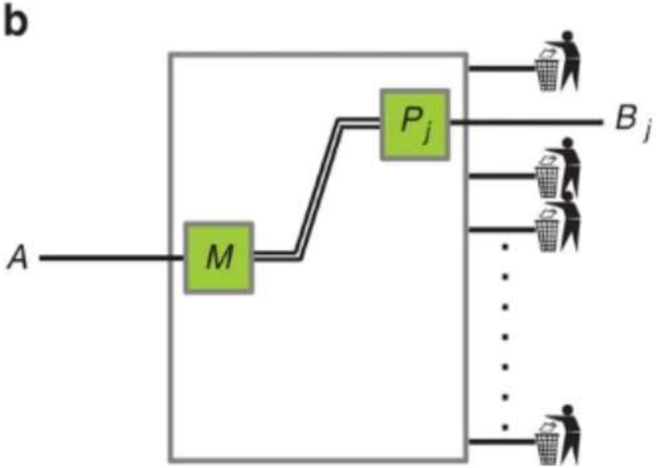
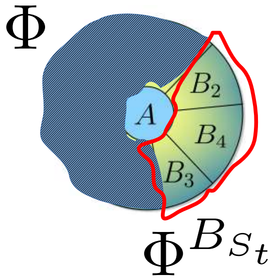


This Thm tells us that EWt-dynamics already leads to emergence of some objectivity: **objectivity of observables**

Quantum Darwinism – BPH's approach



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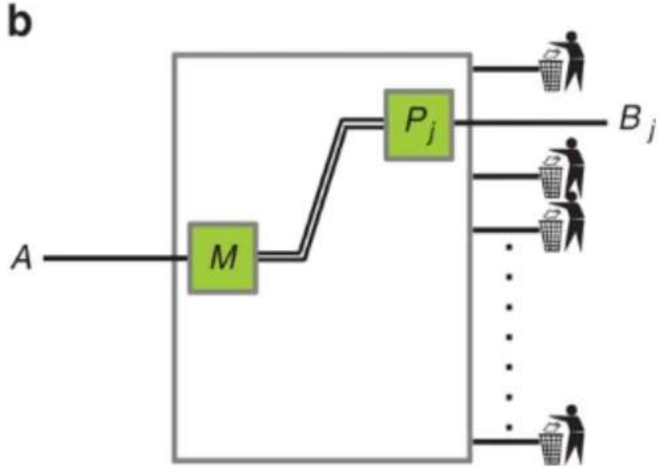

Assumptions:

- We consider portions in which the approximation is valid
- Infinite environment $\approx \to =$

$$\Phi^{B_{S_t}} = \sum_k \mathrm{Tr}\{\tilde{E}_k \rho\} \sigma_k^{B_{S_t}}$$

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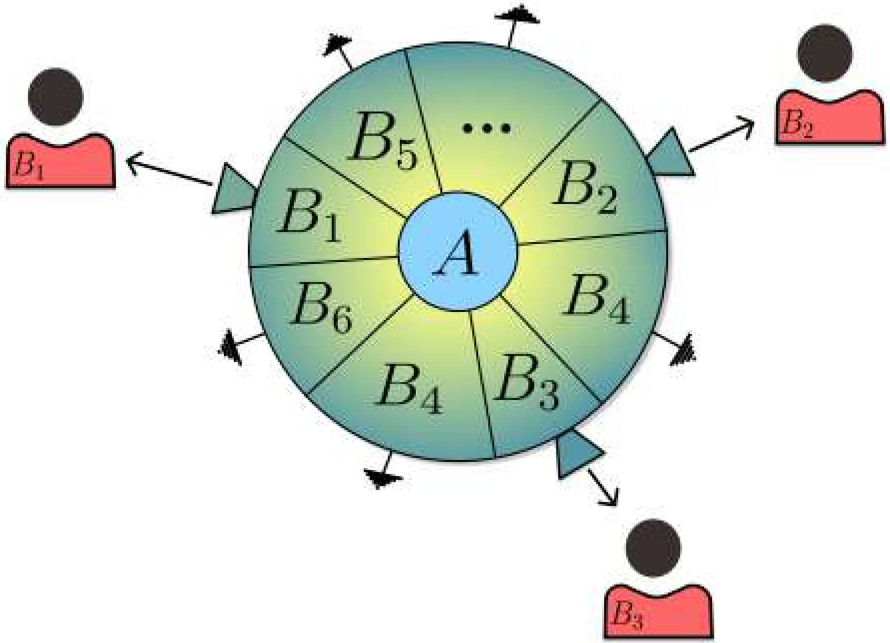
Quantum Darwinism – BPH’s approach



Quantum Darwinism – BPH's approach

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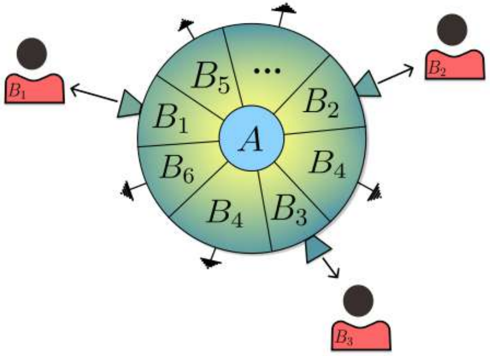

Quantum Darwinism – BPH's approach

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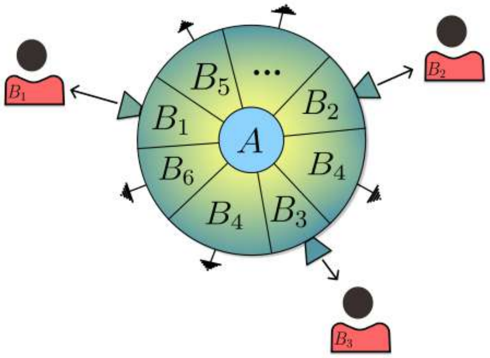

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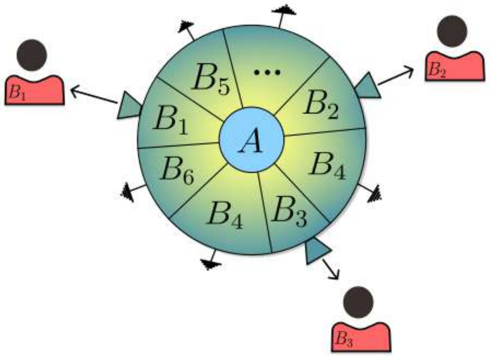

QD assumption: $(\sigma_k)_k$ encode well enough the labels $(k)_k$!

Quantum Darwinism – BPH's approach

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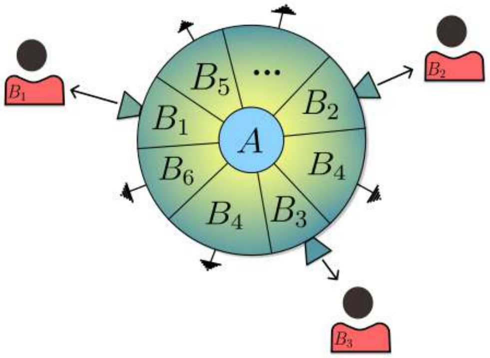
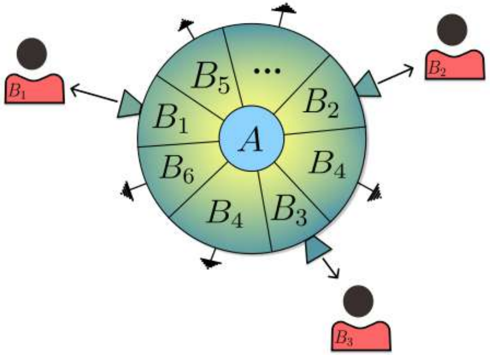


QD assumption: $(\sigma_k)_k$ are sufficiently distinguishable!

Quantum Darwinism – BPH's approach

$$\Phi^{B_j}(\rho^A) = \sum_k \underbrace{\mathrm{Tr}\{\tilde{E}_k \rho^A\}}_{\tilde{p}_k} \sigma_k^{B_j}$$



Quantum Darwinism – BPH’s approach

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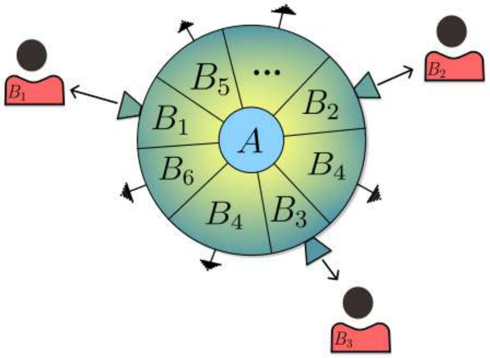

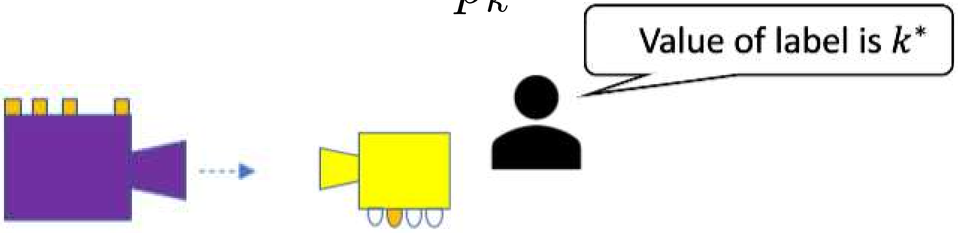

$(\tilde{p}_k, \sigma_k)$

$\{E_k\}, \#\{E_k\} = k_{max}$

Quantum Darwinism – BPH’s approach

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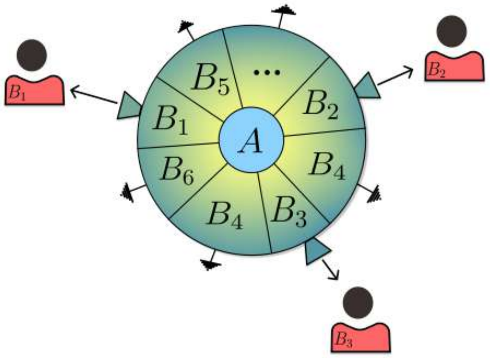
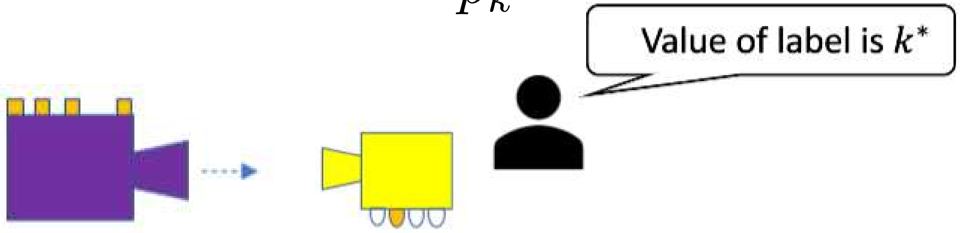
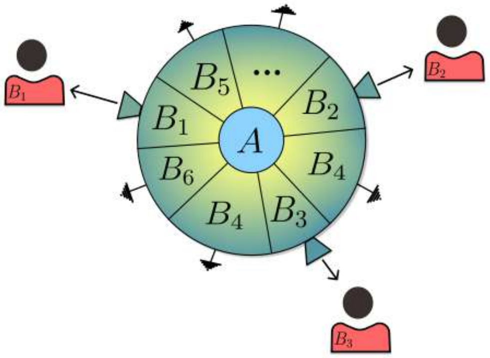




$(\tilde{p}_k, \sigma_k)$ $\{E_k\}, \#\{E_k\} = k_{max}$

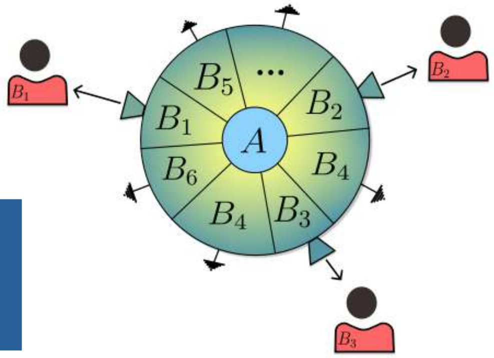
$$p_{\text{guess}}[(\tilde{p}_k, \sigma_k^{B_j})_k] := \max_{\{F_k\}} \sum_k \tilde{p}_k \mathrm{Tr}\{F_k \sigma_k^{B_j}\}.$$

Quantum Darwinism – BPH's approach



Def. Consider an EWt-dynamics $\Phi^{B_{S_t}}$ and t partial traces Φ^{B_j}.
We say that a QD_η process occurs if, for all $B_j \in B_{S_t}$, $\min_{\rho^A} p_{\text{guess}} \geq \eta$.

Quantum Darwinism – BPH's approach



Quantum Darwinism: the environment encodes information regarding system A

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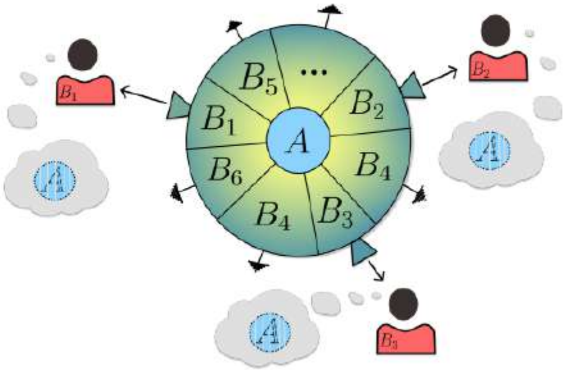
Quantum Darwinism – BPH's approach

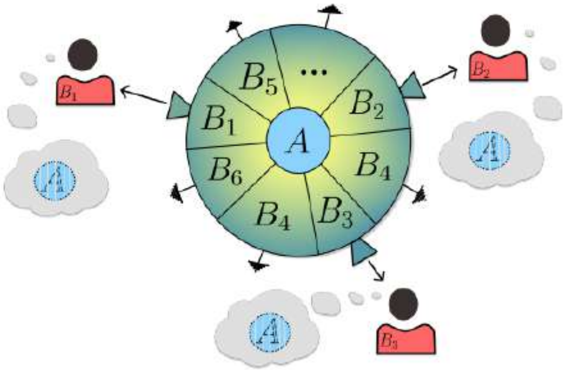


Prop.(adapt. from ref.*)

$$\overbrace{\text{EWt-dynamics} + \text{good enough encoding}}^{QD_\eta} \Rightarrow \text{Bobs are likely to agree!}$$

*F. Brandão,M. Piani,P. Horodecki, Nat. Comm. 6 7908(2015)

Quantum Darwinism – BPH's approach



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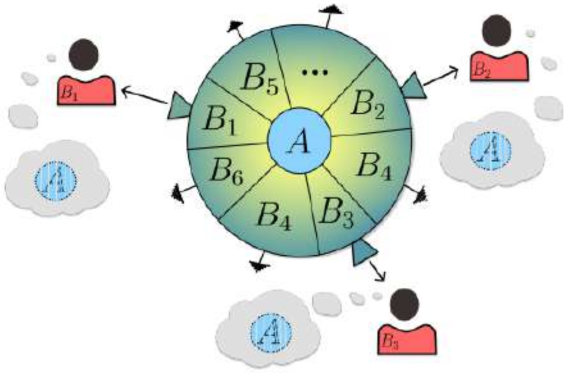
Quantum Darwinism – BPH's approach

EW dynamics describes our generic experience

Specifics of the interaction do not matter!



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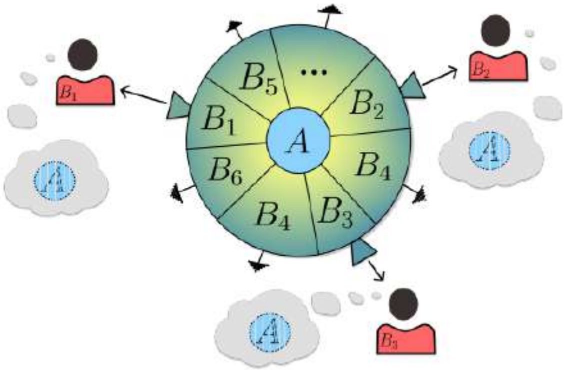


EW dynamics describes our generic experience

Specifics of the interaction do not matter!

There is no restriction on the observable
selected by the dynamics

$(\tilde{E}_k)$ can be an IC-POVM!

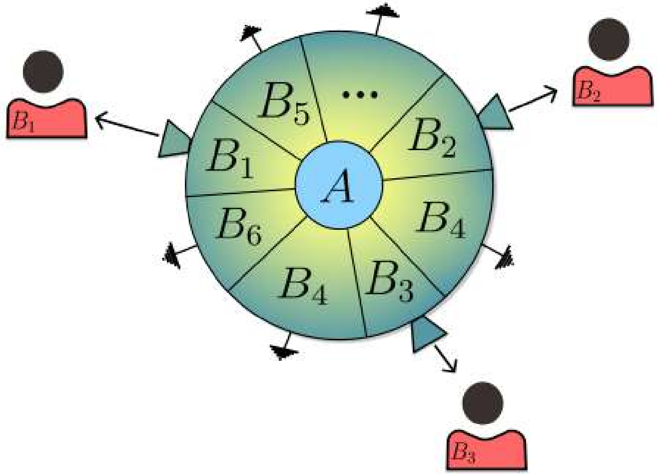
Results

Taking Spekkens' notion of (non)contextuality

RDB, R. Wagner, C. Duarte, B. Amaral, M. Terra Cunha, **Arxiv 2104.05734**

Results

Taking Spekkens' notion of (non)contextuality



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Consider an EWt-dynamics and the reduced $(\Phi^{B_j})_j$.

For those $B_j \in B_{S_t}$ in which $(\sigma_k^{B_j})_k$ are affinely independent, there exists a noncontextual model.

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$$\text{EWt-dynamics} + \text{ affine independence} \Rightarrow \text{noncontextuality}$$

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Not operational...

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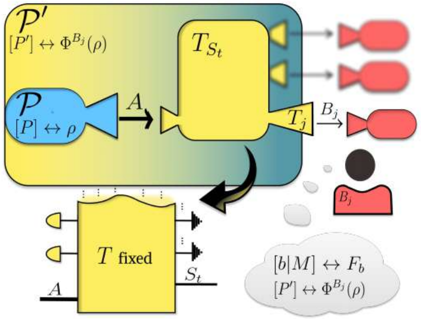

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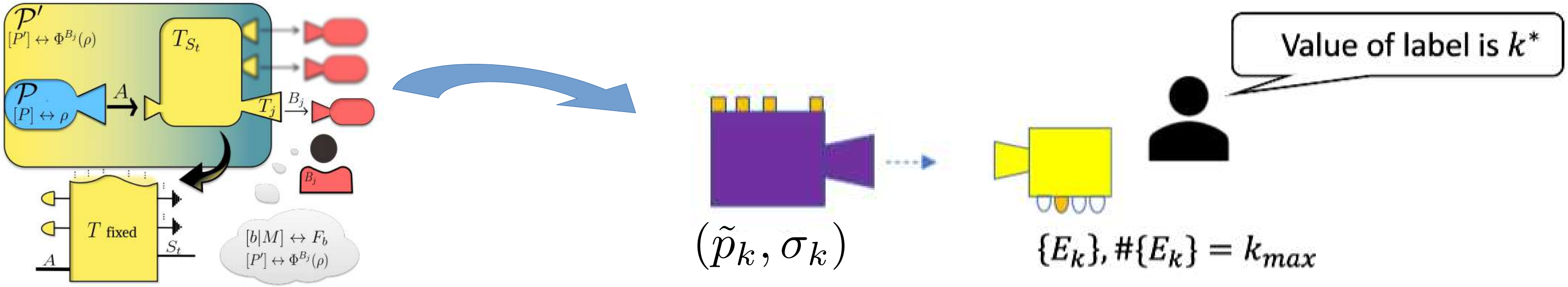

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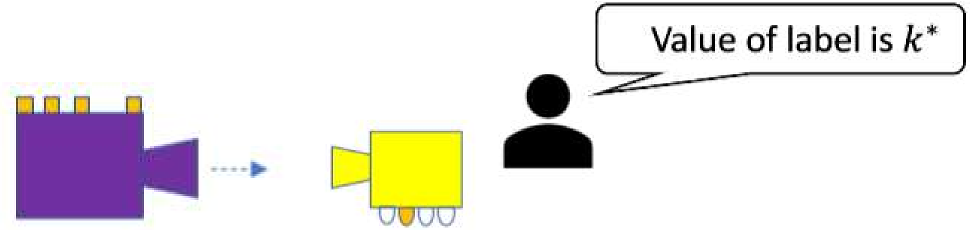
$$p_{\text{guess}} > \hat{P} \Rightarrow \text{affine independent } (\sigma_k^{B_j})$$



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Theorem (Main result)

If a process QD_η occurs with $\eta > \hat{P}[\tilde{E}_k]$, every Bob $B_j \in B_{S_t}$ can construct a noncontextual ontological model

Results

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Lemma 2

Theorem (Main result)

If a process QD_η occurs with $\eta > \hat{P}[\tilde{E}_k]$, every Bob $B_j \in B_{S_t}$ can construct a noncontextual ontological model

$$\mathrm{QD}_\eta \Rightarrow \text{noncontextuality for every Bob}$$

Results – relaxing assumption

Infinite environment $$\Phi^{B_{S_t}} = \sum_k \mathrm{Tr}\{\tilde{E}_k \rho\} \sigma_k^{B_{S_t}}$$

Finite environment $$\Phi^{B_{S_t}} \approx \sum_k \mathrm{Tr}\{\tilde{E}_k \rho\} \sigma_k^{B_{S_t}}$$

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Conclusions and perspectives

- Noncontextuality generically emerges out of quantum Darwinism in infinite environments
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  - It can emerge even if Darwinism fails (EW+affine independence are enough);
  - Objectivity emerging from Darwinism can be considered classical, even in the broader case
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Conclusions and perspectives

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- Relation to non-Markovianity?

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Perspectives

- Do other classical limits (strong QD) allow for emergence of noncontextuality?

- Relation to non-Markovianity?

- If we take the special case of the multiple observers with weak interaction, can we protect contextuality?

THANKS FOR LISTENING!

Appendix

Appendix 1

KS
Contextuality

KS Contextuality

Pic by Konrad Jacobs, Erlangen

Pic from Princeton website

KS Contextuality

Are there answers to all questions, even those that were never asked?

Pic by Konrad Jacobs, Erlangen

Pic from Princeton website

KS Contextuality

Is Quantum Theory compatible with the idea of an underlying reality where measurements play only the role of revealing pre-determined values?

Pic by Konrad Jacobs, Erlangen

Pic from Princeton website

KS Contextuality

Is Quantum Theory compatible with the idea of an underlying reality where measurements play only the role of revealing pre-determined values?

Compatibility with an ontological model assigning deterministic values.

Probabilities arise as a result of our ignorance.

Pic by Konrad Jacobs, Erlangen

Pic from Princeton website

KS Contextuality

Is Quantum Theory compatible with the idea of an underlying reality where measurements play only the role of revealing pre-determined values?

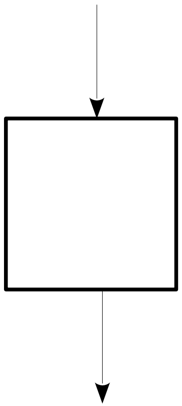
Answer: Yes, but this assignment must be context-dependent.

Pic by Konrad Jacobs, Erlangen

Pic from Princeton website

KS Contextuality

$$A_i, A_j$$



$$o_i, o_j | A_i, A_j$$

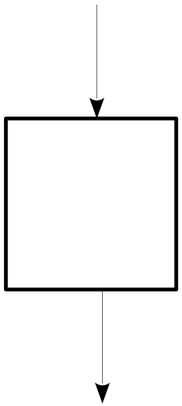
$$p(o_i, o_j | i, j)$$

KS Contextuality

Specify compatibility structure!

$$\Gamma \equiv \{X, \mathcal{C}, O\}$$

$$A_i, A_j$$



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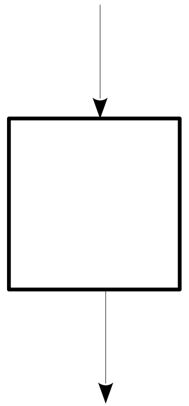

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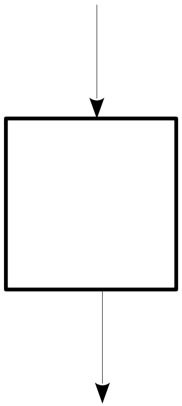

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$$\iff \exists p(o_1, \ldots, o_{|X|}|\lambda)$$

A. Fine Phys. Rev. Lett. 48, 291

S. Abramsky and A. Brandenburger 2011 New J. Phys. 13 113036

KS Contextuality

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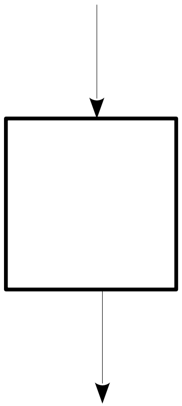

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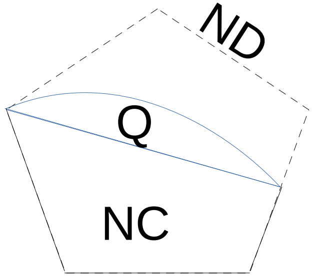

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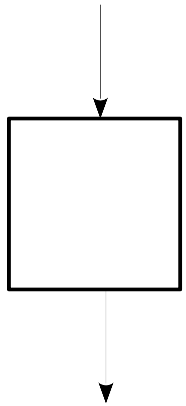

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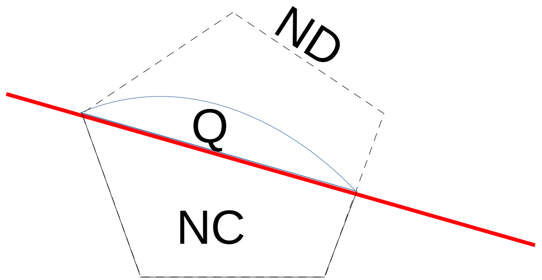


KS noncontextuality inequalities

$$p(o_i, o_j | i, j) \stackrel{?}{=} \sum_{\lambda} p(\lambda) p(o_i | i, \lambda) p(o_j | j, \lambda)$$

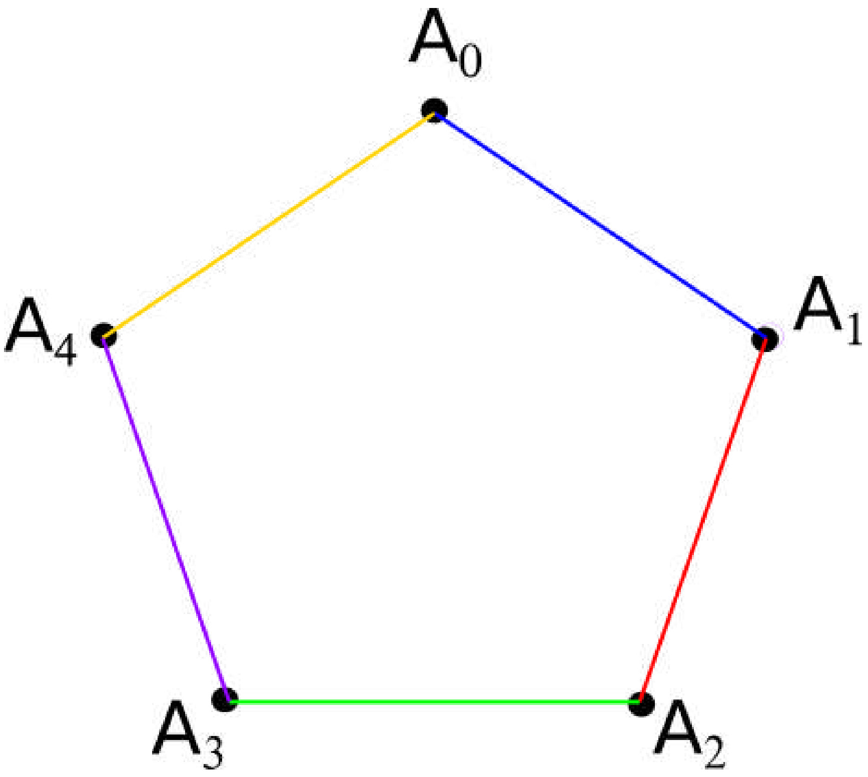
N-cycle inequalities

Understanding the inequality

$$\sum_{i=0}^{N-1} \langle A_i A_{i+1} \rangle \geq -N+2$$

-3

$$p(o_i, o_j | i, j) \stackrel{?}{=} \sum_{\lambda} p(\lambda) p(o_i | i, \lambda) p(o_j | j, \lambda)$$



Understanding the inequality

Foolish non-contextual assignment

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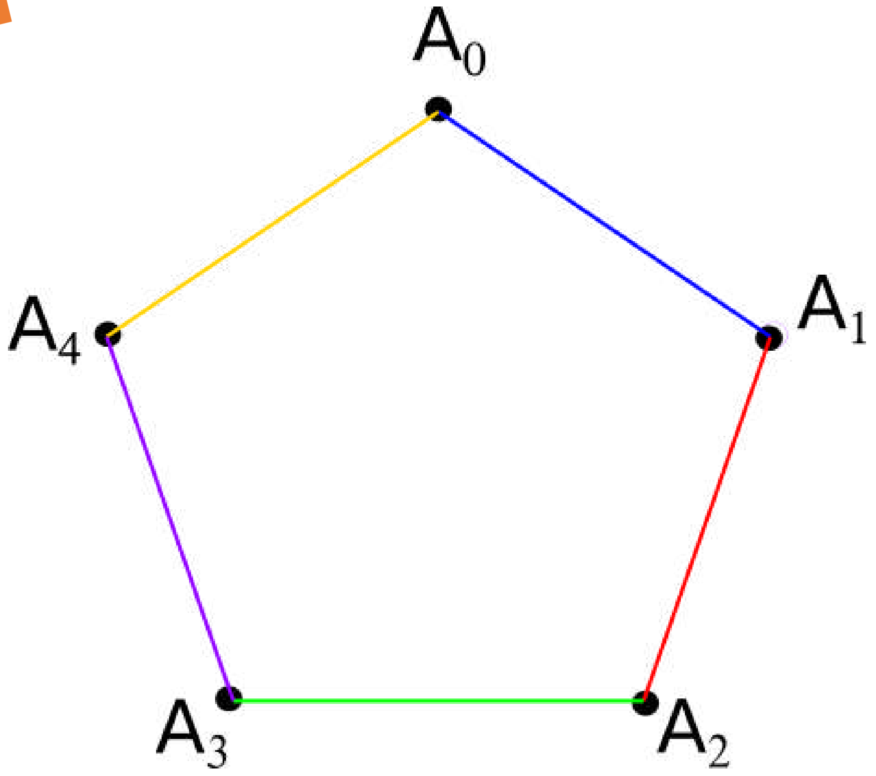

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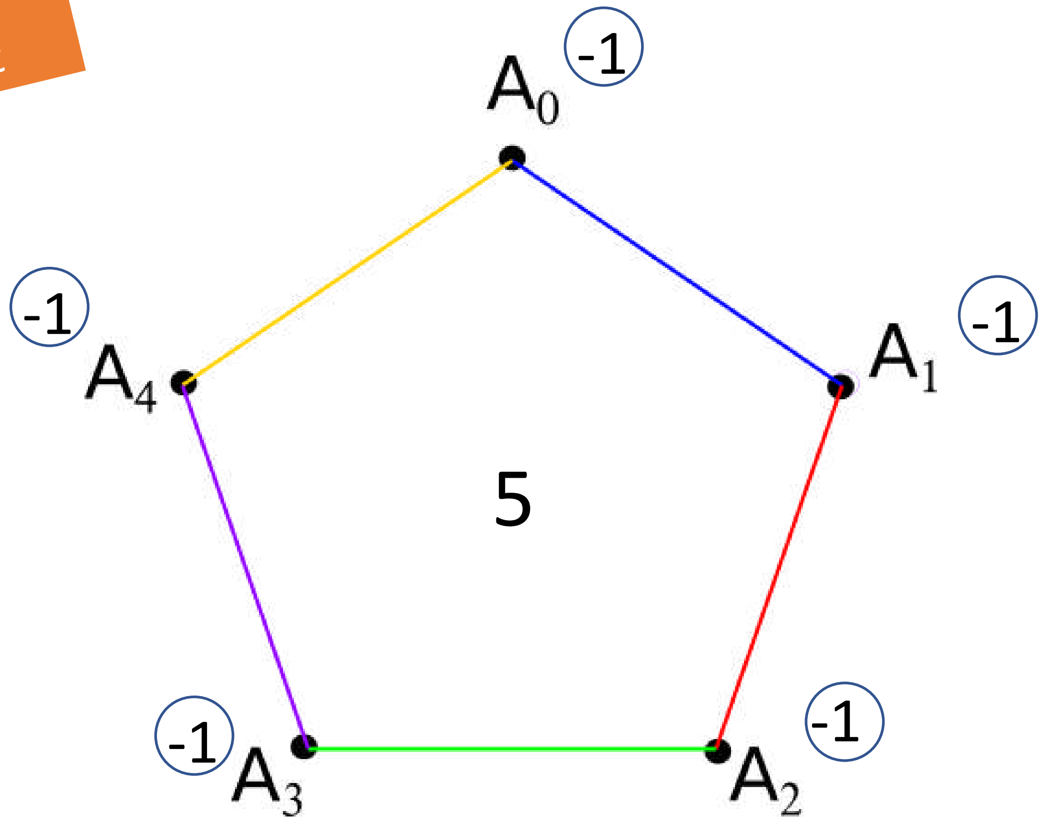

We need to alternate values...

Understanding the inequality

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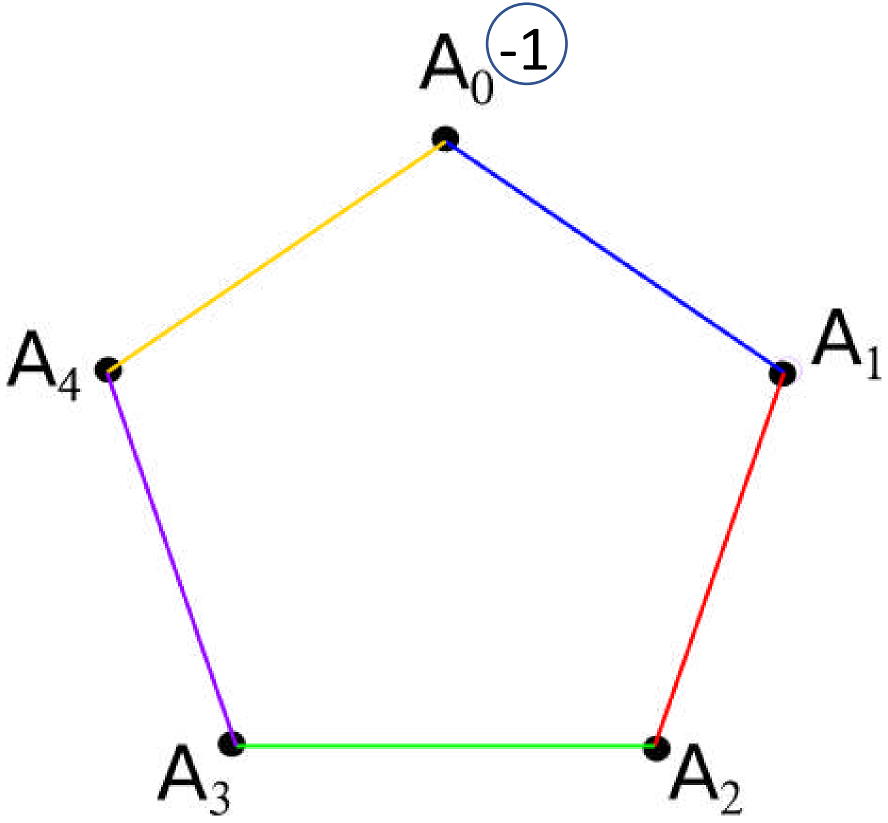
-3

Alternating values, non-contextual



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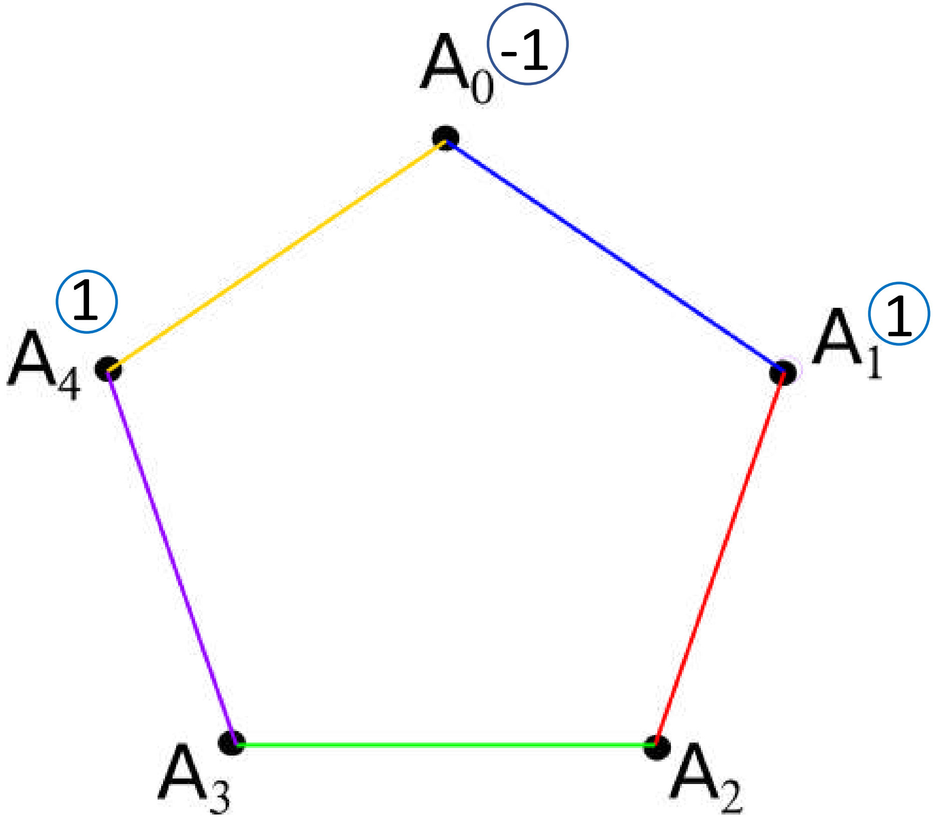

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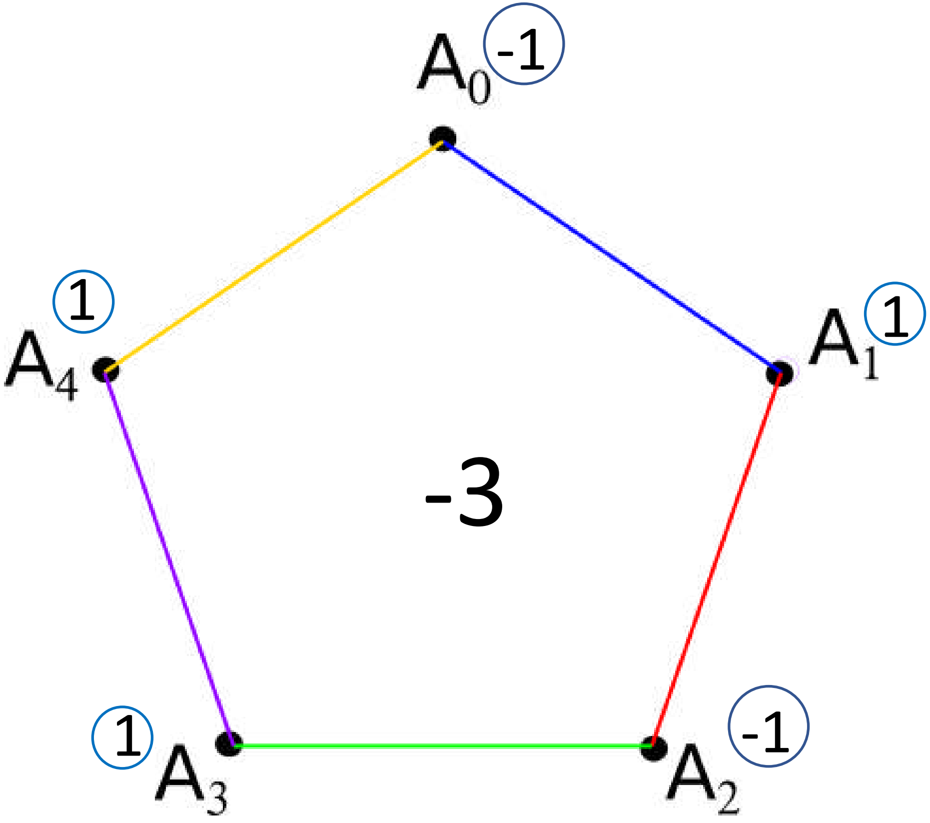

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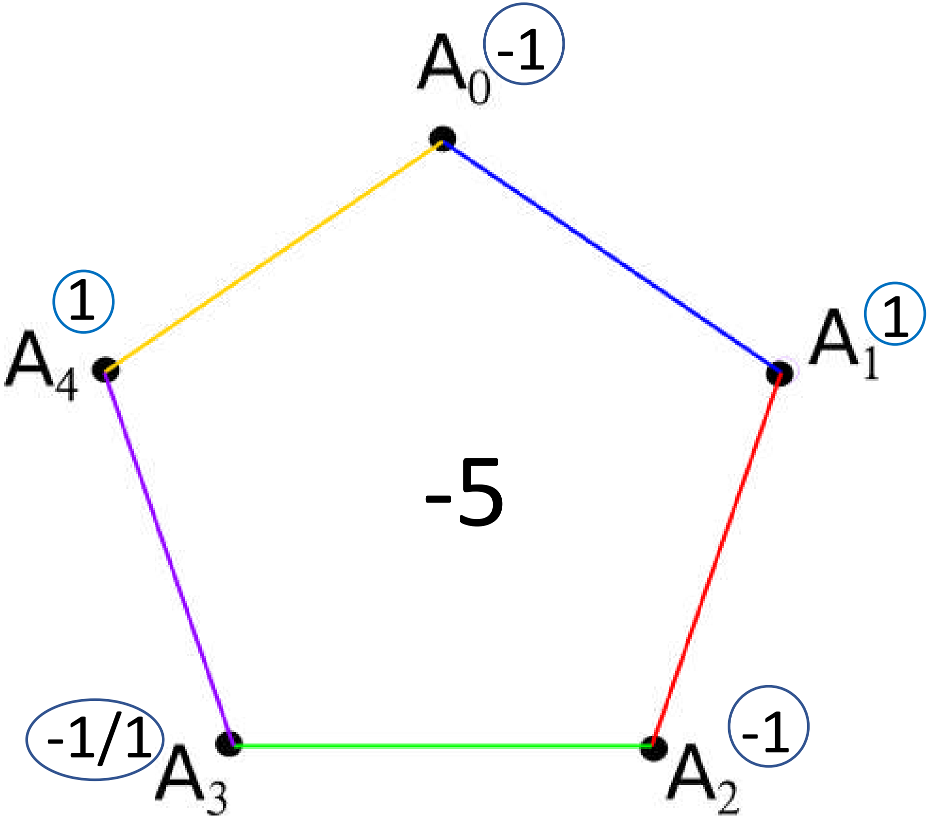

Contextual/Disturbing assignment

Odd N-cycle: new inequalities

$$\sum_{i=0}^{N-1} \langle A_i A_{i+1} \rangle \geq -N+2$$

$$\sum_{i=0}^{N-1} p(o_i = o_{i+1} | i, i+1) \geq 1$$

$$\sum_{i=0}^{N-1} p(o_i \neq o_{i+1} | i, i+1) \leq N-1$$

Odd N-cycle: new inequalities

- Non-Disturbance

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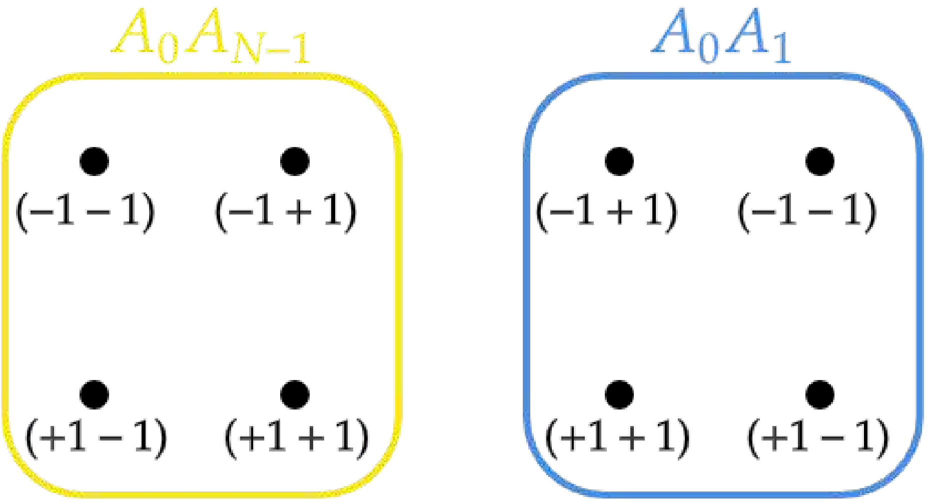

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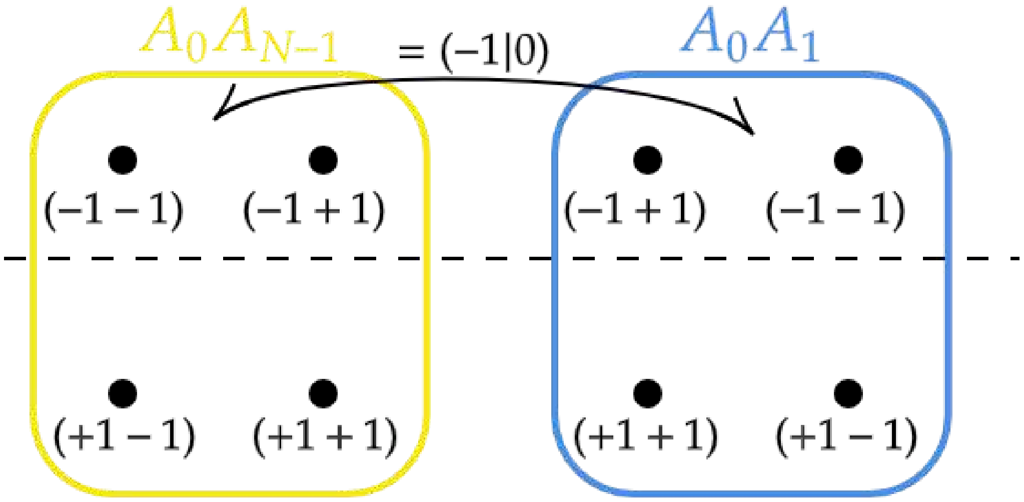

Odd N-cycle: new inequalities

- Non-Disturbance
- Exclusiveness: $p(-1,-1|i,i\pm 1)=0$

$$\sum_{i=0}^{N-1} p(o_i = o_{i+1}|i,i+1) \geq 1$$

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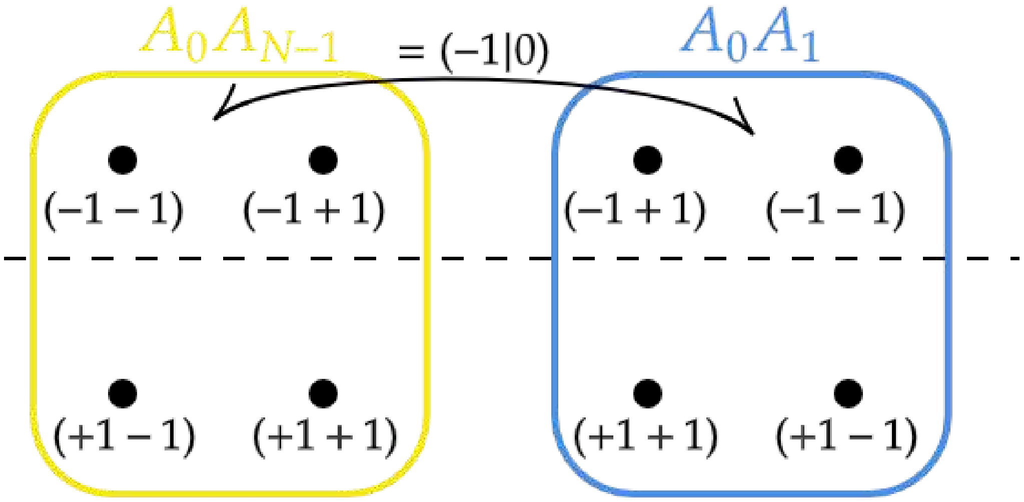

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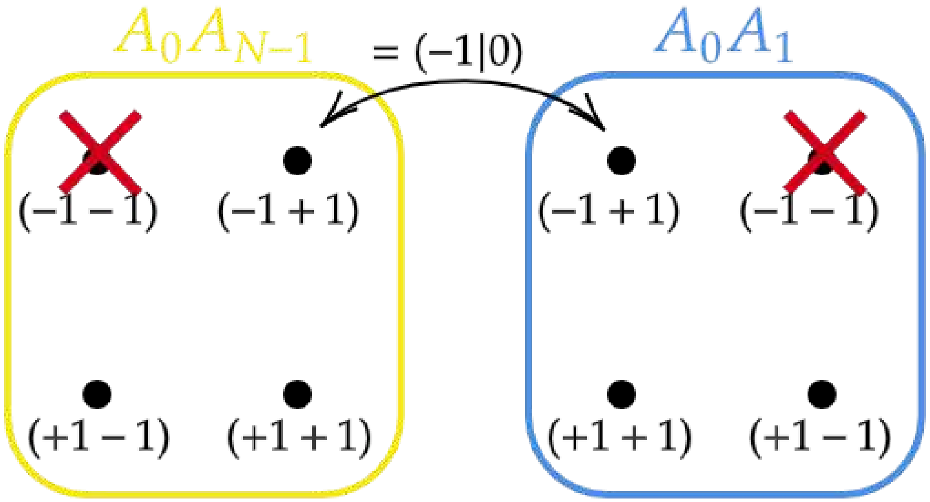

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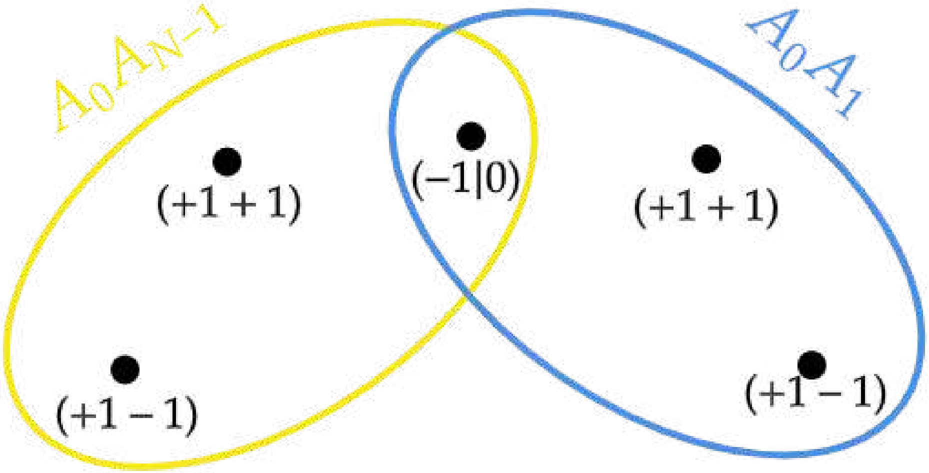

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$$\sum_{i=0}^{N-1} p(o_i = o_{i+1}|i,i+1) \geq 1 \longrightarrow \sum_i p(1,1|i,i+1) \geq 1$$

$$\sum_{i=0}^{N-1} p(o_i \neq o_{i+1}|i,i+1) \leq N-1 \longrightarrow \sum_i p(-1|i) \leq \frac{N-1}{2}$$

Odd N-cycle: new inequalities

- Non-Disturbance
- Exclusiveness: $p(-1,-1|i,i\pm 1)=0$

$$\sum_{i=0}^{N-1} p(o_i = o_{i+1}|i,i+1) \geq 1 \longrightarrow \sum_i \underbrace{p(1,1|i,i+1)}_{b_i} \geq 1$$

$$\sum_{i=0}^{N-1} p(o_i \neq o_{i+1}|i,i+1) \leq N-1 \longrightarrow \sum_i \underbrace{p(-1|i)}_{a_i} \leq \frac{N-1}{2}$$

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$$\sum_{i=0}^{N-1} p(o_i = o_{i+1}|i,i+1) \geq 1 \longrightarrow \boxed{\sum_i p(b_i) \geq 1}$$

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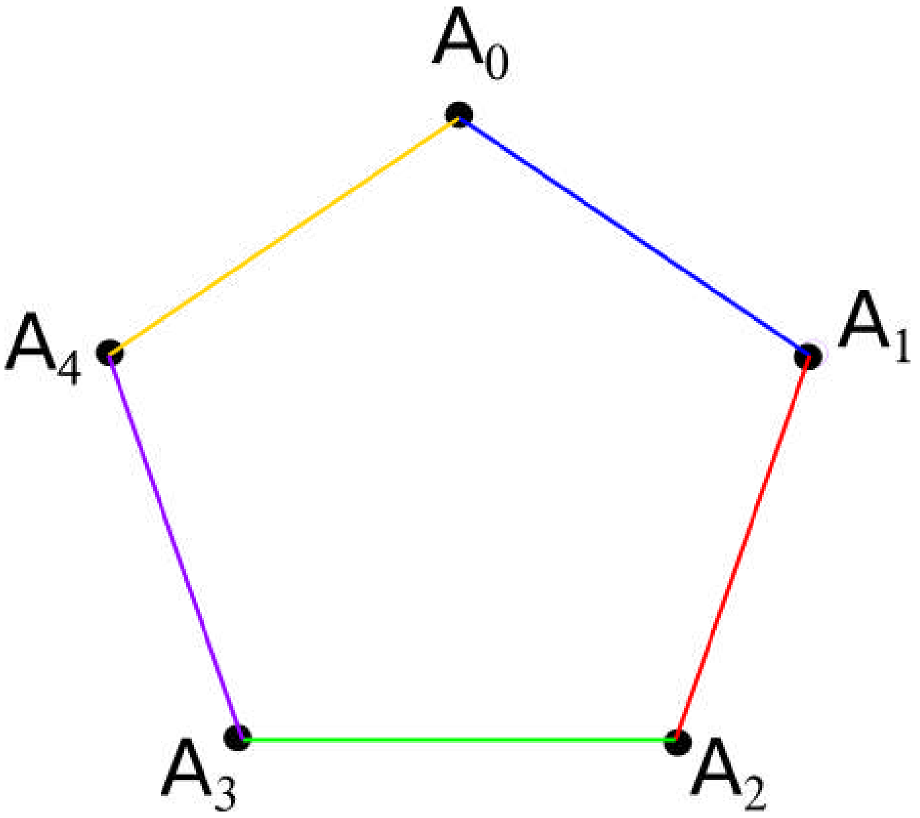

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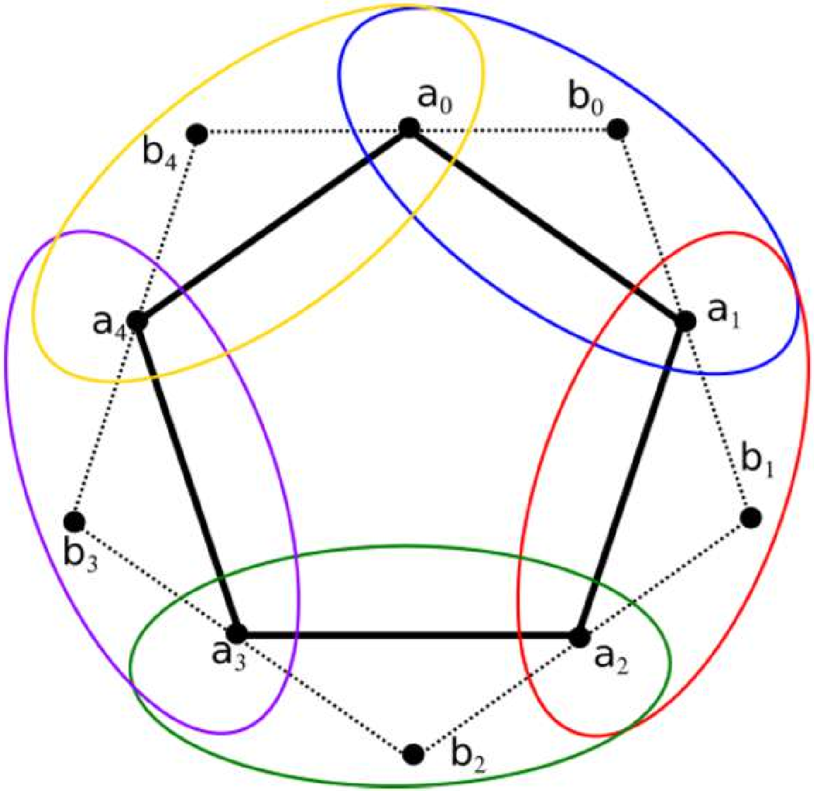

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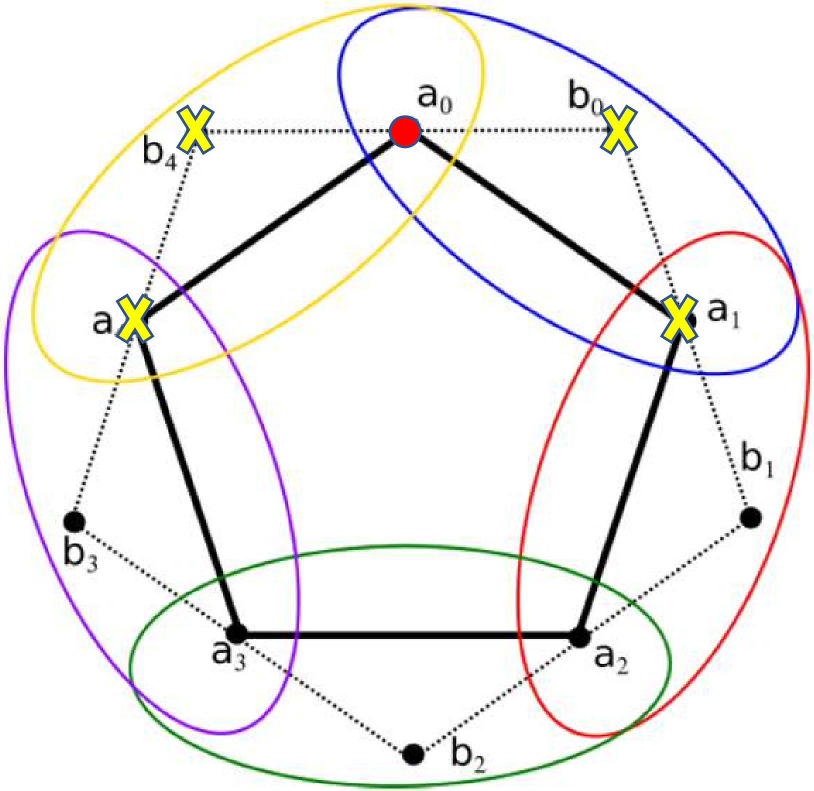

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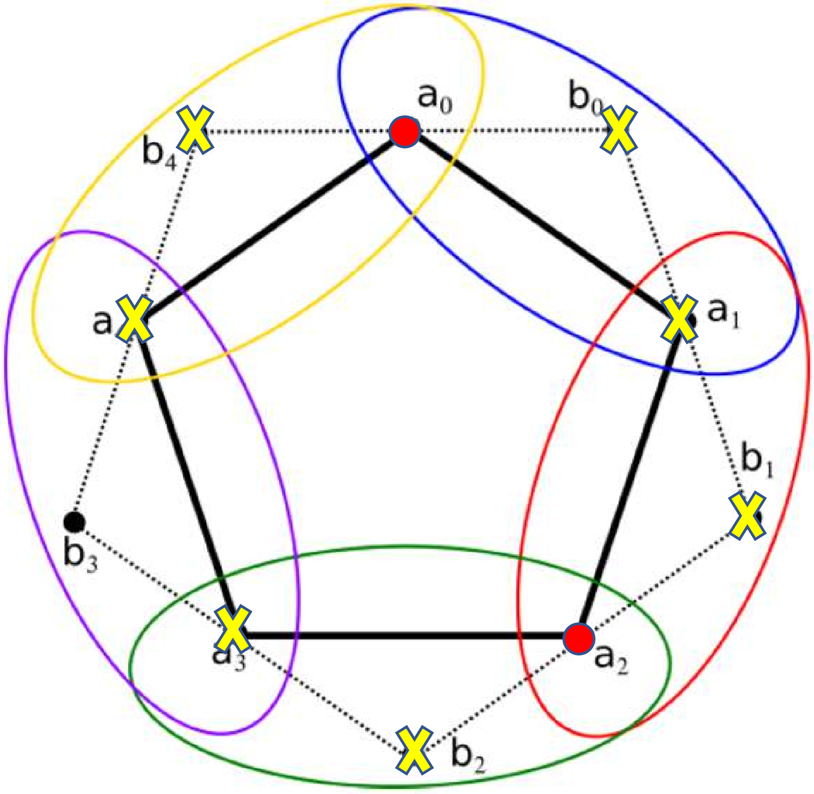

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$$\boxed{\begin{array}{l}\displaystyle\sum_i p(b_i)\geq 1\\[2ex] \displaystyle\sum_i p(a_i)\leq \frac{N-1}{2}\end{array}}$$

$$\sum_i \langle A_i A_{i+1}\rangle \geq -N+2$$

The new inequalities can be violated while the original is obeyed!

Odd N-cycle: new inequalities

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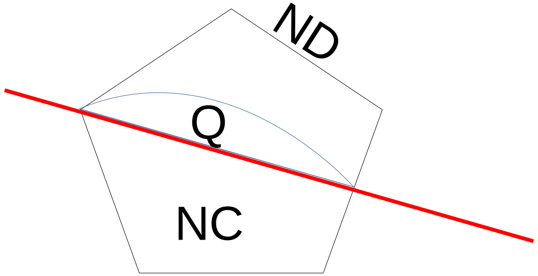
The new inequalities can be violated while the original is obeyed!

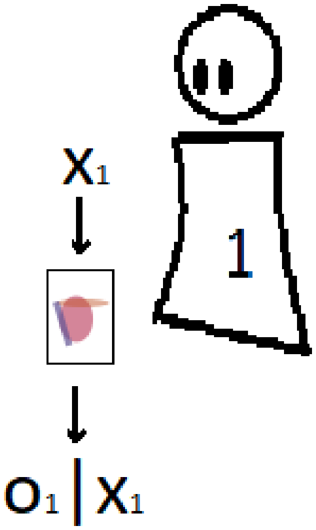
Example: foolish noncontextual assignment...

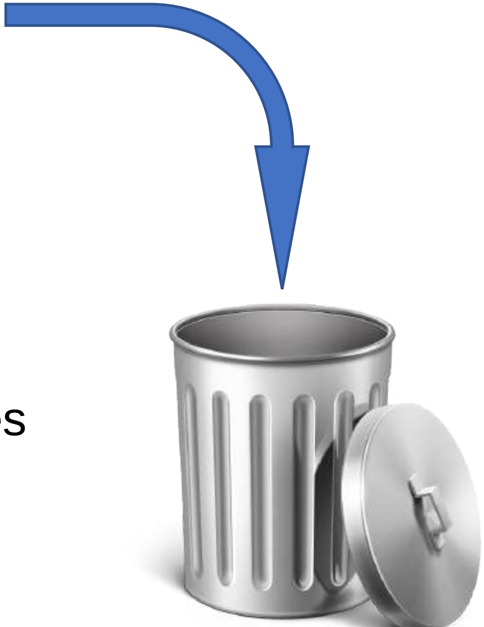
Sequential setup

KS Contextuality



KS noncontextuality inequalities

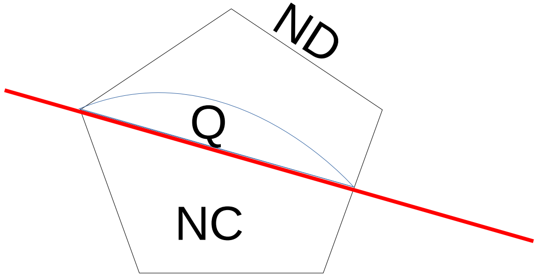




Colocar em outro lugar?

Usual way to test the inequalities

KS Contextuality



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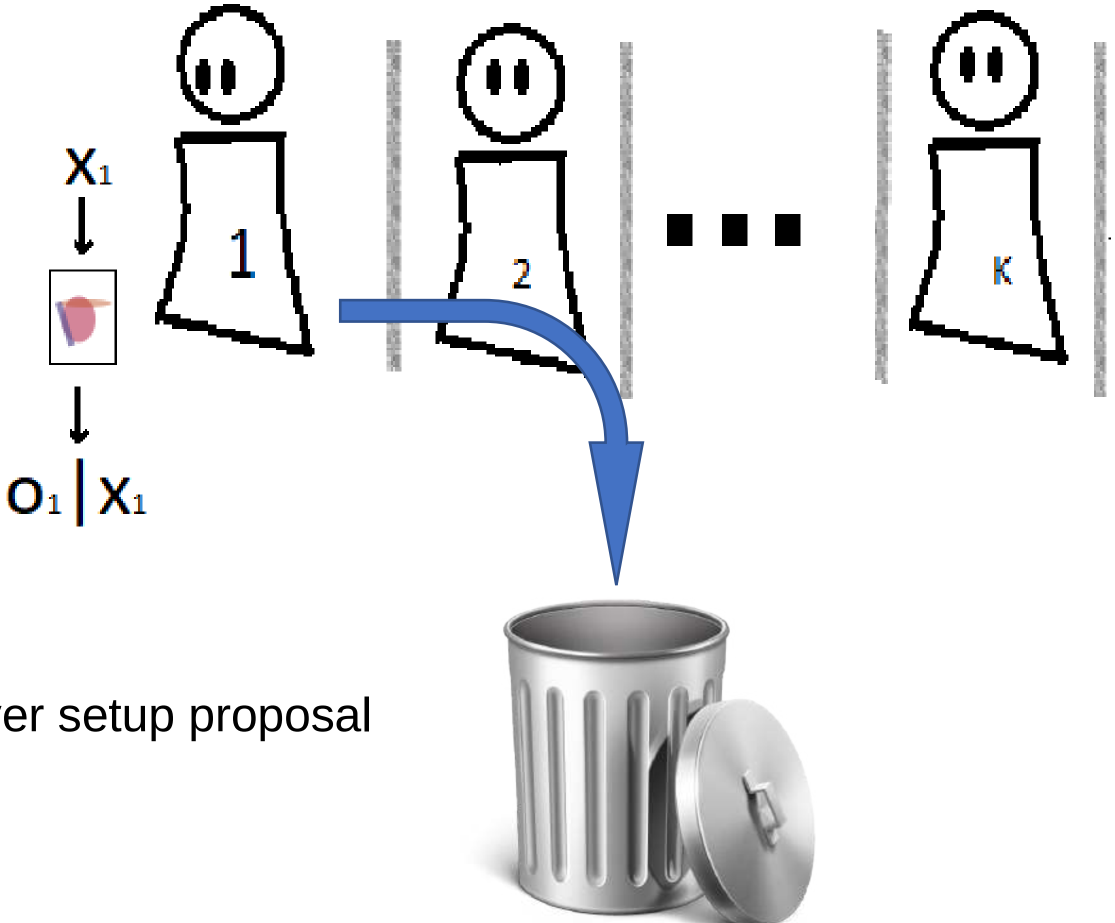
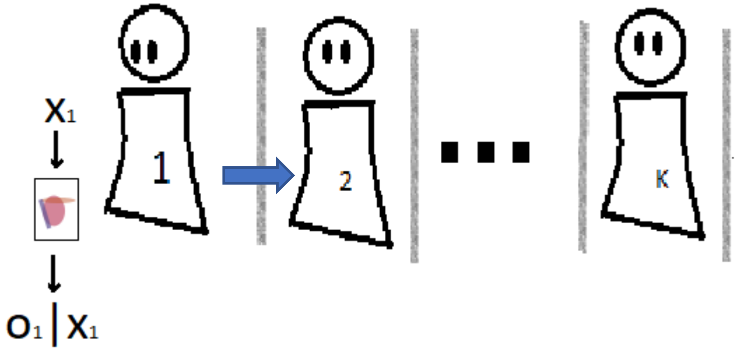


Colocar em outro lugar?

Our multiobserver setup proposal

Adapting to sequential observers



- Which observers still find violations?

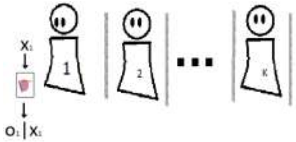
  -Does this depend on N?

  -Does this depend on the measurement protocol?

- Already analized, in different setups, for non-locality and steering, but lacking for Contextuality.

S. Sasmal et al, Phys. Rev. A98, 012305 (2018)
R. Silva et al, Phys. Rev. Lett.114, 250401 (2015)
D. Das et al, Phys. Rev. A99, 022305 (2019).

Results



$$1.\ \mathcal{M}_i = \{a_i, b_i, a_{i+1}\}$$

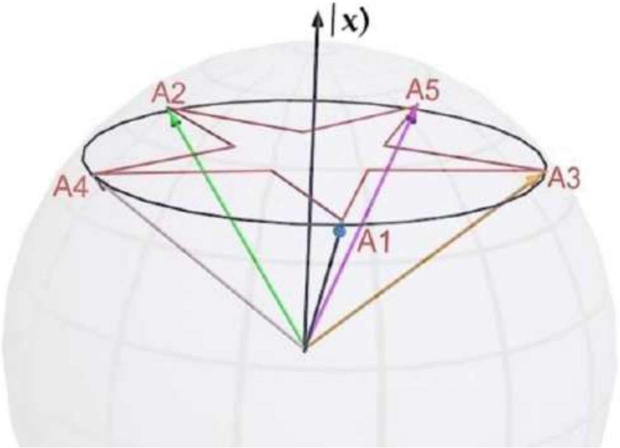
$$2.\ \mathcal{M}_i^a = \{a_i, \neg a_i\}$$

$$3.\ \mathcal{M}_i^b = \{b_i, \neg b_i\}$$

Quantum realizations

- Hilbert space of dimension 3

$$A_i = \mathcal{I} - 2|a_i\rangle\langle a_i|$$



$$\langle a_i | a_{i+1} \rangle = 0$$

Image from ref: Tao Li et al, Experimental Contextuality in Classical Light, Scientific Reports, 2017

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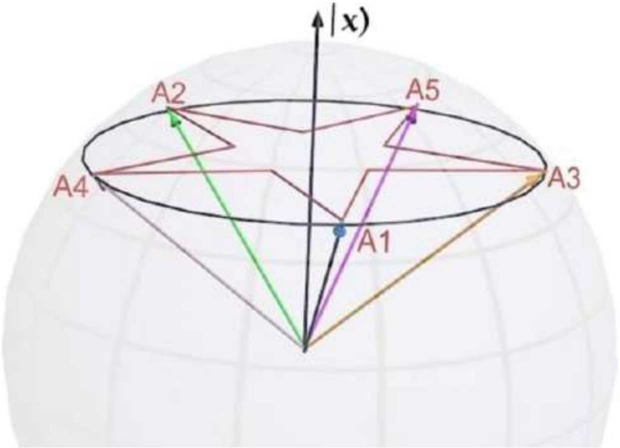
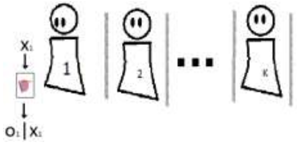


$$|\psi_{handle}\rangle = (0,0,1)^T$$

$$\langle a_i|a_{i+1}\rangle = 0$$

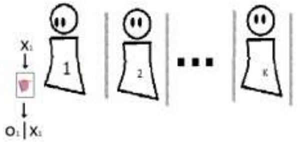
Image from ref: Tao Li et al, Experimental Contextuality in Classical Light, Scientific Reports, 2017

Results



- Protocol 1: Markovian process, preparing the post-measured state

Results

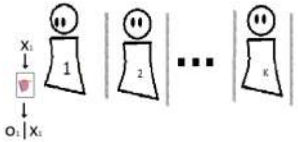


- Protocol 1: Markovian process, preparing the post-measured state

➡ Best result by the handle!

➡ Assymptotic limit given by $\dfrac{N}{3}$

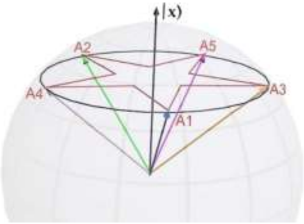
Results

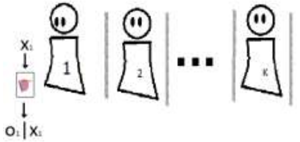


- Measurement protocolos 2 and 3
  - These do not determine completely the post-measured State
  - Using the symmetries of the N-cycle quantum realization...

Results

- Measurement protocolos 2 and 3

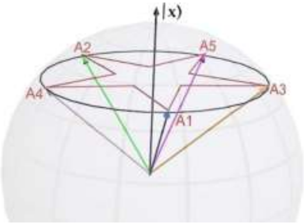


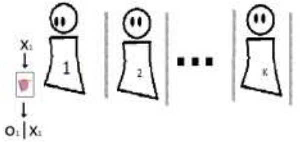


$$\beta = \sum_{i=0}^{N-1} \langle |b_i\rangle\langle b_i| \rangle \geq 1$$

Image from ref: Tao Li et al, Experimental Contextuality in Classical Light, Scientific Reports,

- Measurement protocolos 2 and 3





$$\beta = \sum_{i=0}^{N-1} \langle |b_i\rangle\langle b_i| \rangle \geq 1$$

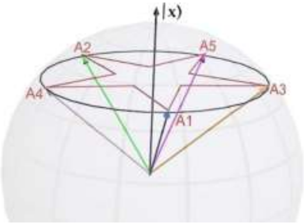
$$\beta_Q^k = B_N \beta_Q^{k-1} + b_N$$

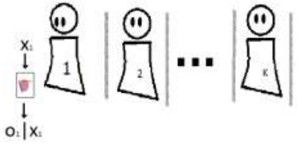
$$B = 1 - \frac{3b}{N}$$

Image from ref: Tao Li et al, Experimental Contextuality in Classical Light, Scientific Reports,

Results

- Measurement protocols 2 and 3





$$\alpha_Q^k = C_N \alpha_Q^{k-1} + c_N \qquad \beta_Q^k = B_N \beta_Q^{k-1} + b_N$$

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Results

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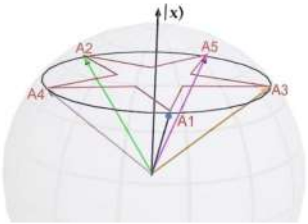
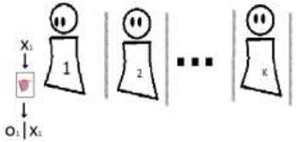




$$\alpha_Q^k = C_N \alpha_Q^{k-1} + c_N \qquad \beta_Q^k = B_N \beta_Q^{k-1} + b_N$$

1.

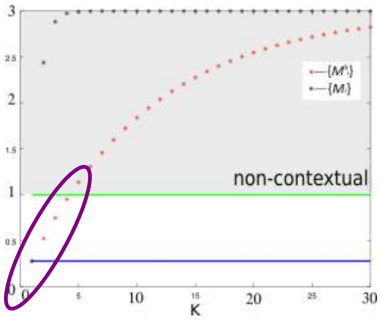
$$|\psi_{handle}\rangle = (0,0,1)^T \longrightarrow$$

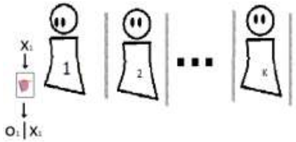
Gives the best result!

2.

$$\alpha^\infty = \beta^\infty = \frac{N}{3}$$

Results

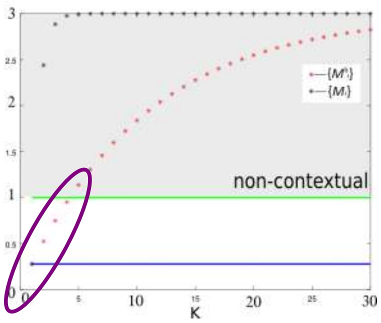


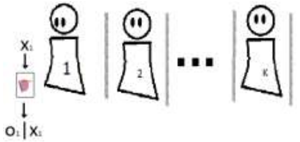


	$K_{\max}$ - Predef. Order		
N	$M(\alpha,\beta)$	$M^a(\alpha)$	$M^b(\beta)$
5	1	1	2
7	1	1	3
9	1	1	4
11	1	1	5
13	1	1	5

- Assyptotic limit without violation!
- Violation disappears quickly
- Protocols behave differently

Results





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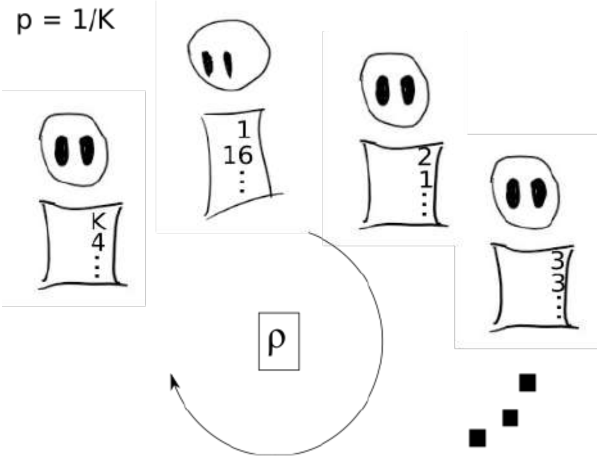

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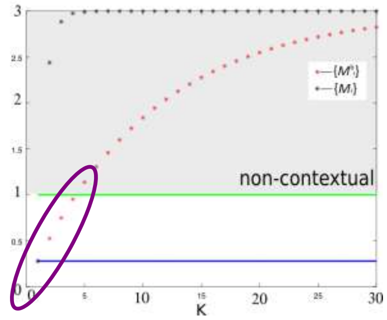

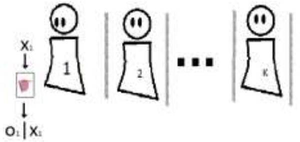

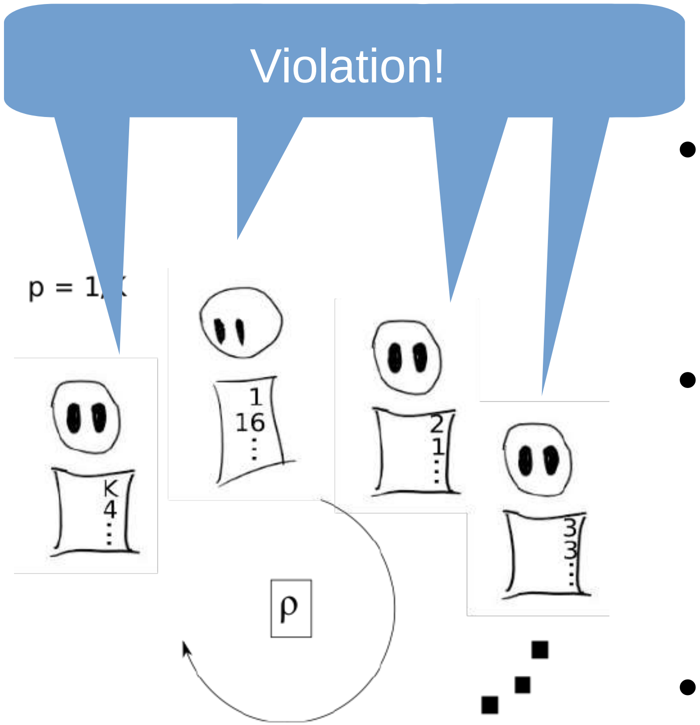

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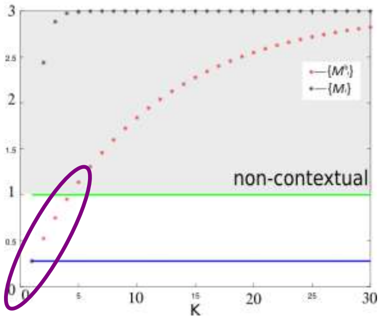

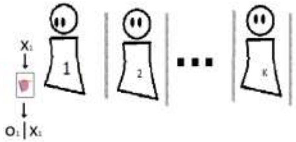

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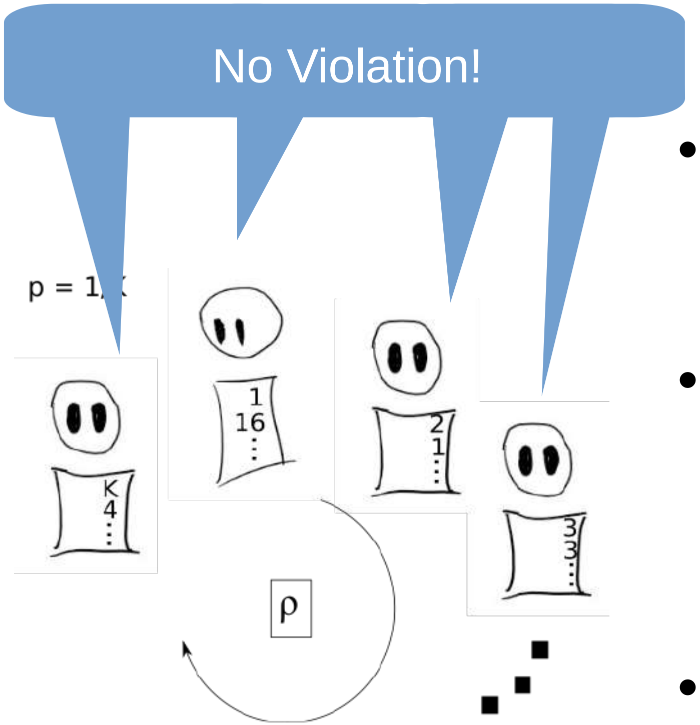

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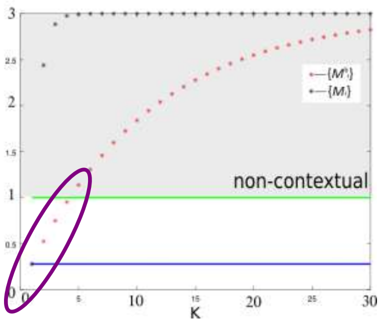

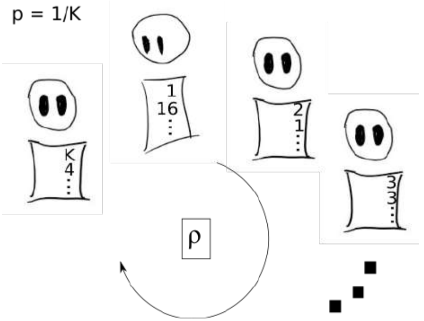

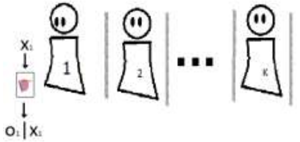

	$K_{\max}$ - Predef. Order			$K_{\max}$ - Unif. Distr.		
N	$M(\alpha,\beta)$	$M^a(\alpha)$	$M^b(\beta)$	$M(\alpha,\beta)$	$M^a(\alpha)$	$M^b(\beta)$
5	1	1	2	1	2	4
7	1	1	3	1	1	6
9	1	1	4	1	1	8
11	1	1	5	1	1	9
13	1	1	5	1	1	11

- Assymptotic limit without violation!
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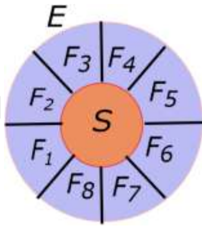
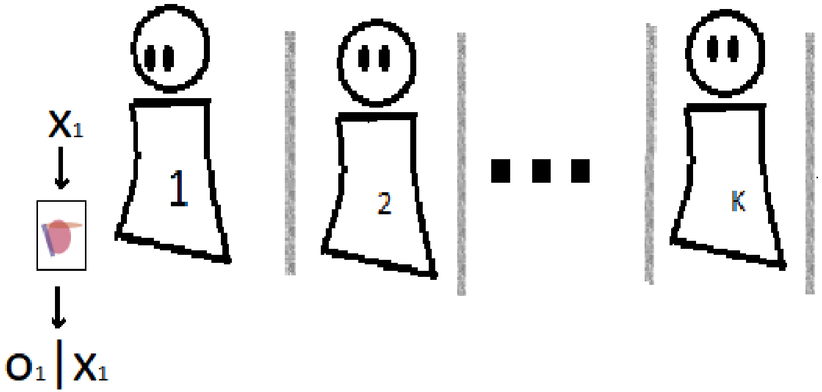
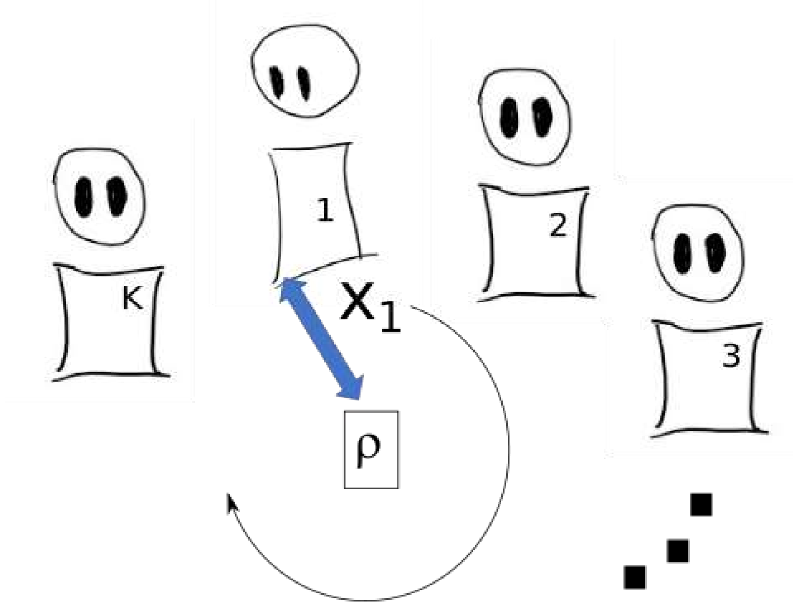
- Protocols behave differently

Interpreting as a classical limit



- Multiplayers setup as collisional models and multisystem environment





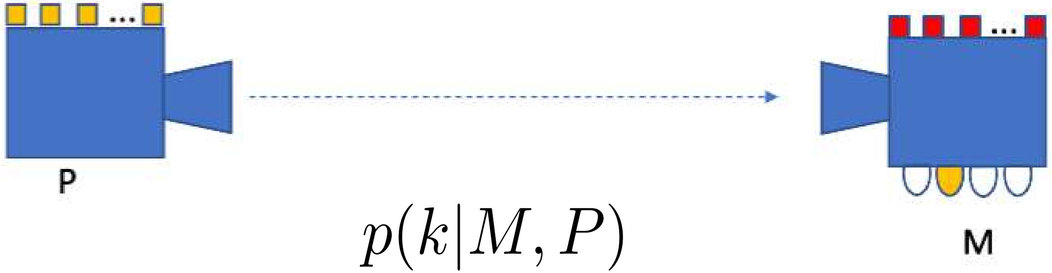
Spekkens Contextuality

Spekkens Contextuality

Operational Theory

Procedures
- Preparations,
- Transformations,
- Measurements;

Rule for obtaining behaviours;



$$p(k|M,P)$$

Ontological Model

Spekkens Contextuality

Operational Theory

Procedures
- Preparations,
- Transformations,
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Rule for obtaining behaviours;
Equivalences

$$P_1 \simeq P_2 \iff p(k|M,P_1) = p(k|M,P_2) \quad \forall k, M$$

Ontological Model

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$$P_1 \simeq P_2 \iff p(k|M, P_1) = p(k|M, P_2)$$
$$\forall k, M$$

$$[P], [k|M]$$

Ontological Model

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Quantum Theory

$$[P] \leftrightarrow [\rho_P]$$

$$[k|M] \leftrightarrow [E_k]$$

Ontological Model

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Ontological Model

$$P \mapsto \mu_P(\lambda)$$
$$k|M \mapsto \xi_M(k|\lambda)$$

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Ontological Model

$$P \mapsto \mu_P(\lambda)$$
$$k|M \mapsto \xi_M(k|\lambda)$$

$$\mu_P(\lambda) \geq 0 \; \forall\lambda \quad \int \mu_P(\lambda) d\lambda = 1$$
$$\xi_M(k,\lambda) \geq 0 \quad \sum_k \xi_M(k|\lambda) = 1 \; \forall\lambda$$

Spekkens Contextuality

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Equivalences

Ontological Model

$$P \mapsto \mu_P(\lambda)$$

$$k|M \mapsto \xi_M(k|\lambda)$$

$$p(k|M,P) = \int \mu_P(\lambda)\xi_M(k|\lambda)d\lambda$$

Spekkens Contextuality

Definition: a context is a label distinction between elements of a given equivalence class

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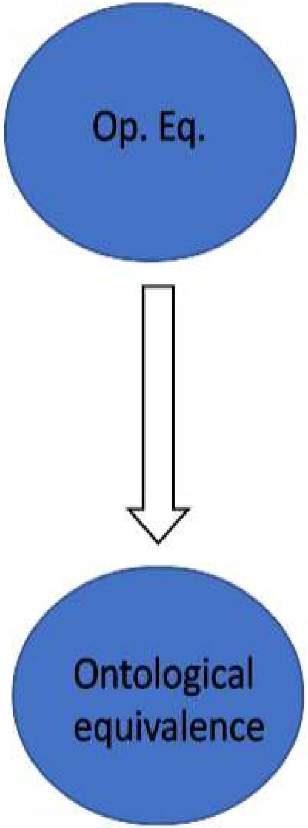
Noncontextual ontological model

Operational Theory



Ontological Models

R. Spekkens, Phys. Rev. A 71, 052108

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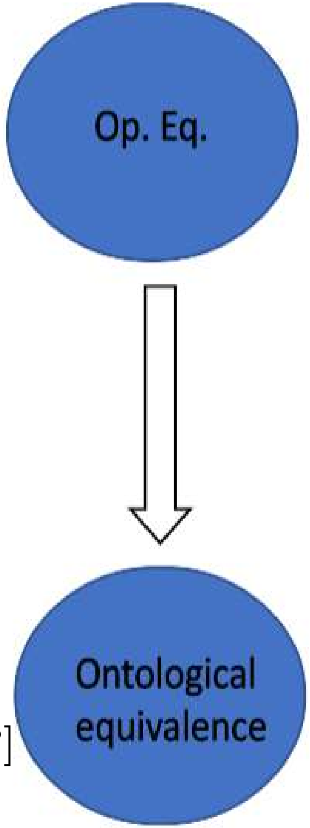
Definition: a context is a label distinction between elements of a given equivalence class

Noncontextual ontological model

Operational Theory

$$P_1, P_2 \in [P]$$



Ontological Models

$$\mu_{P_1} = \mu_{P_2} = \mu_{[P]}$$

R. Spekkens, Phys. Rev. A 71, 052108

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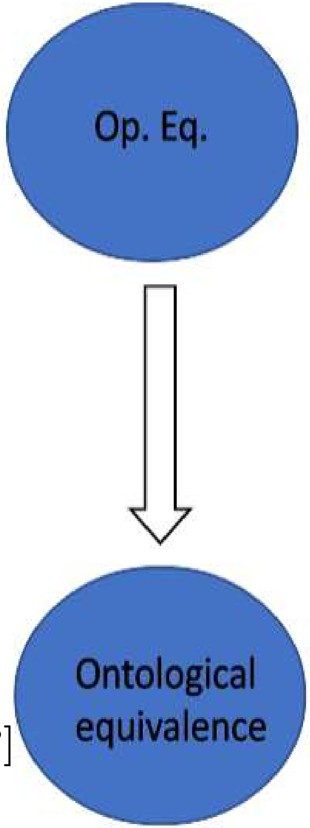


Quantum Theory

$$[\rho],\ [E_k]$$

Ontological Models

$$\mu_{P_1} = \mu_{P_2} = \mu_{[P]}$$

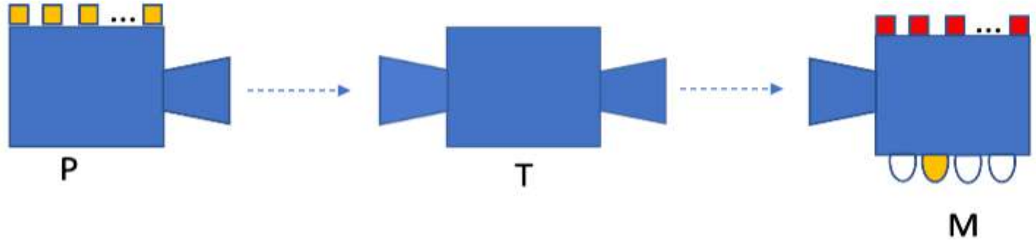
$$\mu_\rho,\ \xi_{E_k}$$

R. Spekkens, Phys. Rev. A 71, 052108

Spekkens Contextuality

Operational Theory

Procedures
- Preparations,
- Transformations,
- Measurements;



$$p(k|M, T, P)$$

Ontological Model

Spekkens Contextuality

Operational Theory

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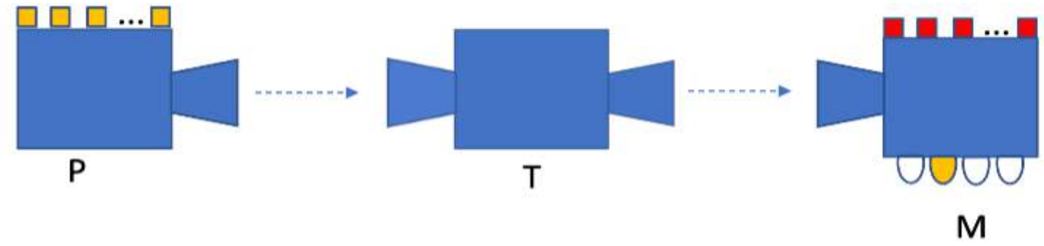
Rule for obtaining behaviours;



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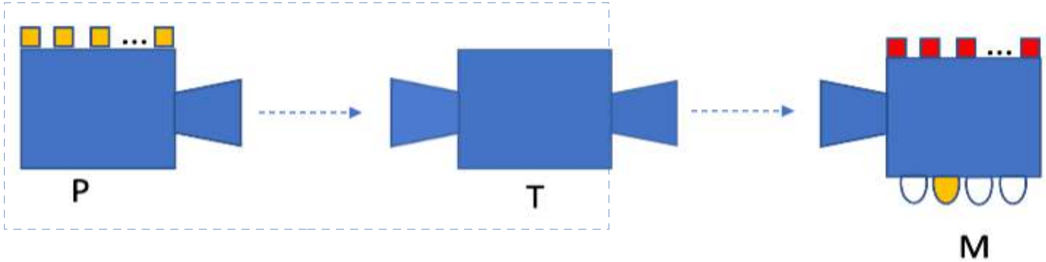

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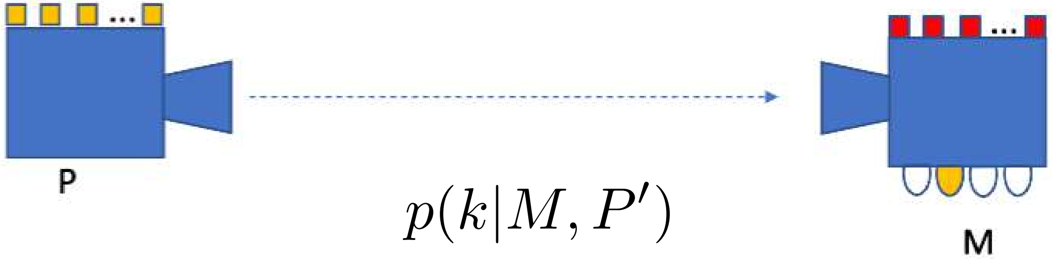

$$p(k|M, P')$$

$$P_1' \simeq P_2' \iff p(k|M, P_1') = p(k|M, P_2') \ \forall k, M,$$

Ontological Model

Spekkens Contextuality

Operational Theory

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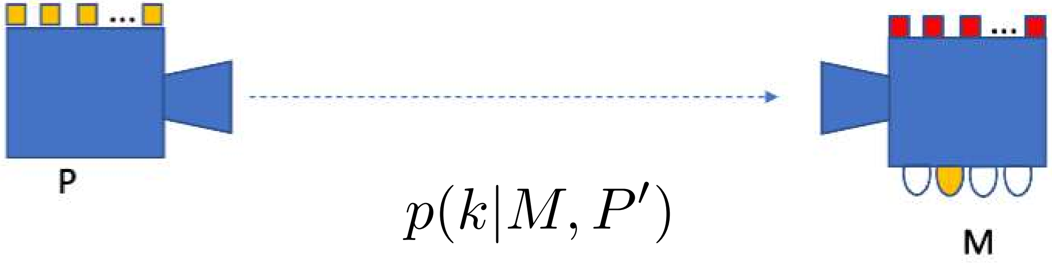

$$p(k|M,P')$$

$$P_1' \simeq P_2' \iff p(k|M,P_1') = p(k|M,P_2') \;\forall k, M,$$

$$\text{Equivalence classes } [P'],[k|M]$$

Ontological Model

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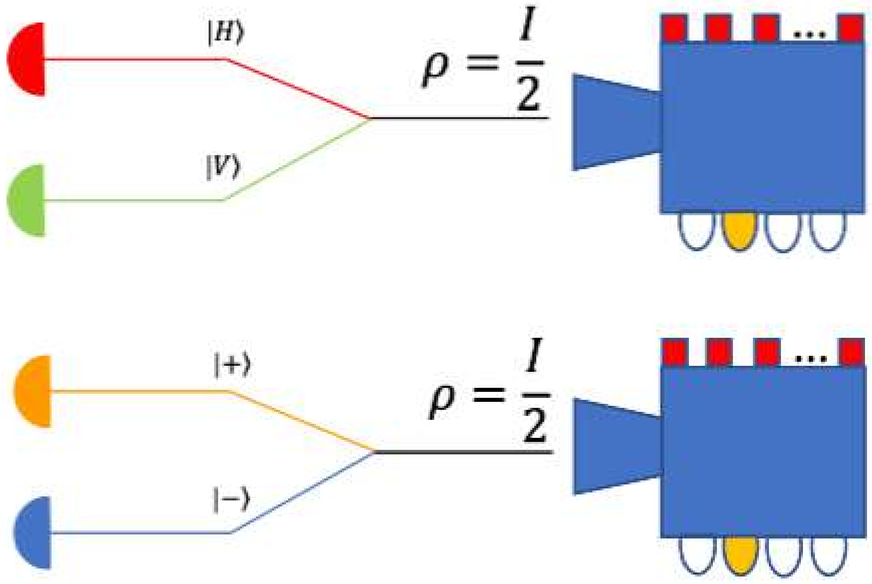

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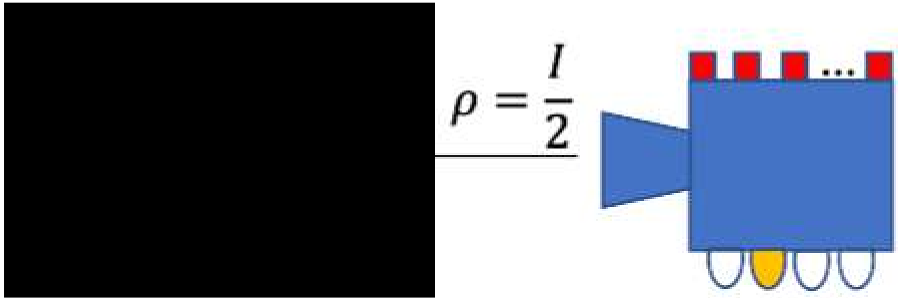

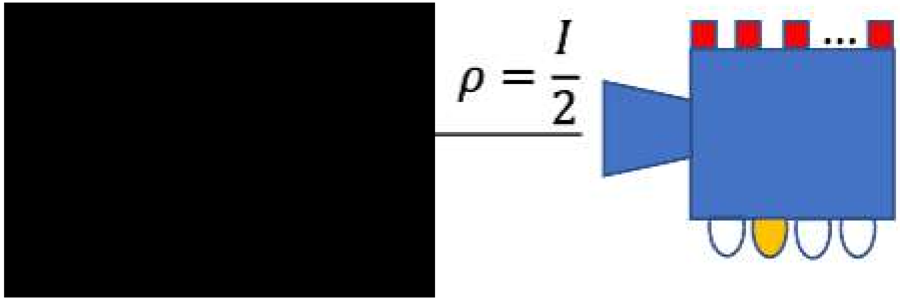

Operational Theory

Procedures

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Equivalences

Ontological Model

$$P'_1 \simeq P'_2 \iff p(k|M, P'_1) = p(k|M, P'_2)\ \forall k, M,$$

Appendix 2

BPH

BPH formal

Theorem 5 (Theorem 2 in ref. [15]). *Let* $\Phi^{S_t} : \mathcal{D}(\mathcal{H}_A) \to \mathcal{D}(\bigotimes_{j\in S_t} \mathcal{H}_{B_j})$ *be an EW t-dynamics, where* $S_t \subset \{1, \ldots, N\}$. *For every* $0 < \delta < 1$ *there exists a POVM* $\{\tilde{E}_k\}_k$ *such that for more than a* $(1-\delta)$ *fraction of the subsets* S_t,

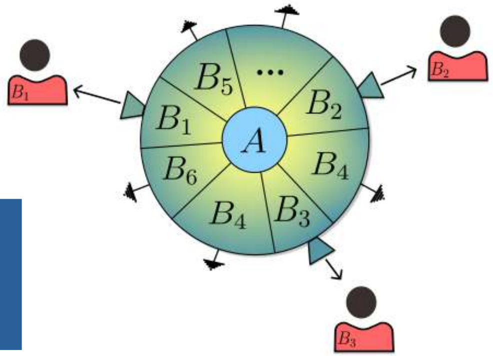
$$\left\|\Phi^{S_t} - \Phi^{S_t}_{obs}\right\|_\diamond \leq \left(\frac{27\ln(2)d_A^6 \log(d_A)t}{N\delta^3}\right)^{\frac{1}{3}}, \qquad \text{(A1)}$$

with $d_A \equiv \dim(\mathcal{H}_A)$, *and where* $\Phi^{S_t}_{obs}$ *is a measure-and-prepare map with respect to the family of states* $(\sigma_k^{S_t})_k$, *meaning that for all* $\rho \in \mathcal{D}(\mathcal{H}_A)$,

$$\Phi^{S_t}_{obs}(\rho) = \sum_k \mathrm{Tr}\{\tilde{E}_k\rho\}\sigma_k^{S_t}. \qquad \text{(A2)}$$

Quantum Darwinism – BPH's approach

*F. Brandão,M. Piani,P. Horodecki, Nat. Comm. 6 7908(2015)



Quantum Darwinism: the environment encodes information regarding system A

Prop.(adapt. from ref.*)

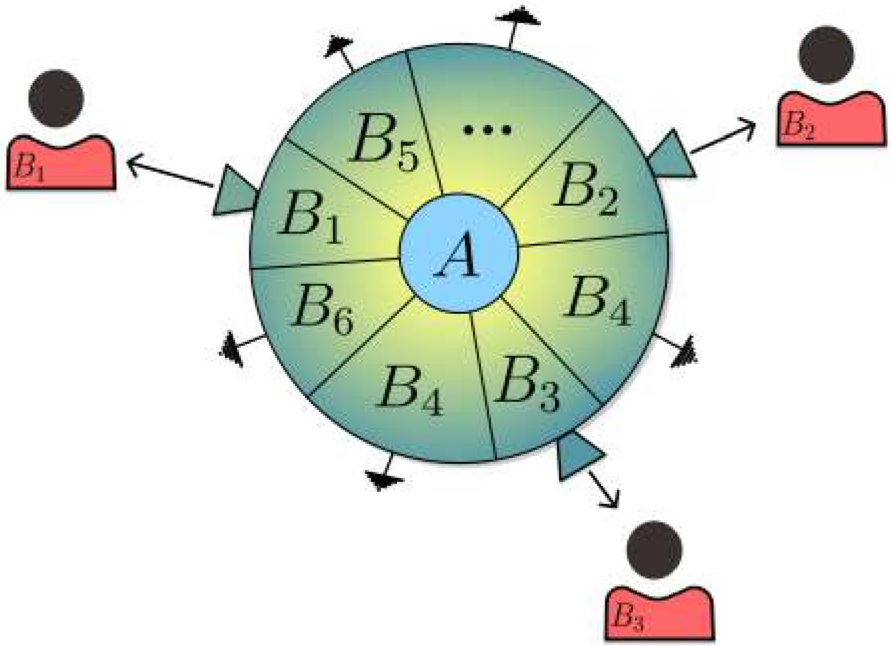
If a QD_η process occurs with high η, Bobs are likely to agree on the outcome they see, i.e.

$$\min_{\rho^A} \sum_k \tilde{p}_k \mathrm{Tr}\{\bigotimes_{j\in S_t} F_k^{B_j} \sigma_k^{B_{S_t}}\} \geq 1 - 6t\delta^{\frac{1}{4}},$$

where $\delta = 1 - \eta$

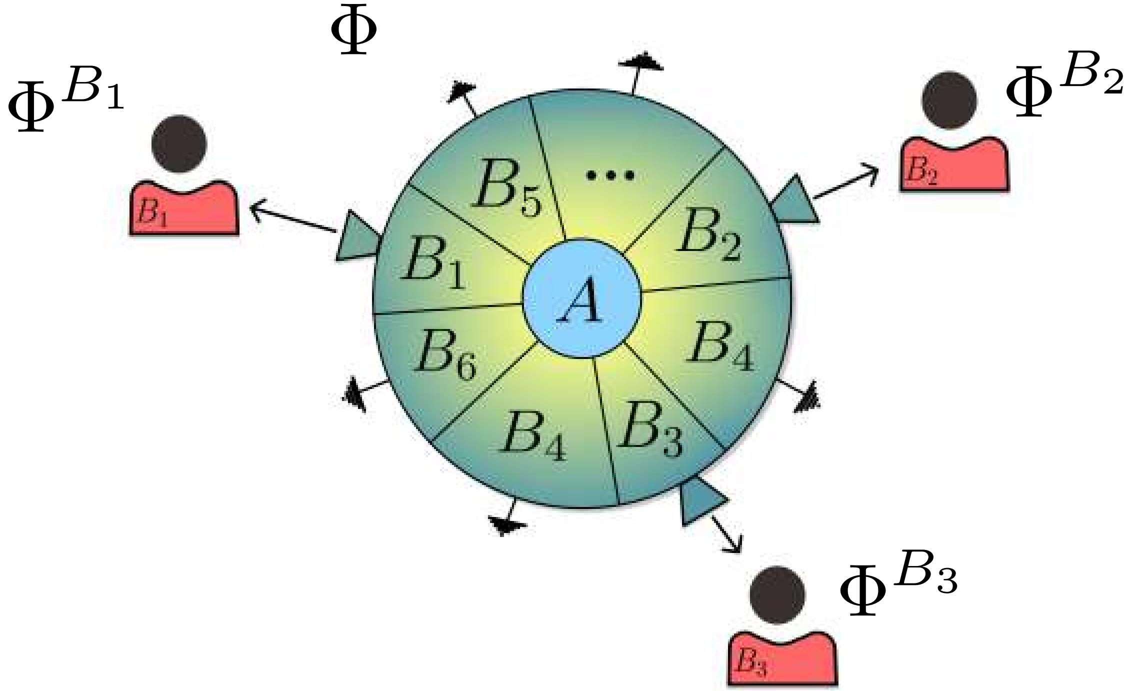
Emergence of noncontextuality in QD

Equivalences in QD settings



Equivalences in QD settings

Only Bobs' perspectives matter!



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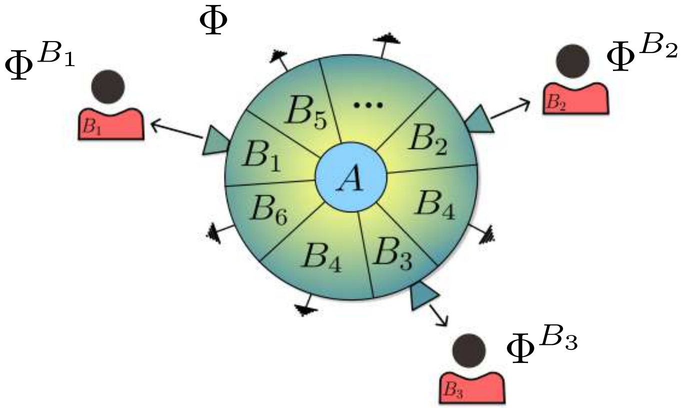

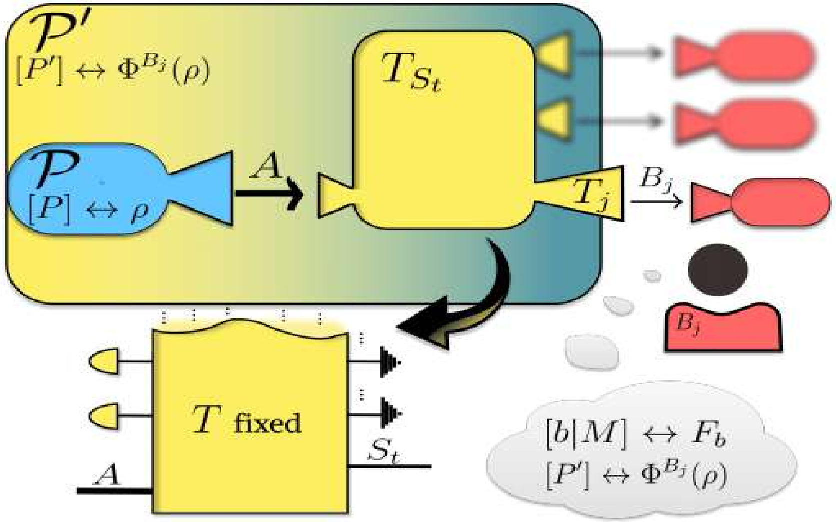

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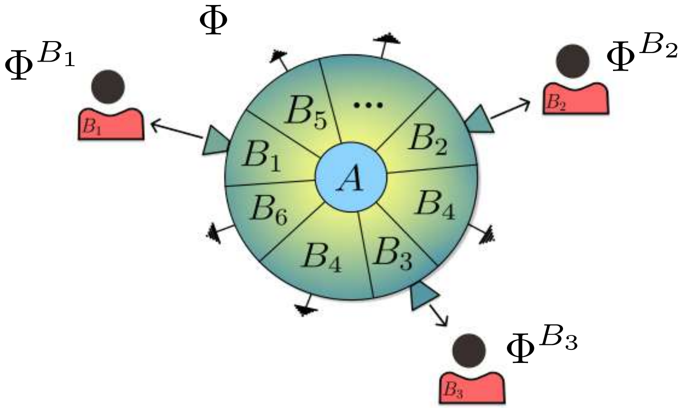

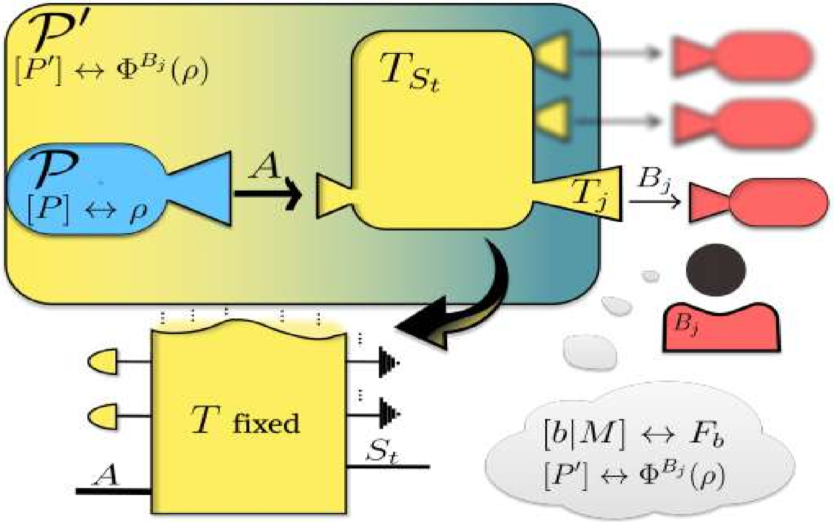

Equivalence classes for Bob B_j are $[\Phi^{B_j}(\rho^A)]$ and $[F_b]$

Results

Results

Equivalence classes for Bob B_j are $[\Phi^{B_j}(\rho^A)]$ and $[F_b]$

Lemma 1

Consider an EWt-dynamics and the reduced $(\Phi^{B_j})_j$.
For those $B_j \in B_{S_t}$ in which $(\sigma_k^{B_j})_k$ are affinely independent, there exists a noncontextual model.

RDB, R. Wagner, C. Duarte, B. Amaral, M. Terra Cunha, **Arxiv 2104.05734**

Results

Equivalence classes for Bob B_j are $[\Phi^{B_j}(\rho^A)]$ and $[F_b]$

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Proof.

$$P'_1 \simeq P'_2 \iff \sigma^{B_j}_{P'_1} = \sigma^{B_j}_{P'_2}$$

Results

Equivalence classes for Bob B_j are $[\Phi^{B_j}(\rho^A)]$ and $[F_b]$

Lemma 1

Consider an EWt-dynamics and the reduced $(\Phi^{B_j})_j$.
For those $B_j \in B_{S_t}$ in which $(\sigma_k^{B_j})_k$ are affinely independent, there exists a noncontextual model.

$$\sigma_P^{B_j} = \sum_k \tilde{p}_k \sigma_k^{B_j}$$

Proof.

$$P_1' \simeq P_2' \iff \sigma_{P_1'}^{B_j} = \sigma_{P_2'}^{B_j}$$

$$\iff \mathrm{Tr}\{\tilde{E}_k \rho_{P_1}^A\} = \mathrm{Tr}\{\tilde{E}_k \rho_{P_2}^A\}$$

Results

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$$p(b|M,P') = \sum_k \mu_{P'}(k)\xi_M(b|k)$$

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$$P_1' \simeq P_2' \iff \sigma^{B_j}_{P_1'} = \sigma^{B_j}_{P_2'}$$

$$\iff \mathrm{Tr}\{\tilde{E}_k \rho^A_{P_1}\} = \mathrm{Tr}\{\tilde{E}_k \rho^A_{P_2}\}$$

$$\Lambda := \{k\}_k;$$

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$\longrightarrow$ Noncontextual OM!

Results

Proof of Lemma 2

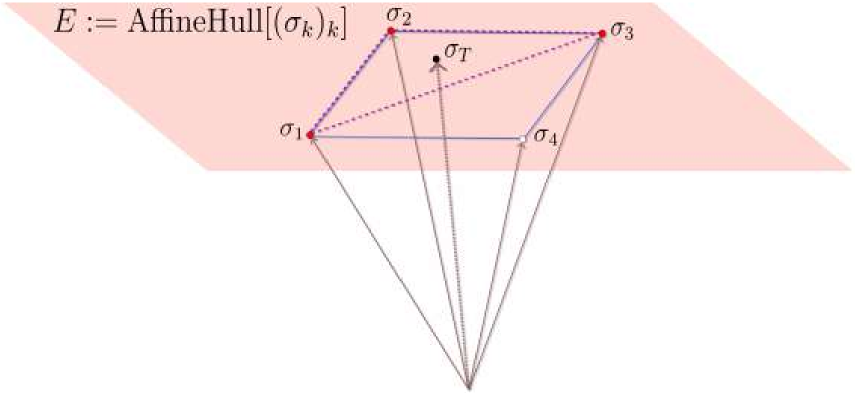
Definition 4 (Distinguishability Bound). *Consider a measure-and-prepare channel defined by* $(\{\tilde{E}_k, \sigma_k\})_k$, *with* $\tilde{E}_k \neq 0$ *for all* k. *Then, there will be states* $\rho^A \in \mathcal{D}(\mathcal{H}_A)$ *such that* $\mathrm{Tr}\{\tilde{E}_k \rho^A\} \neq 0$ *for all* k. *Denote the set of such states by* $\mathcal{S}$. *Now, assume (w.l.g.) that* $\mathrm{Tr}\{\tilde{E}_1 \rho^A\} \geq \mathrm{Tr}\{\tilde{E}_2 \rho^A\} \geq \ldots \geq \mathrm{Tr}\{\tilde{E}_{k_{\max}} \rho^A\} > 0$ *(otherwise, relabel* $(\tilde{E}_k)$ *so that it does). We define the* ***distinguishability bound*** $\hat{P}$ *as*

$$\hat{P}[(\tilde{E}_k)_k] := \min_{\rho^A \in \mathcal{S}} \left[\sum_{i=1}^{k_{\max}-1} \mathrm{Tr}\{\tilde{E}_k \rho^A\} + \frac{\mathrm{Tr}\{\tilde{E}_{k_{\max}} \rho^A\}}{2} \right]$$
$$= 1 - \frac{1}{2} \max_{\rho^A \in \mathcal{S}} \mathrm{Tr}\{\tilde{E}_{k_{\max}} \rho^A\}. \quad \text{(B1)}$$

Results

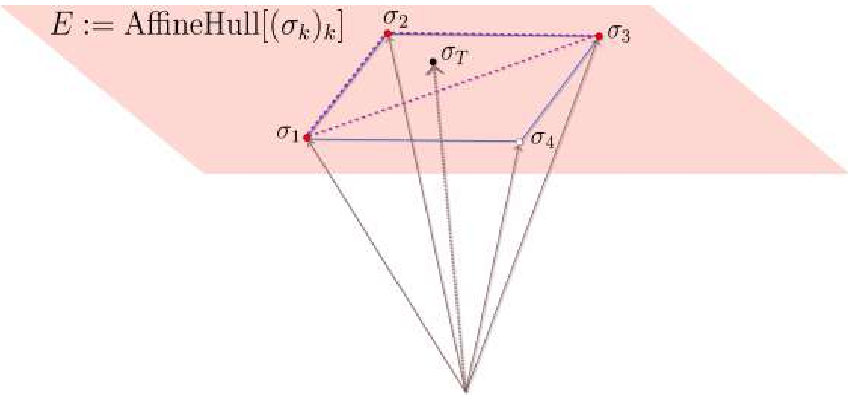
Proof of Lemma 2

Theorem 8 (Carathéodory, adapted from version of ref. [29]). *Given any affine space E of dimension n, for any (non-void) family $f = (\sigma_k)_{k=1}^{k_{\max}}$ in E, the set* $\mathrm{ConvHull}(f)$ *is equal to the set of convex combinations of families of $n+1$ points of f.*



Results

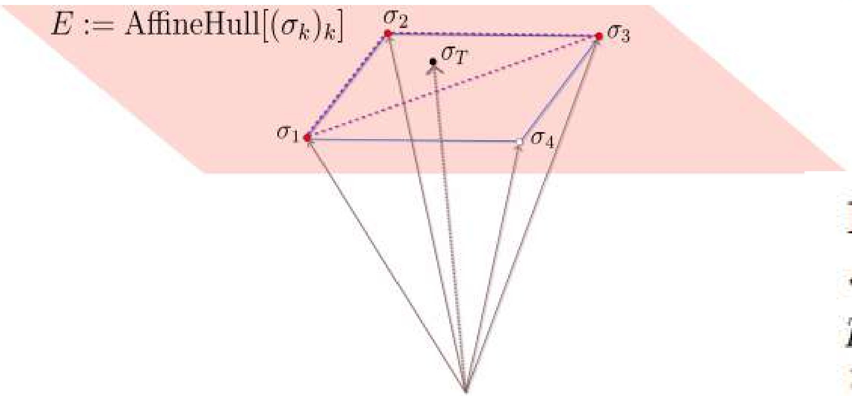
Proof of Lemma 2



Corollary 9. *Consider any state of the form* $\sigma_T = \sum_k \tilde{p}_k \sigma_k$ *with* $\tilde{p}_k > 0$ *for* $k \in \{1, \ldots, k_{\max}\}$*. Then, if* $(\sigma_k)_k$ *is an affinely dependent set, there exists a set of convex coefficients* $\{q_k\}_k$*, with* $q_k = 0$ *for at least one value of* k*, such that* $\sigma_T = \sum_k q_k \sigma_k$*.*

Results

Proof of Lemma 2



Corollary 9. *Consider any state of the form $\sigma_T = \sum_k \tilde{p}_k \sigma_k$ with $\tilde{p}_k > 0$ for $k \in \{1, \ldots, k_{\max}\}$. Then, if $(\sigma_k)_k$ is an affinely dependent set, there exists a set of convex coefficients $\{q_k\}_k$, with $q_k = 0$ for at least one value of k, such that $\sigma_T = \sum_k q_k \sigma_k$.*

Remark 1. *Assume $(\sigma_k)_k$ to be an affinely dependent set. Consider a state $\sigma_T = \sum_k \tilde{p}_k \sigma_k = \sum_k q_k \sigma_k$ with $\tilde{p}_1 \geq \tilde{p}_2 \ldots \geq \tilde{p}_{k_{\max}} > 0$ and $q_k = 0$ for some k. Define the statistical distance between $\{q_k\}_k$ and $\{\tilde{p}_k\}_k$ as*

$$D(\{q_k\}) = \frac{1}{2} \sum_k |q_k - \tilde{p}_k|. \tag{B2}$$

Then, $D(\{\tilde{p}_k\}, \{q_k\}) \geq \tilde{p}_{k_{\max}}$.

Results

Proof of Lemma 2

Lemma 10 (Violation of $p_{\text{guess}} \leq \hat{P}$ for all ρ^A implies $(\sigma_k)_k$ is an affinely independent set). *Consider an EW dynamics defined by the measure-and-prepare channel $(\tilde{E}_k, \sigma_k)_k$, with $\tilde{E}_k \neq 0$ for all k. Consider the associated distinguishability bound $\hat{P}[(\tilde{E}_k)_k]$ as per definition 4. Then, if for all $\rho^A \in \mathcal{D}(\mathcal{H}_A)$ the inequality*

$$p_{\text{guess}}[(\text{Tr}\{\tilde{E}_k \rho^A\}, \sigma_k)_k] > \hat{P}[(\tilde{E}_k)_k] \tag{B3}$$

holds, the states $(\sigma_k)_k$ are affinely independent.

$$\text{Tr}\{F_k^* \sigma_k\} > \frac{1}{2} \; \forall k$$

$$p_{\text{guess}}[(\tilde{p}_k, \sigma_k)_k] = \sum_k \tilde{p}_k \text{Tr}\{F_k^* \sigma_k\}$$

$$\leq \sum_{k \neq b} \tilde{p}_k \overbrace{\text{Tr}\{F_k^* \sigma_k\}}^{\leq 1} + \frac{\tilde{p}_b}{2}$$

$$\leq \sum_{k \neq b} \tilde{p}_k + \frac{\tilde{p}_b}{2} \leq \sum_{k \neq k_{\max}} \tilde{p}_k + \frac{\tilde{p}_{k_{\max}}}{2} = \hat{P},$$

Results

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Lemma 10 (Violation of $p_{\text{guess}} \leq \hat{P}$ for all ρ^A implies $(\sigma_k)_k$ is an affinely independent set)**.** *Consider an EW dynamics defined by the measure-and-prepare channel $(\tilde{E}_k, \sigma_k)_k$, with $\tilde{E}_k \neq 0$ for all k. Consider the associated distinguishability bound $\hat{P}[(\tilde{E}_k)_k]$ as per definition 4. Then, if for all $\rho^A \in \mathcal{D}(\mathcal{H}_A)$ the inequality*

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holds, the states $(\sigma_k)_k$ are affinely independent.

$$P_{\text{err}} = \sum_k \sum_{b \neq k} \tilde{p}_b \text{Tr}\{F_k^* \sigma_b\}$$

$$\begin{aligned}
P_{\text{err}} &= \sum_k \overbrace{\text{Tr}\{F_k^*(\sigma_T - \tilde{p}_k \sigma_k)\}}^{\geq 0} \\
&\geq \sum_{k \in K} \text{Tr}\{F_k^*(\sigma_T - \tilde{p}_k \sigma_k)\} \\
&= \sum_{k \in K} \text{Tr}\{F_k^*(\sum_i q_i \sigma_i - \tilde{p}_k \sigma_k)\} \\
&\geq \sum_{k \in K} \text{Tr}\{F_k^*(q_k \sigma_k - \tilde{p}_k \sigma_k)\}.
\end{aligned}$$

Results

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$$p_{\text{guess}}[(\text{Tr}\{\tilde{E}_k \rho^A\}, \sigma_k)_k] > \hat{P}[(\tilde{E}_k)_k] \qquad \text{(B3)}$$

holds, the states $(\sigma_k)_k$ *are affinely independent.*

$$\begin{aligned} P_{\text{err}} &\geq \sum_{k \in K} (q_k - \tilde{p}_k) \overbrace{\text{Tr}\{F_k^* \sigma_k\}}^{>\frac{1}{2}} > \frac{1}{2} \sum_{k \in K} (q_k - \tilde{p}_k) \\ &= \frac{1}{2} D(\{\tilde{p}_k\}, \{q_k\}) \geq \frac{p_{k_{\max}}}{2}, \end{aligned} \qquad \text{(B9)}$$

Results

Proof of relaxation

$$\mathbf{d}(p) := \min_{q \in NC(\mathfrak{S}_j)} \max_{\substack{M \in \mathcal{M}^{B_j} \\ P' \in \mathcal{P}'}} \sum_b |p(b|M,P') - q(b|M,P')|.$$

$$\mathbf{d}(p) \leq \max_{\substack{M \in \mathcal{M}^{B_j} \\ P' \in \mathcal{P}'}} \sum_b |p(b|M,P') - q_*(b|M,P')|.$$

$$\begin{aligned}\mathbf{d}(p) &\leq \max_{\substack{M \in \mathcal{M}^{B_j} \\ P \in \mathcal{P}}} \sum_b \left| \mathrm{Tr}\{F_b^M \Phi^{B_j}(\rho_P)\} - \mathrm{Tr}\{F_b^M \Phi_{obs}^{B_j}(\rho_P)\} \right| \\ &= \max_{\substack{M \in \mathcal{M}^{B_j} \\ P \in \mathcal{P}}} \sum_b \left| \mathrm{Tr}\{F_b^M (\Phi^{B_j} - \Phi_{obs}^{B_j})(\rho_P)\} \right| \\ &= \max_{\substack{M \in \mathcal{M}^{B_j} \\ P \in \mathcal{P}}} \sum_b \left| \mathrm{Tr}\{F_b^M \mathcal{R}_j(\rho_P)\} \right|.\end{aligned}$$

Results

Proof of relaxation

Lemma 6 (Hölder's inequality). *Let A, B be any $n \times n$ complex matrices. Then,*

$$\left|\mathrm{Tr}\, A^\dagger B\right| \le (\mathrm{Tr}\,|A|^l)^{\frac{1}{l}} (\mathrm{Tr}\,|B|^s)^{\frac{1}{s}} \tag{A4}$$

such that $1 \le l, s \le \infty$ *with* $\frac{1}{l} + \frac{1}{s} = 1$.

$$\left|\mathrm{Tr}\{F_b^M \mathcal{R}_j(\rho_P)\}\right| \le \sqrt{\mathrm{Tr}\{|\mathcal{R}_j(\rho_P)|^2\}} \sqrt{\mathrm{Tr}\{|F_b^M|^2\}}$$

$$\begin{aligned}
\mathbf{d}(p) &\le \max_{\substack{M \in \mathcal{M}^{B_j} \\ P \in \mathcal{P}}} \sum_b \left|\mathrm{Tr}\{F_b^M \mathcal{R}_j(\rho_P)\}\right| \\
&\le \max_{\substack{M \in \mathcal{M}^{B_j} \\ P \in \mathcal{P}}} \sum_b \sqrt{\mathrm{Tr}\{|\mathcal{R}_j(\rho_P)|^2\}} \sqrt{\mathrm{Tr}\{|F_b^M|^2\}} \\
&= \max_{M \in \mathcal{M}^{B_j}} \sum_b \sqrt{\mathrm{Tr}\{|F_b^M|^2\}} \max_{P \in \mathcal{P}} \sqrt{\mathrm{Tr}\{|\mathcal{R}_j(\rho_P)|^2\}} \\
&= C \max_{P \in \mathcal{P}} \sqrt{\mathrm{Tr}\{|\mathcal{R}_j(\rho_P)|^2\}}
\end{aligned}$$

Results

Proof of relaxation

After relating the 2-norm to the 1-norm,...

$$\begin{aligned}\mathbf{d}(p) &\leq \frac{C}{d_A} \max_{\mathbb{1}_A \otimes \rho \in \mathcal{D}(\mathcal{H}_A^{\otimes 2})} \left\| \mathrm{id}_A \otimes \left(\Phi^{B_j} - \Phi^{B_j}_{obs} \right) (\mathbb{1}_A \otimes \rho) \right\|_1 \\ &\leq \frac{C}{d_A} \max_{\sigma \in \mathcal{D}(\mathcal{H}_A^{\otimes 2})} \left\| \mathrm{id}_A \otimes \left(\Phi^{B_j} - \Phi^{B_j}_{obs} \right) (\sigma) \right\|_1 \\ &= \frac{C}{d_A} \left\| \Phi^{B_j} - \Phi^{B_j}_{obs} \right\|_\diamond .\end{aligned}$$