Local certification of programmable quantum devices of arbitrary high dimensionality

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(Joint work with M. Ray, A. Varvitsiotis, A. Cabello, L.C. Kwek)

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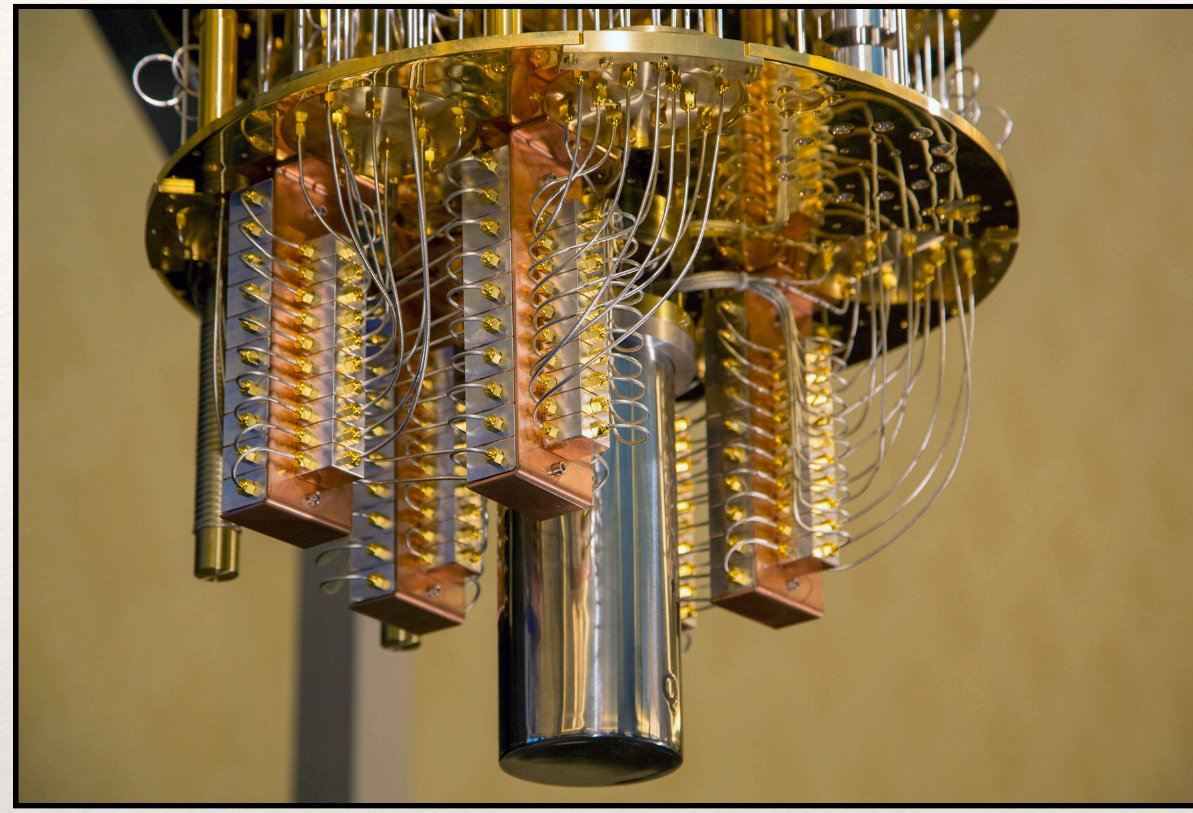
The 4th workshop on Quantum Contextuality in Quantum Mechanics and Beyond (QCQMB)

Motivation

Goal: To characterise untrusted devices



Bell non-locality based *self-testing* can be employed to characterise quantum devices via measurement statistics.

Computation typically happens in a local fashion!

Is it possible to extend the notion of self-testing to *local* contextuality scenarios?

Results

- A graph-theoretic framework to render local self-testing statements.
- The generalized KCBS inequalities admit robust self-testing.

Robust Self-Testing of Quantum Systems via Noncontextuality Inequalities

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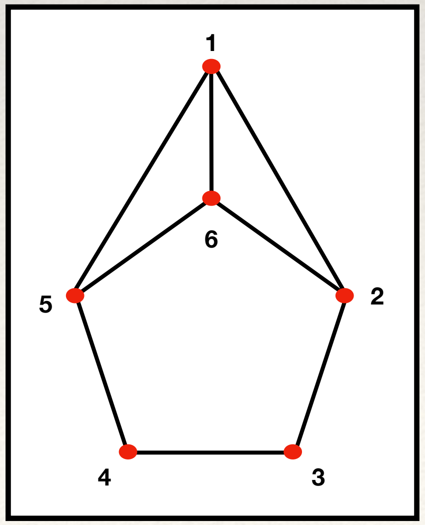
Results

- The non-contextuality inequalities corresponding to the family of odd anti-cyclic graphs with at least five vertices are self-testable.
- Given an anti-cycle non-contextuality inequality with an odd number of n measurement events, the quantum system achieving the optimal quantum bound must be at least (n−2) dimensional.
- Not all graphs with a non-zero gap between NCHV bound (given by the independence number) and the maximum quantum bound (given by the Lovász theta number) for the corresponding canonical non-contextuality inequality admits self-testing.

Local certification of programmable quantum devices of arbitrary high dimensionality

Kishor Bharti,[1,*] Maharshi Ray,[1,*] Antonios Varvitsiotis,[2] Adán Cabello,[3,4] and Leong-Chuan Kwek[1,5,6]



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Results

1. Presented a graph-theoretic framework to render Bell scenario-based self-testing statements.
2. Recovered some old self-testing statements in our framework.
3. Additionally presented a self-testing statement for a previously unknown case.
4. In the process of proving self-testing statements, we furnished proof for a conjecture in discrete mathematics employing ideas from quantum foundations

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Graph-Theoretic Framework for Self-Testing in Bell Scenarios

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Robust Self Testing in Bell Scenario

- Self Testing: Given access to only the statistics of a quantum test, the measurement settings and state can be uniquely determined up to local isometry.

$$\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle \leq 2$$

± 1 $\qquad\qquad\qquad\qquad$ ± 1

$$Q \leftarrow Z \otimes I \qquad\qquad I \otimes \frac{-Z - X}{\sqrt{2}} \rightarrow S$$

$$R \leftarrow X \otimes I \qquad\qquad I \otimes \frac{Z - X}{\sqrt{2}} \rightarrow T$$

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

$$\text{Quantum bound} = 2\sqrt{2}$$

If a CHSH test achieves the quantum bound, it admits self testing.

Robust Self Testing in Bell Scenario

Bell Inequality

$$\sum_{a,b,x,y} B_{xy}^{ab} p(ab|xy) \leq B_l$$

Geometrically, a Bell inequality corresponds to a hyperplane that separates the set of local behaviours from nonlocal behaviours.

$$p(ab|xy) = \langle\psi| A_{x|a} \otimes B_{y|b} |\psi\rangle$$

$$|\psi\rangle \in \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$$

Quantum Behaviours

We will denote the maximum quantum value by B_q

Robust Self Testing in Bell Scenario

Self-Testing

$\left(\mathcal{H}_{\mathcal{A}}, \mathcal{H}_{\mathcal{B}}, |\psi\rangle, \{A_{x|a}\}, \{B_{y|b}\}\right)$ $\left(\mathcal{H}'_{\mathcal{A}}, \mathcal{H}'_{\mathcal{B}}, |\psi'\rangle, \{A'_{x|a}\}, \{B'_{y|b}\}\right)$ B_q

$V = V_A \otimes V_B$ $|junk\rangle$

$$V|\psi'\rangle = |junk\rangle \otimes |\psi\rangle,$$
$$V(A'_{x|a} \otimes B'_{y|b})|\psi'\rangle = |junk\rangle(A_{x|a} \otimes B_{y|b})|\psi\rangle$$

Robust Self Testing in Bell Scenario

Robust Self-Testing

$$\left(\mathcal{H}_{\mathcal{A}}, \mathcal{H}_{\mathcal{B}}, |\psi\rangle, \{A_{x|a}\}, \{B_{y|b}\}\right) \qquad \left(\mathcal{H}'_{\mathcal{A}}, \mathcal{H}'_{\mathcal{B}}, |\psi'\rangle, \{A'_{x|a}\}, \{B'_{y|b}\}\right)$$

$$B_q \qquad B_q - \epsilon$$

$$\|V(A'_{x|a} \otimes B'_{y|b})|\psi'\rangle - |junk\rangle(A_{x|a} \otimes B_{y|b})|\psi\rangle\| \leq \mathcal{O}(\epsilon^r)$$

Contextuality

Context: a set of compatible observables

$$\mathbb{A}, \mathbb{B}, \mathbb{C}$$

- **Context 1:** $\{\mathbb{A}, \mathbb{B}\}$
- **Context 2:** $\{\mathbb{A}, \mathbb{C}\}$

Given a theory, if the value assigned to an observable does not depend on the context in which it has been measured, then the theory is called non-contextual. Otherwise, the theory is contextual.

Quantum mechanics is a contextual theory. **KS67**

Exclusivity Graph Approach

CSW14

Measurement events: $e_1, \ldots, e_n$

Mutually exclusive events: Same measurement but correspond to different outcomes

Exclusivity graph:

Events denoted by nodes

Exclusive events share an edge

Example:

$$e_1 \sim e_2$$

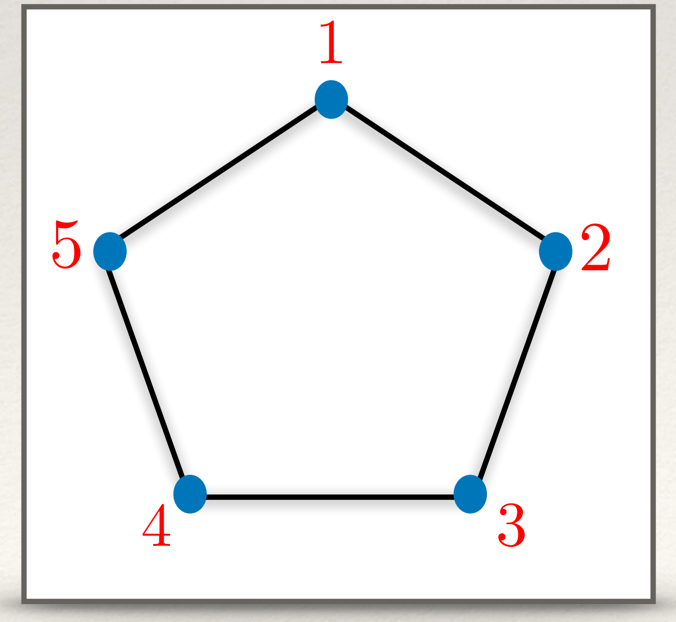
$$e_2 \sim e_3$$

$$e_3 \sim e_4$$

$$e_4 \sim e_5$$

$$e_5 \sim e_1$$

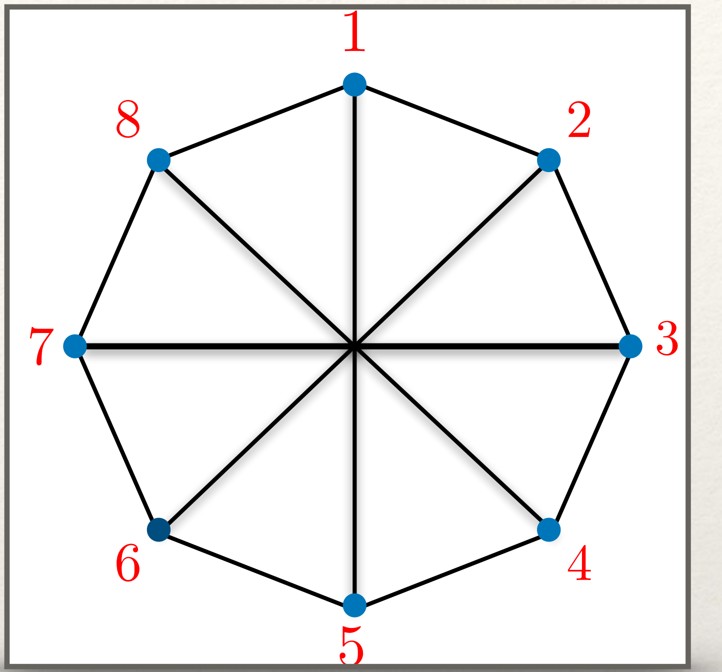
$\sim$: Exclusivity



Exclusivity Graph Approach

CSW14

Another example: Exclusivity Graph for CHSH Scenario



$$\{1, 2, \cdots n\} \equiv [n]$$

$$p : [n] \to [0, 1]$$

Behaviour

$$p_i + p_j \leq 1 \quad i \sim j$$

A deterministic noncontextual behavior p is a mapping $p : [n] \to \{0, 1\}$, where $p_i + p_j \leq 1$, for all $i \sim j$. The polytope of noncontextual behaviors, denoted by $B_{nc}(G)$, is the convex hull of all deterministic noncontextual behaviors. Behaviors that do not lie in $B_{nc}(G)$ are contextual.

$$\sum_{i \in [n]} w_i p_i \leq B_{nc} \qquad p \in B_{nc}(G) \qquad w \in \mathbb{R}$$

Non-contextuality inequalities

Exclusivity Graph Approach

CSW14

Quantum Behaviour

$$p : [n] \to [0,1]$$

ρ Quantum description for the preparation

$\Pi_1, \ldots \Pi_n$ Projectors acting the Hilbert space $\mathcal{H}$

$$p_i = \mathrm{Tr}(\rho\Pi_i), \forall i \in [n] \text{ and } \mathrm{Tr}(\Pi_i\Pi_j) = 0, \text{ for } i \sim j$$

$\rho, \{\Pi\}_{i=1}^{n}$ Ensemble for the behaviour p

$B_q(G)$ The set of quantum behaviours

B_{qc} The quantum value corresponding to the non-contextuality inequality

Robust Self-Testing

Self-Testing

$$\{|u_i\rangle\}_{i=0}^{n} \qquad \{|u'_i\rangle\}_{i=0}^{n} \qquad B_{qc}$$

$$V|u_i\rangle\langle u_i|V^\dagger = |u'_i\rangle\langle u'_i| \quad 0 \leq i \leq n$$

Robustness

$$\sum_{i=1}^{n} |\langle u'_i||u'_0\rangle|^2 \geq B_{qc} - \epsilon$$

$$\| V|u_i\rangle\langle u_i|V^\dagger - |u'_i\rangle\langle u'_i| \| \leq \mathcal{O}\left(\epsilon^r\right), \quad 0 \leq i \leq n$$

How to Show Robust Self-Testing

$$\max\left\{\sum_{i=1}^{n} w_i p_i : p \in B_q(G)\right\}$$

$$\begin{aligned} \vartheta(G,w) = \max \quad & \sum_{i=1}^{n} w_i X_{ii} \\ \text{subject to} \quad & X_{ii} = X_{0i}, \quad 1 \le i \le n, \\ & X_{ij} = 0, \quad i \sim j, \\ & X_{00} = 1, \quad X \in \mathcal{S}_+^{1+n} \end{aligned}$$

Primal

$$\min t \ge 0$$

$$tE_{00} + \sum_{i=1}^{n}(\lambda_i - w_i)E_{ii} - \sum_{i=1}^{n}\lambda_i E_{0i} + \sum_{i\sim j}\mu_{ij}E_{ij} \succeq 0$$

Dual

How to Show Robust Self-Testing

Consider a noncontextuality inequality $\sum_{i=1}^{n} w_i p_i \leq B_{nc}$, and let $\{|v_i\rangle\}_{i=0}^{n}$ be a quantum ensemble achieving the corresponding quantum value B_{qc}, and moreover $\langle v_0|v_i\rangle \neq 0$, for all $1 \leq i \leq n$. Say that there exists a dual optimal solution Z^* for the SDP such that the homogeneous linear system

$$MZ^* = \langle M, E_{0i}\rangle = \langle M, E_{ii}\rangle = \langle M, E_{ij}\rangle = 0,$$

in the symmetric matrix variable M only admits the trivial solution $M = 0$. Then, the noncontextuality inequality is an $(\epsilon, \frac{1}{2})$-robust self-test for $\{|v_i\rangle\}_{i=0}^{n}$.

Proof Technique

Non-contextuality Inequality ⟷ Exclusivity Graph

$$\vartheta(G) = \max\left\{\sum_{i=1}^{n} X_{ii} : \; X \in \mathbb{S}_{+}^{1+n}, \; X_{00} = 1, \; X_{ii} = X_{0i}, \; X_{ij} = 0, \forall i,j \in E\right\}$$

$$\mathrm{TH}(G) = \{x \in \mathbb{R}_{+}^{n} : X \in \mathbb{S}_{+}^{1+n}, X_{i,i} = x_i, X_{00} = 1, \; X_{ii} = X_{0i}, \; X_{ij} = 0, \forall ij \in E\}$$

↓

Uniqueness of optimal X

↓

Error bound analysis for SDPs

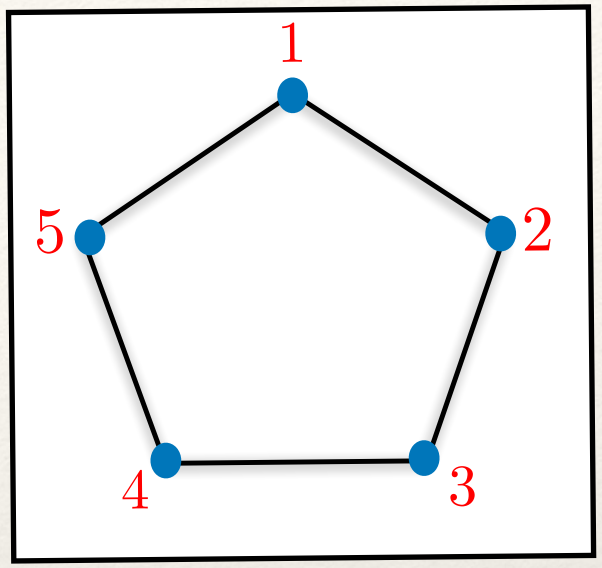
↓

The non-contextuality inequality admits robust self testing.

Example: Generalized KCBS Inequalities

Odd number of measurement events $e_1, \ldots, e_n$

e_i and e_{i+1} are exclusive, where indices are taken modulo n



$$\max\left\{\sum_{i=1}^{n} p_i : p \in B_{nc}(C_n)\right\} = \frac{(n-1)}{2}$$

$$\max\left\{\sum_{i=1}^{n} p_i : p \in B_q(C_n)\right\} = \frac{n\cos\pi/n}{1+\cos\pi/n}$$

$$p_i^{(n)} = \frac{\cos\pi/n}{1+\cos\pi/n}, \; 1 \le i \le n$$

For any odd integer n, the $KCBS_n$ inequality is an $(\epsilon, \frac{1}{2})$-robust self-test for the ensemble $|v_0\rangle = (1,0,0)$ and $|v_j\rangle = (\cos(\theta), \sin(\theta)\sin(\phi_j), \sin(\theta)\cos(\phi_j))$, where $\cos^2(\theta) = \frac{\cos(\pi/n)}{1+\cos(\pi/n)}$ and $\phi_j = \frac{j\pi(n-1)}{n}$ for $1 \le j \le n$.

Anti-Cycles

Theorem: For any odd *n*, the non-contextuality inequality corresponding to the **anti-*n*-cycle** graph admits $\left(\epsilon, \frac{1}{2}\right)$-robust *self-test.*

Strategy

Idea : Relate primal solution of cycles to dual solution of anti-cycles.

P_{C_n} (Primal : Cycle)

$P_{\overline{C_n}}$ (Primal : Anti-cycle)

D_{C_n} (Dual : Cycle)

$D_{\overline{C_n}}$ (Dual : Anti-cycle)

Theorem : Let $X^* = \text{Gram}(v_0, v_1, \cdots, v_n)$ be the unique optimal solution for P_{C_n}. Then, $Z^* = \vartheta(\overline{C_n})\text{Gram}(-v_0, v_1, \cdots, v_n)$ is a dual optimal solution for $D_{\overline{C_n}}$.

Z^* can also be expressed as : $Z^* = \left[\begin{array}{c|c} \vartheta(\overline{C_n}) & -e^T \\ \hline -e & circ(u) \end{array}\right]$

where $u = (1, \vartheta(\overline{C_n})\langle v_1 \mid v_2 \rangle, \cdots, \vartheta(\overline{C_n})\langle v_1 \mid v_n \rangle)$

Application - Certify high dim

Claim : For all odd *n,* the dimension in which the quantum realisations corresponding to the anti-*n*-cycle graph achieves the maximum (Lovasz theta) is *n*-2.

Idea :

$$X^* = \left[\begin{array}{c|c} 1 & \frac{\bar{\vartheta}_n}{n} e^T \\ \hline \frac{\bar{\vartheta}_n}{n} e & circ(u) \end{array}\right] \quad \text{with} \quad u = \left(\frac{\vartheta(\overline{C_n})}{n}, \frac{n-\vartheta(C_n)}{2\vartheta(C_n)^2}, 0, 0, \cdots, 0, 0, \frac{n-\vartheta(C_n)}{2\vartheta(C_n)^2}\right)$$

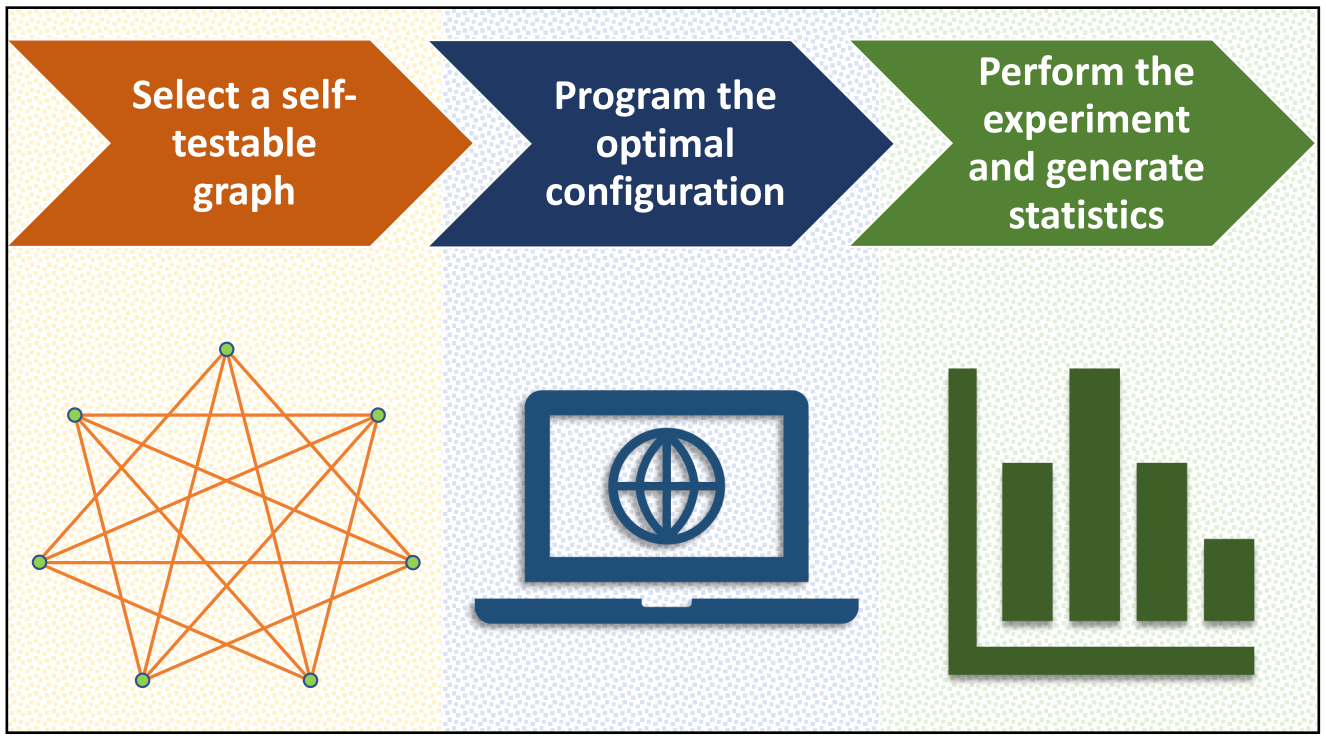
is the unique primal optimal.

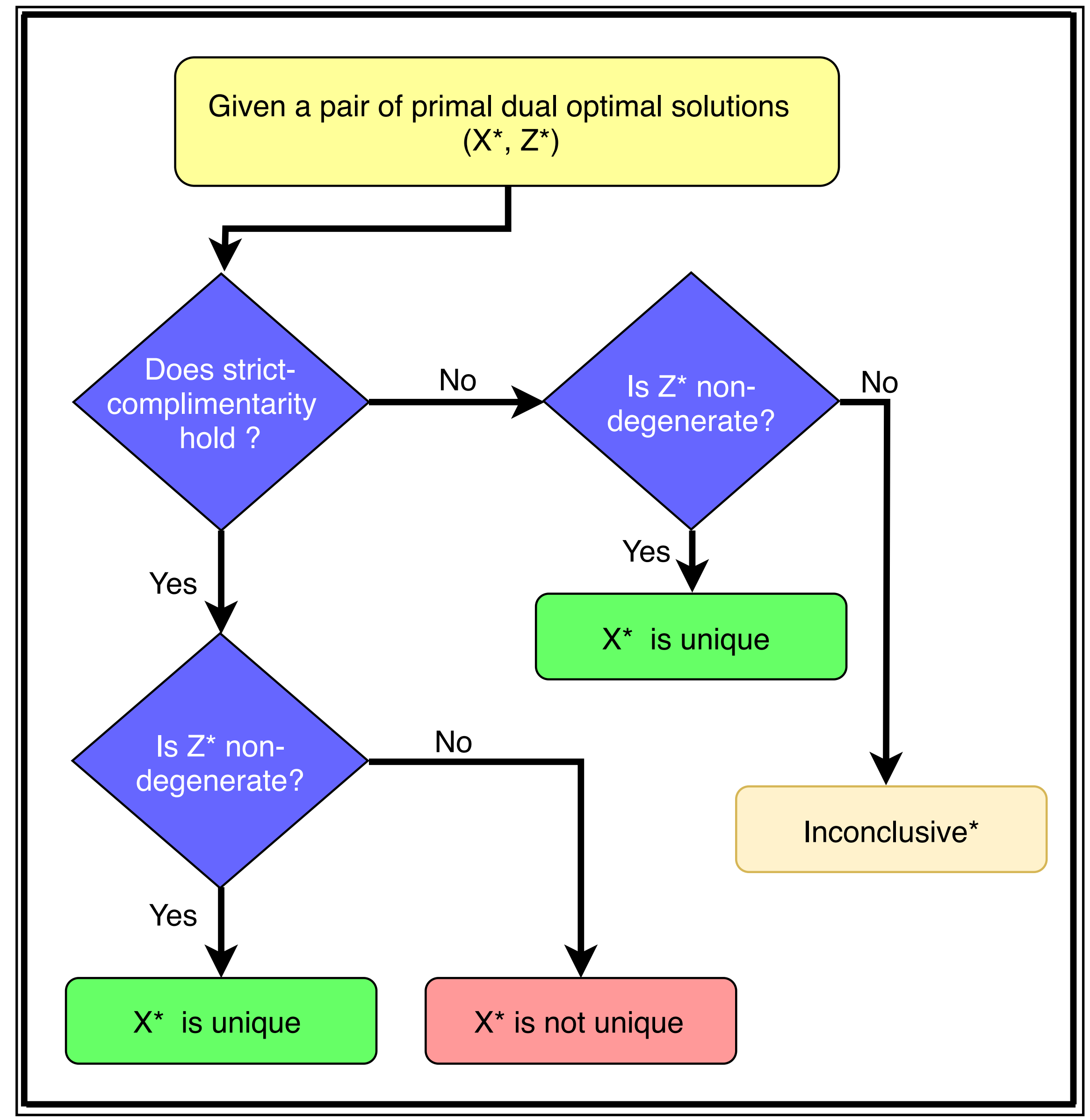
$$Eig(circ(u)) = \left\{\frac{1}{\vartheta_n} + \frac{n-\vartheta_n}{\vartheta_n}\cos\left(\frac{2\pi j}{n}\right) : j \in [n]\right\} \neq 0 \quad unless \quad j = \frac{n-1}{2} \quad or \quad \frac{n+1}{2}$$

$$Rank(X^*) \geq n-2$$

Explicit construction exists for dim = *n-2*

Device Certification





- *K Bharti, M Ray, A Varvitsiotis, NA Warsi, A Cabello, LC Kwek,* Physical Review Letters 122 (25), 250403.
- *K Bharti, M Ray, A Varvitsiotis, A Cabello, LC Kwek,* arXiv preprint arXiv:1911.09448.

Discussion

1. Not all graphs can be self-tested.
2. A complete characterisation of all self-testable graphs ?
3. Explicit robustness bounds.
4. Large gaps between $\vartheta(G)\, and\, \alpha(G)$?
5. Possible direction : Results on verifying quantum computation by classical verifier leverages self-testing results. Can local self-testing schemes help ?