Bounds on overlaps give us coherence, contextuality and non-locality inequalities

[EG, Brod, PRA 101, 062110 (2020)]
and work in progress

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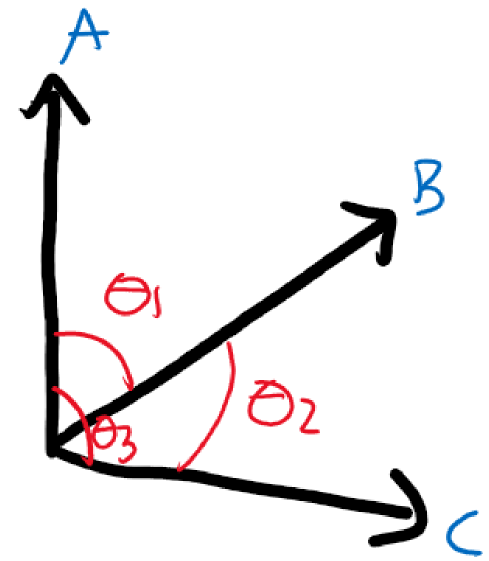
Outline

- Relational quantities among a set of states – overlaps
- Overlap inequalities for general states
- Overlap inequalities for coherence-free states – coherence witnesses
- Relationship between inequalities and non-contextuality/locality
- Some examples
- Conclusion

Projective-unitary invariant properties of a set of quantum states

- Properties that are invariant under:
  - unitary transformations
  - physically meaningless choice of global phases (gauge degree of freedom in QM)

- Geometrical in character – pertain to the **relative orientation of the states**



- Mathematical result: projective-unitary invariant properties only depend on ***k*-state Bargmann invariants:**

$$\Pi_{ABC\cdots K} = \langle A|B\rangle\langle B|C\rangle\langle C|D\rangle\cdots\langle K|A\rangle$$

[Chien, Waldron. SIAM J. DISCRETE MATH. 30 (2), 976 (2016)]

- Bargmann invariants related to geometric phases, photonic indistinguishability

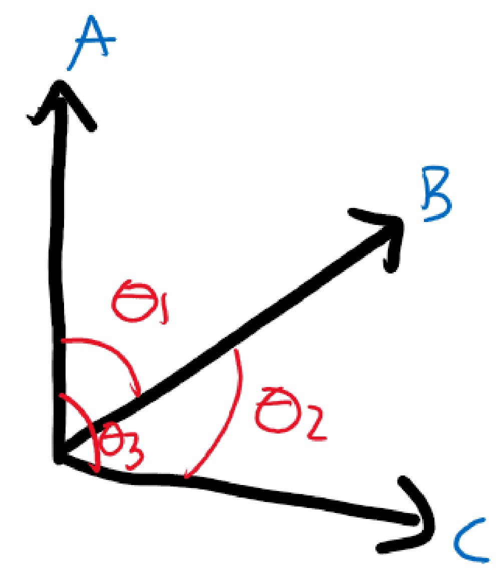
[Bargmann, J. Math. Phys. 5, 862 (1964)] [Simon, Mukunda, PRL 70, 880 (1993)]

[Menssen et al., Phys. Rev. Lett. 118, 153603 (2017)]

Overlaps

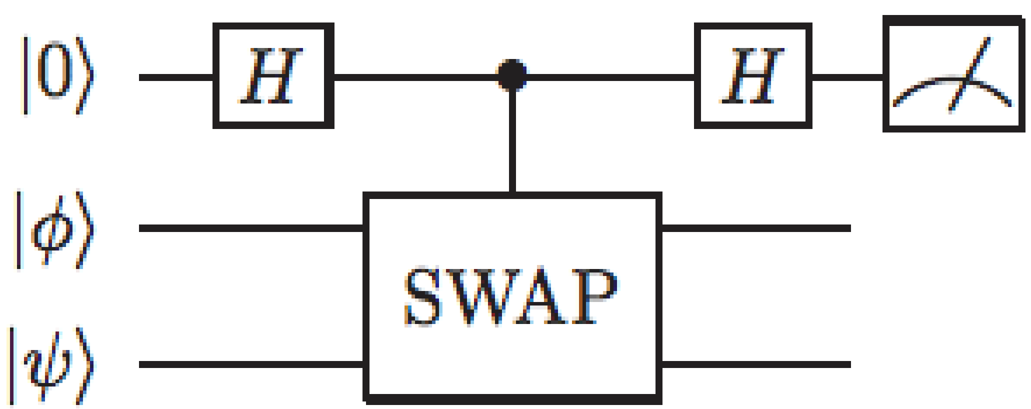
- Here we're interested in the **two-state overlap**:

$$\Pi_{AB} = |\langle A|B\rangle|^2 = \mathrm{Tr}(\rho_A \rho_B)$$



- Equals the probability of preparing A, projecting onto B (and vice-versa)

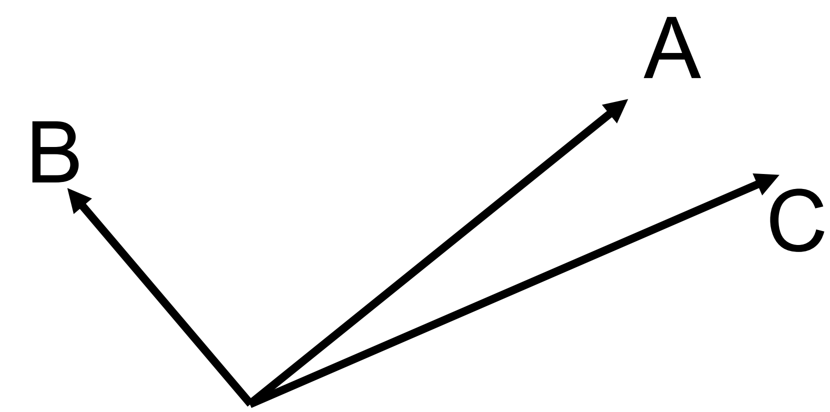
- Can be measured using SWAP test circuit:



$$p(0) = \frac{1 + |\langle \varphi | \psi \rangle|^2}{2}$$

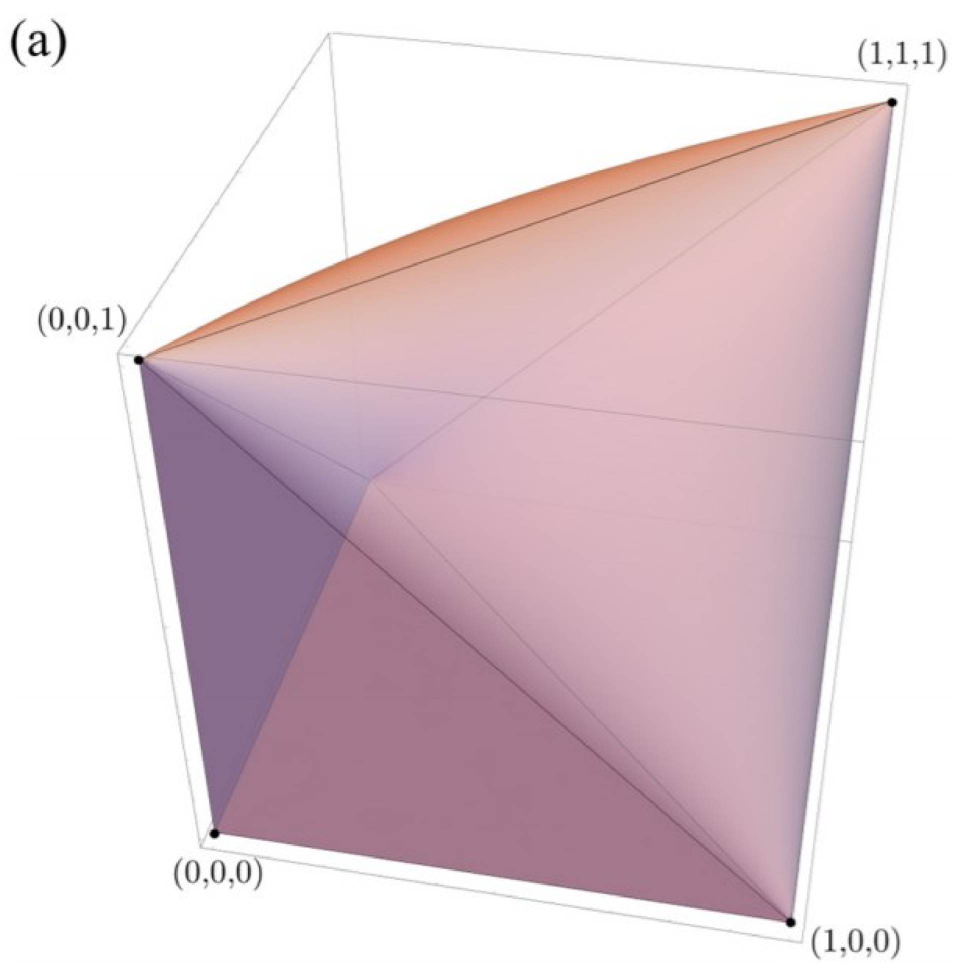
Overlaps among 3 arbitrary quantum states

- Let's consider a set of 3 arbitrary pure quantum states:



- If we have sources of states A, B, C, we can use SWAP tests to estimate overlaps, writing the triple

$$\vec{r} = (r_{AB}, r_{AC}, r_{BC})$$
$$r_{ij} = |< i|j > |^2$$



- Non-trivial boundaries of quantum set:

$$r_{AB} + r_{BC} + r_{AC} - 2\sqrt{r_{AB} r_{BC} r_{AC}} \leqslant 1$$

[EG, Brod, PRA 101, 062110 (2020)]

- What can we compare these bounds to?

Classical states: coherence-free states, diagonal in a single reference basis

Classical = incoherent states

- Our definition of **classical states** = diagonal, incoherent mixtures of states in a fixed, reference basis:
  - GLOBAL ρ: DIAGONAL
  - LOCAL $\rho_i = \mathrm{Tr}_{\text{all } j \neq i}(\rho)$: DIAGONAL
  - OVERLAP $r_{ij} = \mathrm{tr}(\rho_i \rho_j)$ = PROB. OF EQUAL OUTCOMES OF REFERENCE OBSERVABLE

- Example: $\rho = \begin{pmatrix} \rho_{11} & 0 & 0 \\ 0 & \rho_{22} & 0 \\ 0 & 0 & \rho_{33} \end{pmatrix} \quad \sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$

$$r_{\rho\sigma} = Tr(\rho\sigma) = \sum_i \rho_{ii}\sigma_{ii}$$

= probability of equal outcomes from measurements of reference observable on the two subsystems

- Note that diagonal density matrices are just a quantum way of parameterizing a general joint probability distribution of measurement outcomes

Overlaps among 3 arbitrary classical states

- Let

$$\vec{r} = (r_{AB}, r_{AC}, r_{BC})$$

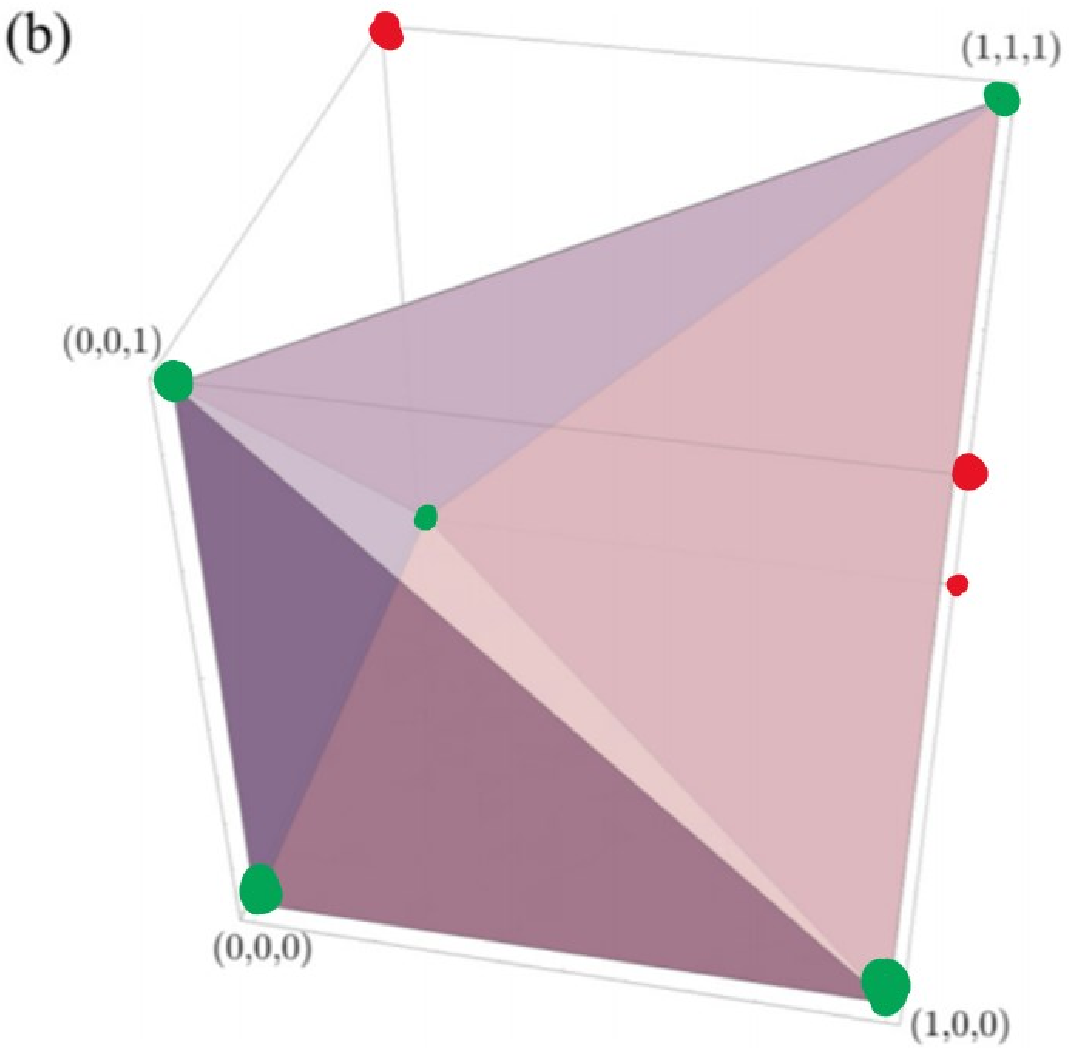
with $r_{AB} := p(A{=}B)$, etc.

- In $\vec{r}$-space, we obviously cannot have vertices

$(1,1,0), (1,0,1), (0,1,1)$

- So the only logically allowed states are convex combinations of the remaining 5 extremal states:

$(0,0,0), (1,1,1),$
$(0,0,1), (0,1,0), (1,0,0)$



That's the polyhedron above

Now we have 3 non-trivial facets:

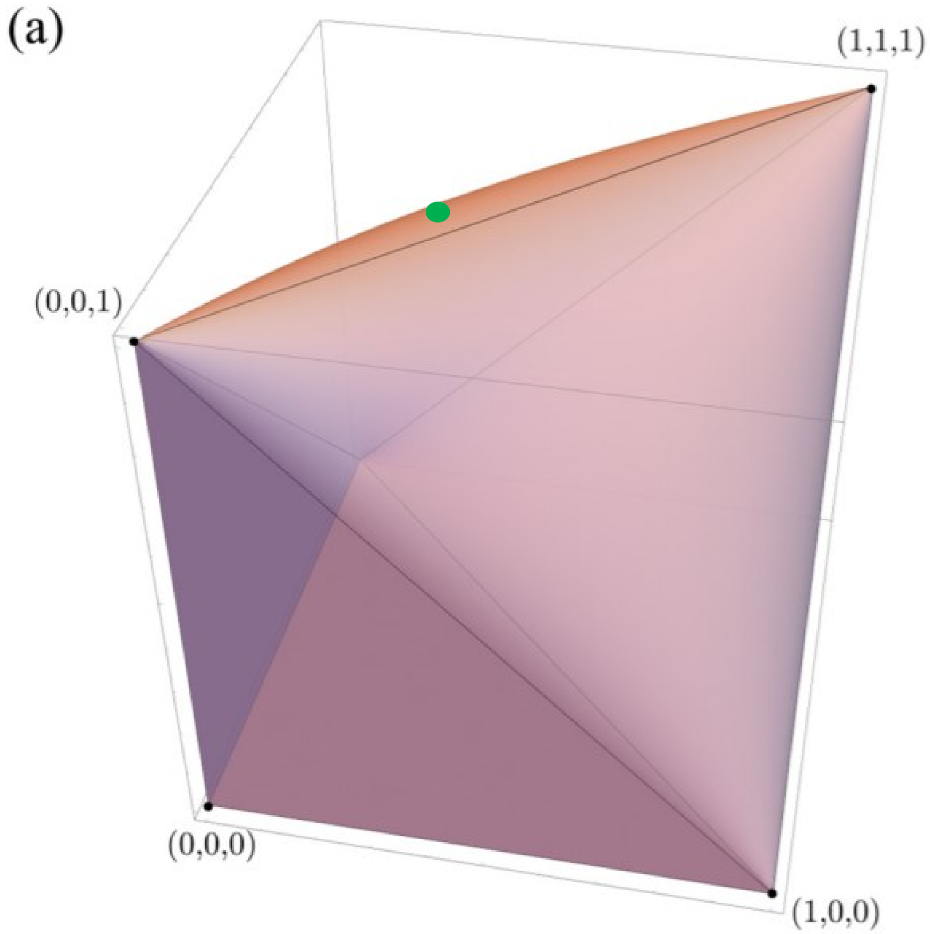
$$r_i + r_j - r_k \leq 1$$

Overlap measurements give us coherence witnesses

- If we measure ***r*** and get a point outside the classical set, we know the three states cannot be diagonal in any single basis.

Basis-independent coherence witness

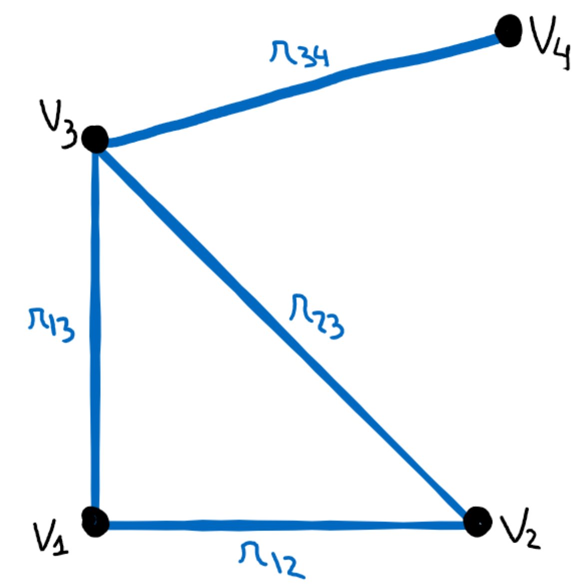
[EG, Brod, PRA 101, 062110 (2020)]



- These witnesses have been measured experimentally in a photonic set-up

[Giordani et al., Phys. Rev. Res. 3, 023031 (2021)]

Overlap inequalities are contextuality inequalities

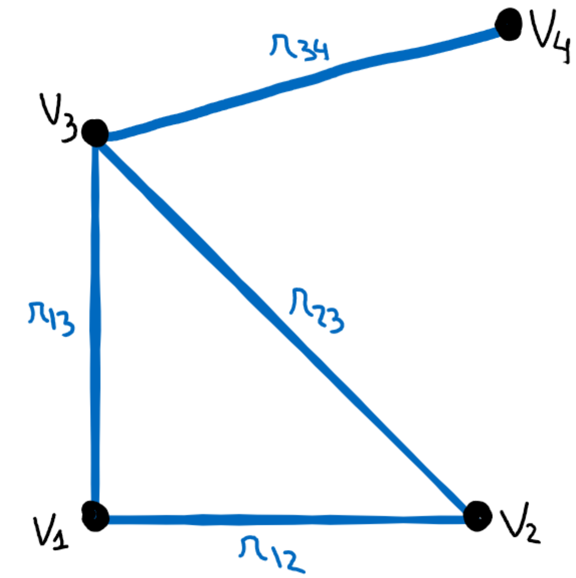


- Weighted graph describing general scenario:
  - Vertex v_i: probabilistic process yielding outcomes o_{ik} with probability p_{ik}
  - Edge weight r_{ij} = probability that v_i and v_j yield equal outcomes
- Classical model:
  - Global pdf for all v_i, with correct marginals for single vertices and and two-vertex context pdfs => correct overlaps r_{ij}
- Quantum realization of classical model: diagonal density matrices, reference observables reveal pre-existing properties
- Note that the **classical model is non-contextual** – quantum realization with diagonal states is a way of parameterizing general non-contextual model

➡ Classical overlap inequalities are contextuality/non-locality inequalities

Overlap facet inequalities

- Weighted graph describing general scenario:
  - Vertex v_i: probabilistic process yielding outcomes o_{ik} with probability p_{ik}
  - Edge weight r_{ij} = probability that v_i and v_j yield equal outcomes



- Overlap inequalities for the k-cycle scenario:

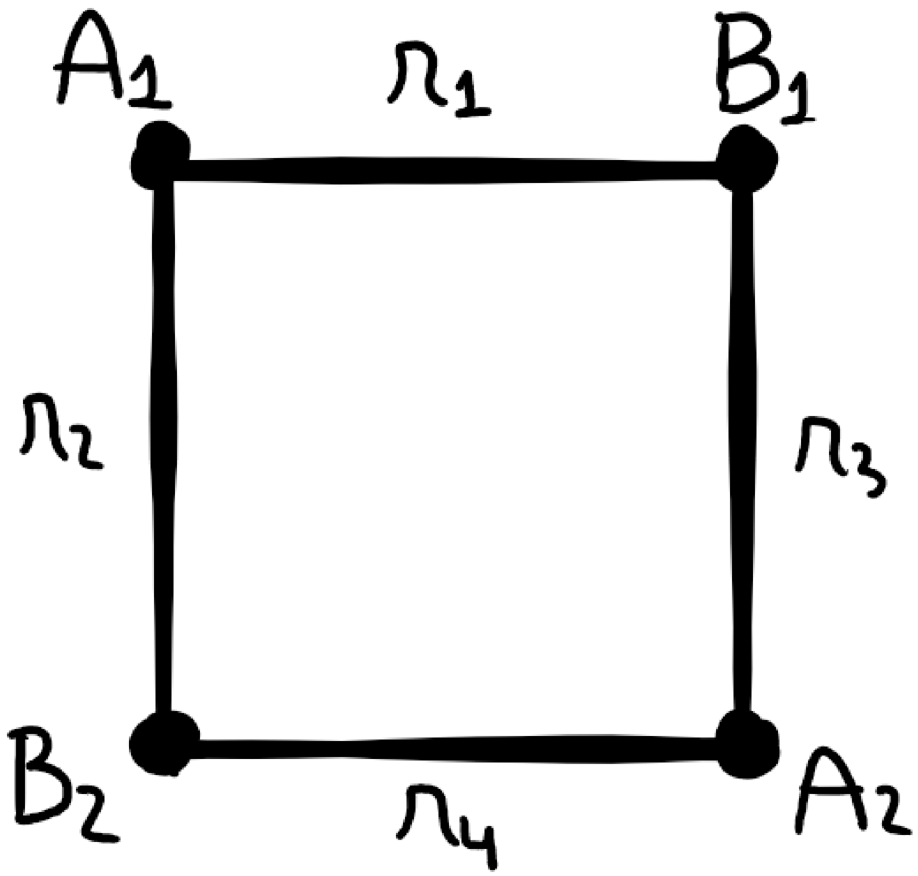
$$\left(\sum_{i=1}^{k-1} r_i\right) - r_k \leq k-2$$

[EG, Brod, PRA 101, 062110 (2020)]
Check [Hardy, Abramsky, PRA **85**, 062114 (2012)], [Araújo et al., PRA 88, 022118 (2013)]

- Computationally obtaining all facet inequalities for general scenarios:
  - List all sets of deterministic 0/1 assignments for entries of overlap m-tuple $\boldsymbol{r} = (r_1, r_2, r_3, \ldots, r_m)$;
  - Delete m-tuples forbidden by transitivity of equality;
  - Determine facets of convex hull of remaining, allowed deterministic m-tuples.

- Violation of inequalities witnesses coherence/contextuality/non-locality

Examples: 4-cycle

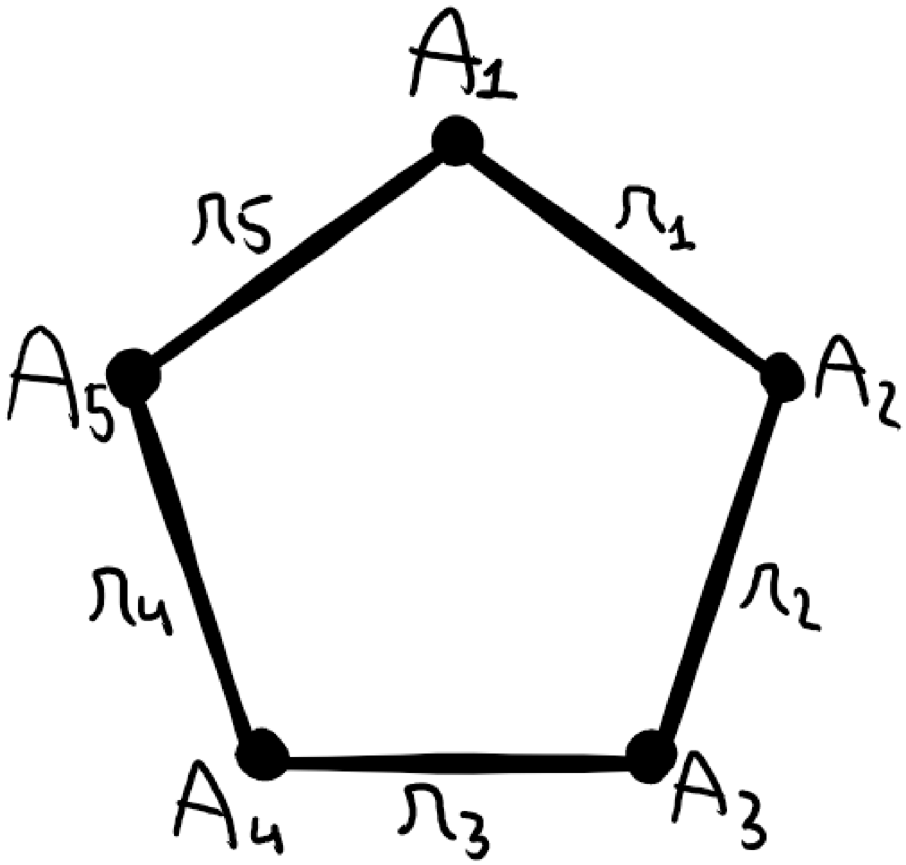


- A_i, B_j DICHOTOMIC: ± 1 OUTCOMES
- $r_{A_i B_j} = \dfrac{\langle A_i B_j \rangle + 1}{2}$

$$r_1 + r_2 + r_3 - r_4 \leq 1 \iff \left| \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \right| \leq 2$$

4-cycle overlap inequality ⇔ **CHSH inequality**

Examples: 5-cycle



- A_i DYCHOTOMIC: ± 1 OUTCOMES
- $\Omega = \dfrac{1 + \langle A_i A_j \rangle}{2}$

$$\Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 - \Omega_5 \leq 3$$

5-cycle overlap inequality

$$\Updownarrow$$

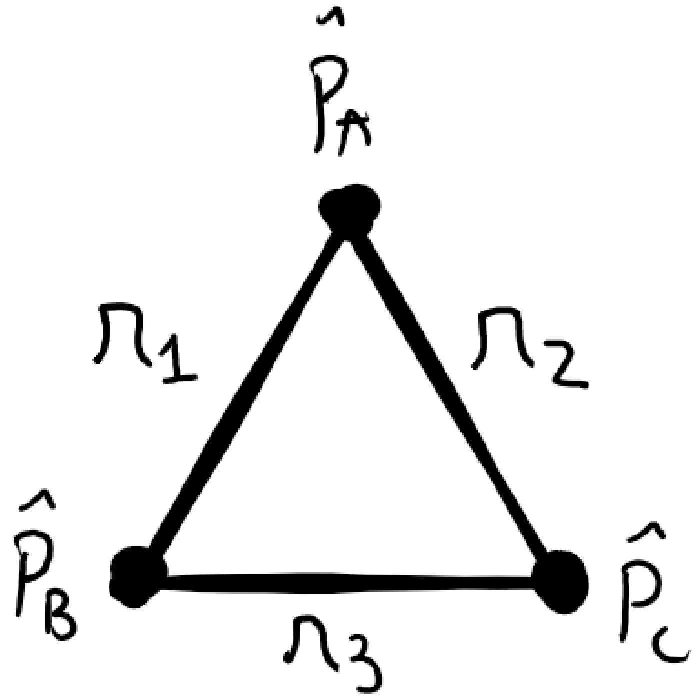
$$\langle A_1 A_2 \rangle + \langle A_2 A_3 \rangle + \langle A_3 A_4 \rangle + \langle A_4 A_5 \rangle - \langle A_5 A_1 \rangle \leq 3$$

KCBS inequality

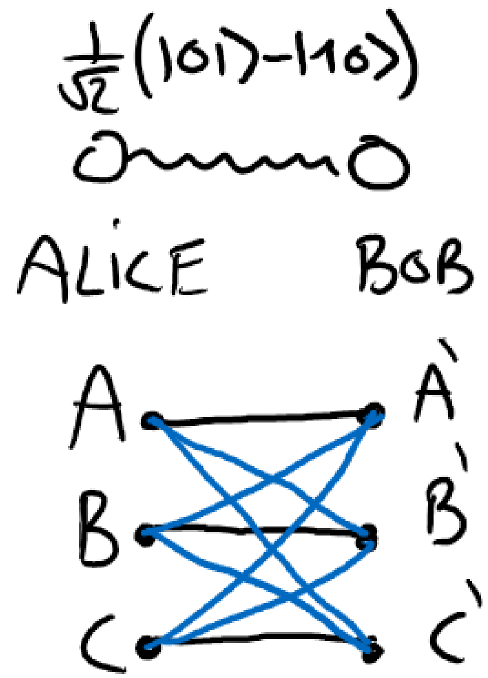
Klyachko et al., PRL101, 020403 (2008)]

Examples: 3-cycle

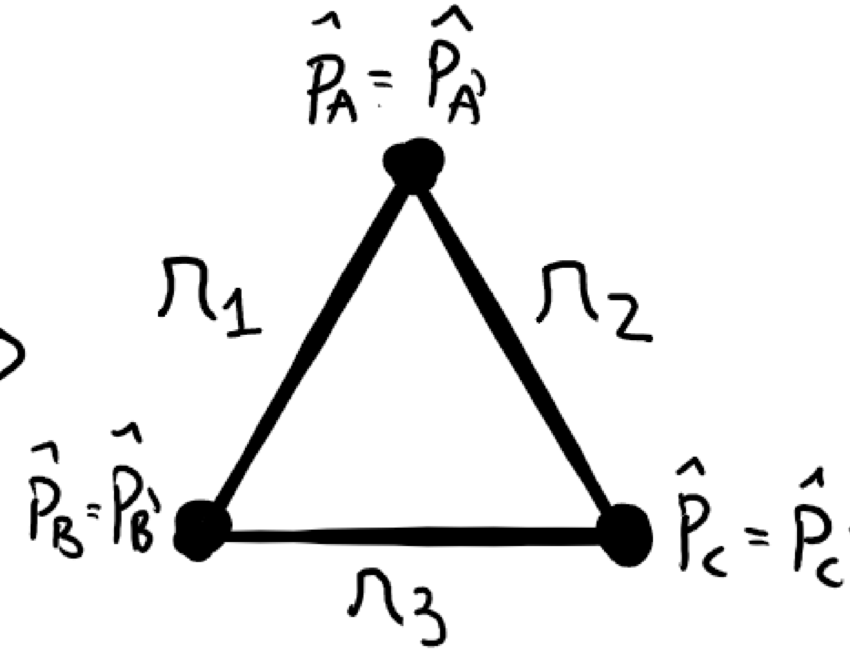
- Simplest non-trivial overlap scenario: 3-cycle



$\Rightarrow$



$\Rightarrow$



$\Omega_{ij'}$ = PROB. OUTCOMES ARE DIFFERENT

$$= \frac{1 - \langle i j' \rangle}{2}$$

in terms of usual +1,-1 outcomes

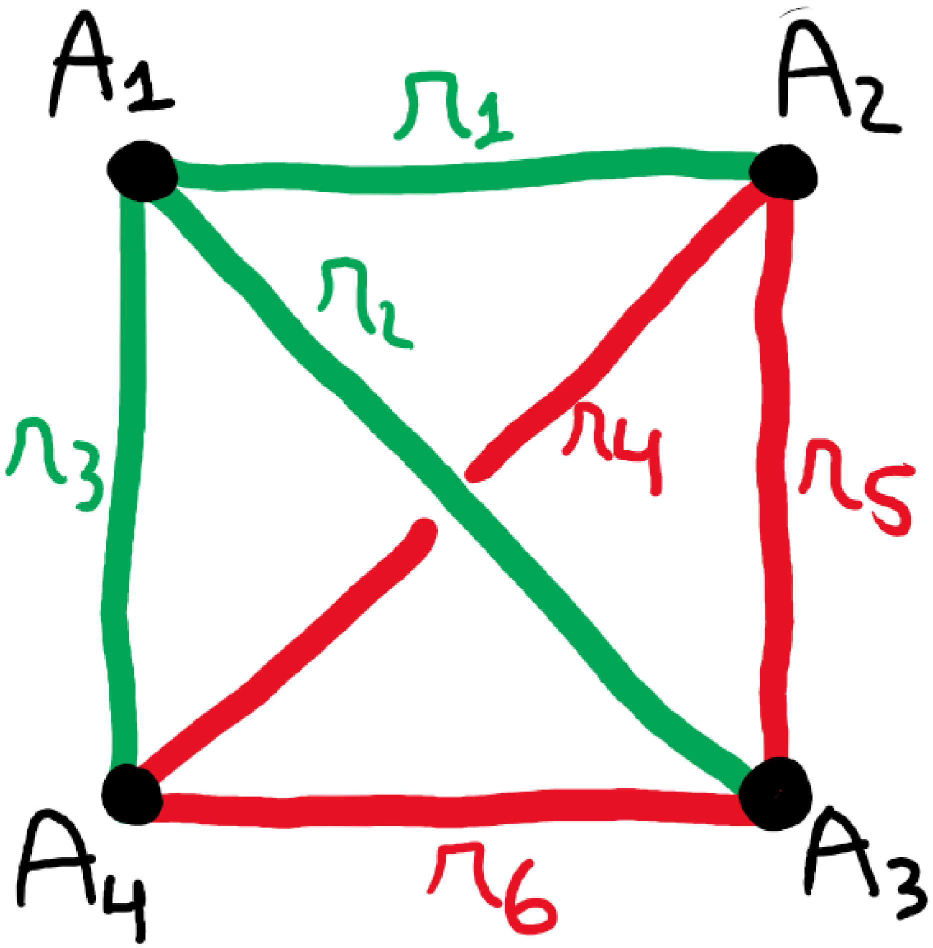
$$\Omega_1 + \Omega_2 - \Omega_3 \leq 1$$

$$\Longleftrightarrow$$

$$\langle BC \rangle - \langle AC \rangle - \langle AB \rangle \leq 1$$

- 3-cycle overlap inequalities equivalent to the **original 3-setting Bell inequality**

Examples: K_4 - complete graph with 4 vertices



- Only new type of facet of K_4 that is not a cycle inequality:

$$(\pi_1+\pi_2+\pi_3)-(\pi_4+\pi_5+\pi_6)\leq 1$$

QM: $\left(\frac{5}{9}+\frac{5}{9}+\frac{5}{9}\right)-\left(\frac{1}{9}+\frac{1}{9}+\frac{1}{9}\right)=\frac{4}{3}\not\leq 1$

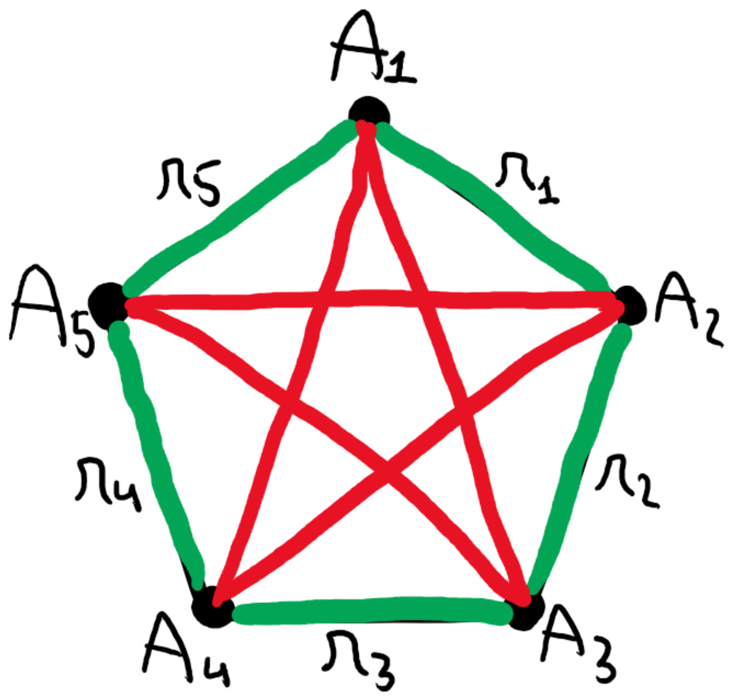
$$|A_1\rangle=|0\rangle$$

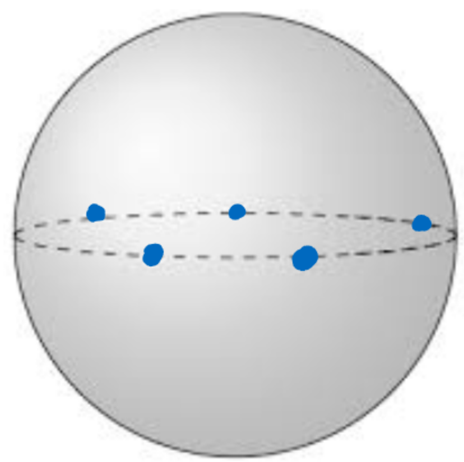
$$|A_2\rangle=\sqrt{\tfrac{5}{9}}|0\rangle+\sqrt{\tfrac{4}{9}}|1\rangle$$

$$|A_3\rangle=\sqrt{\tfrac{5}{9}}|0\rangle-\sqrt{\tfrac{1}{9}}|1\rangle+i\sqrt{\tfrac{1}{3}}|2\rangle$$

$$|A_4\rangle=\sqrt{\tfrac{5}{9}}|0\rangle-\sqrt{\tfrac{1}{9}}|1\rangle-i\sqrt{\tfrac{1}{3}}|2\rangle$$

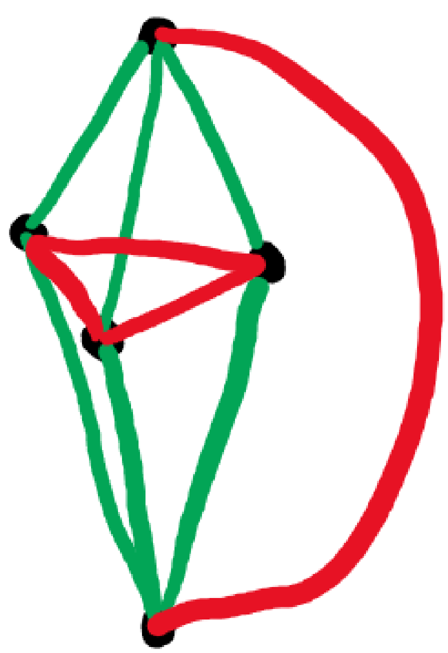
Examples: two facets from K_5

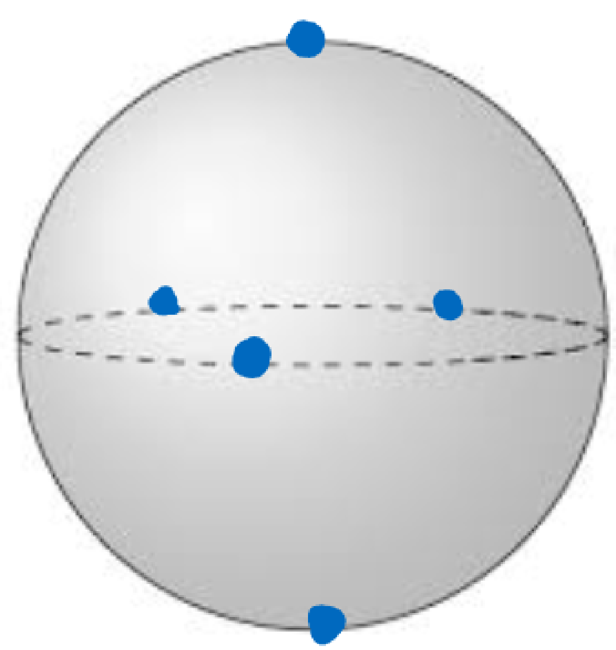




$$\underbrace{\sum \Pi}_{\text{green}} - \underbrace{\sum \Pi}_{\text{red}} \leq 2$$

$$\Leftarrow \text{QM: } 5\cdot\left(\frac{3+\sqrt{5}}{8}\right) - 5\cdot\left(\frac{3-\sqrt{5}}{8}\right) \simeq 2.795 \not\leq 2$$





$$\sum \Pi - \sum \Pi \leq 2$$

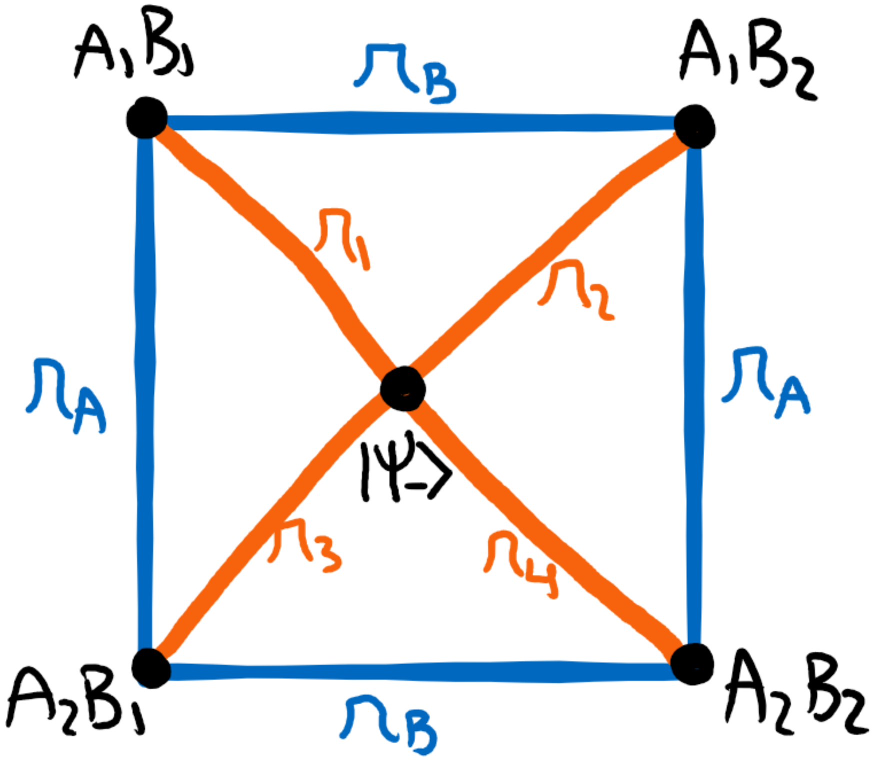
$$\Leftarrow \text{QM: } \frac{9}{4} \not\leq 2$$

Unifying non-classicality: contextuality and coherence

- This approach promises to unify two notions of non-classicality: coherence, and contextuality/non-locality

- Overlap inequalities are quite broad – we can use them to represent compatibility and probabilities in QM.

  - Example: a different derivation of the CHSH inequality



- Center vertex: singlet state

- Other vertices: projective measurements jointly measured by Alice and Bob

- Settings at A and B define r_A, r_B.

- 3-cycle inequalities yield the CHSH inequality.

- There's plenty to explore: Tsirelson bounds, equivalences between protocols, unified resource theories…

Conclusions

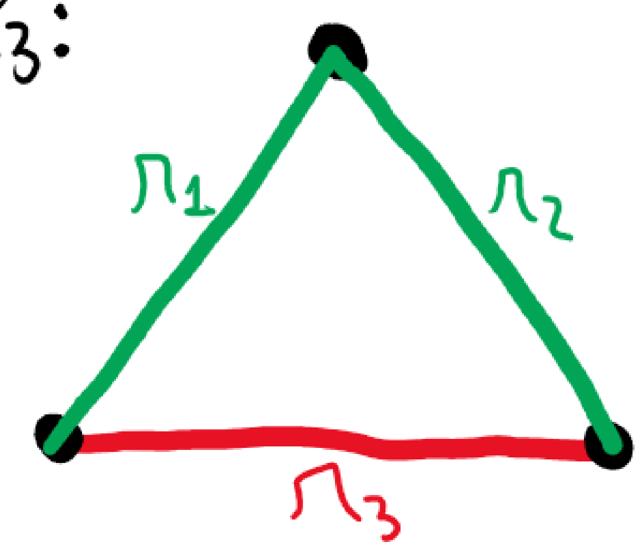
- We’ve introduced basis-independent coherence witnesses based on overlaps
- Bounds on overlaps for coherence-free states = non-contextuality inequalities
- Contextuality and coherence described in a single framework – helpful to discuss resources for quantum computational advantage
- Some thoughts for the workshop:
  - Relationship with PBR theorem?
  - Describing this in the CbD framework?
  - Finding new Bell/contextuality inequalities and their quantum bounds
  - Foundational importance of higher-order Bargmann invariants

Thank you for your attention!

Extra slides

Logically impossible deterministic assignments

K_3:

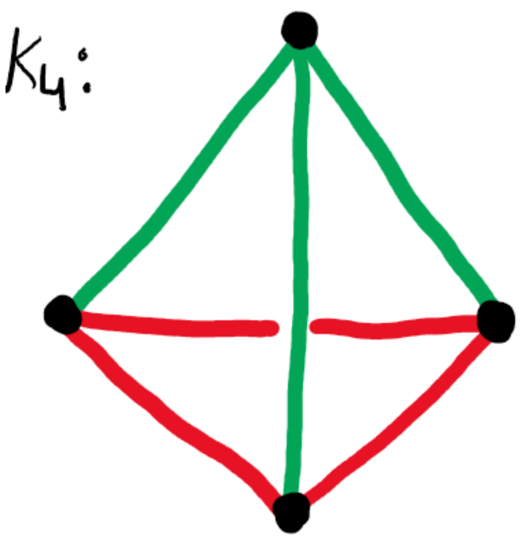


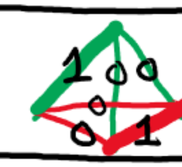
# 1's	# 1's	
1	0	
2	1	

FORBIDDEN: 

$$\Rightarrow r_1 + r_2 - r_3 \leq 1$$

K_4:



# 1's	# 1's	
1	0,1	 
2	1	
3	3	

$$\Rightarrow \sum r - \sum r \leq 1$$