Quantum contextuality: From logical contradictions to experimental tests

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Outline

- Kochen-Specker as a logical contradiction
- Operational definition (two perspectives based on):
  - Effects (statistical identification of effects)
  - Observables (compatible sequential measurements)
- Ideal and non-ideal cases
- A simple example: Peres-Mermin square
- Conclusions and outlook

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Quantum Contextuality

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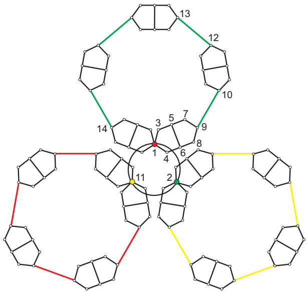
A central result in the foundations of quantum mechanics is the Kochen-Specker theorem. In short, it states that quantum mechanics is in conflict with classical models in which the result of a measurement does not depend on which other compatible measurements are jointly performed. Here, compatible measurements are those that can be performed simultaneously or in any order without disturbance. This conflict is generically called quantum contextuality. In this article, we present an introduction to this subject and its current status. We review several proofs of the Kochen-Specker theorem and different notions of contextuality. We explain how to experimentally test some of these notions and discuss connections between contextuality and nonlocality or graph theory. Finally, we review some applications of contextuality in quantum information processing.

Kochen-Specker contextuality



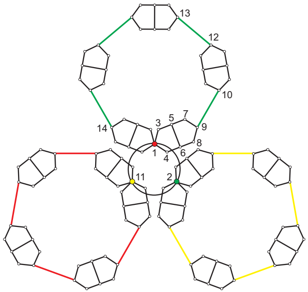
Hilbert space of dimension $d \geq 3$, for each set of d orthogonal directions (**a context**), we associated 1-dim projections $P_1, \ldots, P_d$, s.t.

O $P_i P_j = 0$ if $i \neq j$ (Orthogonality);

C $\sum_i P_i = \mathbb{1}$ (Completeness).

Kochen-Specker considered 117 directions in $d = 3$.

Kochen-Specker contextuality



Interpret each projection as a **proposition**, assign a "truth value" s.t. in each context $P_1, \ldots, P_d$:

O' P_i and P_j cannot be both "true" for $i \neq j$;

C' $P_1, \ldots, P_d$ they cannot be all "false".

Context: set of "jointly-decidable" propositions (orthogonal proj.).
Assignments must be *context-independent*.

Kochen-Specker contextuality

KS-theorem

Truth-value assignements to propositions associated with projectors, with **O** and **C** rules: **Impossible**

- **Logical impossibility proof**.
- Can we pass from logical argument to statistical one and test contextuality in the lab?

Kochen-Specker contextuality

Abstract definition

Some version of the marginal problem (many of the previous talks)

- $\mathcal{M} = \{X_1, \ldots, X_n\}$ set of measurements
- $\mathcal{C} \subset 2^{\mathcal{M}}$ set of contexts
- p_C joint observation, for all $C \in \mathcal{C}$.
- **Noncontextuality**:
  There exists $p_{\mathcal{M}}$ such that p_C is a marginal of $p_{\mathcal{M}}$, for all $C \in \mathcal{C}$.

Not the only one

- Spekkens' definition[1]: no joint measurements, no marginals, etc.

[1] R. W. Spekkens, Phys. Rev. A 71 (2005)

Kochen-Specker contextuality

So far no operational approach

- What to measure? How?
- How to identify contexts?
- Context-independence implies the *identification* of the "same measurement" in "different contexts". How to do that?

Two interpretations of KS

Observable and Effect Perspectives:

OP P_1, P_2, P_3 represent three commuting observables, with effects $\{P_i, \mathbb{1} - P_i\}$. I measure them together: Joint measurement has effects $\{P_1, P_2, P_3\}$. I can measure the same observable in different "contexts", e.g., P_1 with P_2' and P_3'.

EP $\{P_1, P_2, P_3\}$ are the effects of a joint measurements. The same effect P_1 may appear in different measurements, e.g., $\{P_1, P_2', P_3'\}$.

OP is (arguably) the original formulation ("something about simultaneously decidable propositions"),
EP is how KS theorem became popular ("something about orthogonal vectors").

Equivalent interpretations for **ideal** measurements.

Two interpretations of KS

Observable and Effect Perspectives (abstract version):

OP The basic objects of contextuality are observables and their compatibility (joint measurability) relations. A context is defined by a set of compatible observables. A noncontextual hidden variable theory is one that assigns values to each observable independently of which joint measurement they appear in.

EP The basic objects of contextuality are effects and their relation of being part of the same observable. A context is defined by a single observable. A noncontextual hidden variable theory is one that assigns values to each effect independently of which observable they appear in.

Experimental tests of contextuality (EP)

Effect perspective (ideal case):

- Each **context** corresponds to a **measurement** (PVM) $\mathcal{M} = \{P_i\}_i$
- We want to **identify effects** in **different contexts**, e.g., $P_i \in \mathcal{M}, P'_i \in \mathcal{M}'$ with $P_i = P'_i$.
- In QM: $P_i = P'_i \Leftrightarrow \mathrm{tr}[\rho P_i] = \mathrm{tr}[\rho P'_i]$ for all states ρ.
- We extract an operational rule for identifying "the same effect in different contexts": **same statistics** $\Rightarrow$ **same effect**.

Experimental tests of contextuality (EP)

Possible operational definition via statistical identification:
Measurement noncontextuality[2]

$$\xi(k|\lambda, \mathcal{M}) = \xi(k|\lambda, \mathcal{M}') \ \forall\lambda \text{ if } p(k|\mathcal{P}, \mathcal{M}) = p(k|\mathcal{P}, \mathcal{M}') \ \forall\mathcal{P}$$

Where classical theories (ontological models) compute probabilities as

$$p(k|\mathcal{P}, \mathcal{M}) := \sum_{\lambda} \mu(\lambda|\mathcal{P})\xi(k|\lambda, \mathcal{M})$$

[2] R. W. Spekkens, Phys. Rev. A 71 (2005)

Experimental tests of contextuality (EP)

Measurement noncontextuality

$$\xi(k|\lambda,\mathcal{M}) = \xi(k|\lambda,\mathcal{M}')\ \forall\lambda \text{ if } p(k|\mathcal{P},\mathcal{M}) = p(k|\mathcal{P},\mathcal{M}')\ \forall\mathcal{P}$$

Can we use this definition to experimental test Kochen-Specker?

In this language value assignements for $\mathcal{M} = \{P_1, P_2, P_3\}$ satisfy

$$\begin{aligned}
&\xi(i|\lambda,\mathcal{M}) = 0, 1;\\
&\xi(i|\lambda,\mathcal{M})\xi(j|\lambda,\mathcal{M}) = 0 \text{ for } i \neq j;\\
&\sum_i \xi(i|\lambda,\mathcal{M}) = 1
\end{aligned}$$

Experimental tests of contextuality (EP)

Measurement noncontextuality (MNC)

$$\xi(k|\lambda, \mathcal{M}) = \xi(k|\lambda, \mathcal{M}') \ \forall\lambda \text{ if } p(k|\mathcal{P}, \mathcal{M}) = p(k|\mathcal{P}, \mathcal{M}') \ \forall\mathcal{P}$$

Problem with determinism

Assuming MNC, if measurements are not ideal (i.e., they contain noise) the functions ξ will not be in $\{0, 1\}$. We are no longer comparing $\{0, 1\}$-valued assignments following **O**, **C** rules[3].

[Idea: noisy effect convex mixture of projectors, statistical identification implies same mixture at the HV level. No longer $\{0, 1\}$-valued response function.]

We cannot experim. test KS contradiction with this assumption!

[3] R. W. Spekkens, Found. Phys. 44, 1125 (2014).

Experimental tests of contextuality (EP)

Spekkens solution: different notion of contextuality, not only for measurements, but also for preparations.

$$P \sim P' \Longleftrightarrow p(k|P,M) = p(k|P',M),$$
$$\text{for all measurements and outcomes } k, M,$$
$$(M,k) \sim (M',k') \Longleftrightarrow p(k|P,M) = p(k'|P,M'),$$
$$\text{for all preparations } P.$$

Assumption on HV [$p(k|P,M) = \sum_\lambda \mu_P(\lambda)\xi_{M,k}(\lambda)$]

- $P \sim P' \Rightarrow \mu_P = \mu_{P'}$ *preparation noncontextuality*
- $(M,k) \sim (M',k') \Rightarrow \xi_{M,k} = \xi_{M',k'}$ *measurement noncontextuality*

From this conditions it is possible to derive noncontextuality inequalities (of the Spekkens type)[4]

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[4]e.g., Mazurek *et al.*, Nat. Commun. 7, 11780 (2016)

Experimental tests of contextuality (OP)

Observable perspective (math. formulation)

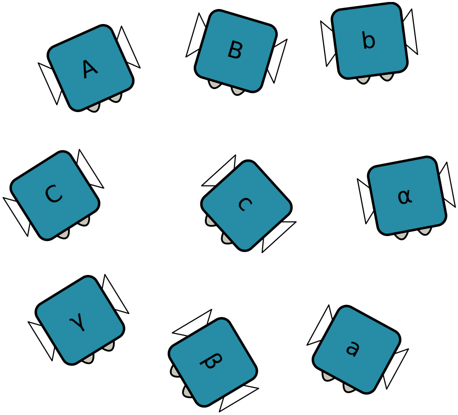
- $\mathcal{M} = \{X_1, \ldots, X_n\}$ set of measurements
- $\mathcal{C} \subset 2^{\mathcal{M}}$ set of contexts
- p_C joint observation, for all $C \in \mathcal{C}$.
- **Noncontextuality**:
  There exists $p_{\mathcal{M}}$ such that p_C is a marginal of $p_{\mathcal{M}}$, for all $C \in \mathcal{C}$.
- Contexts normally "given" or identified assuming QM ("commuting operators")

Operational definition?

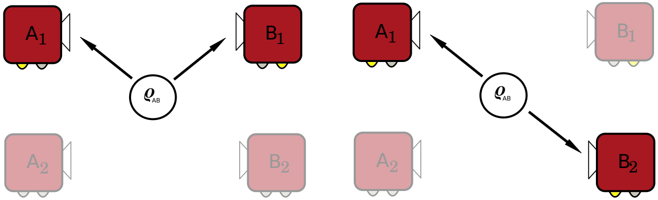
- Test of classical vs. quantum theory $\rightarrow$ remove any QM assumption

Experimental tests of contextuality (OP)

Given some measurement boxes, how do we find contexts?
(with no QM assumptions)



Experimental tests of contextuality (OP)



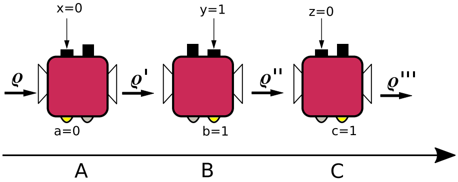
Easy example: Bell scenario

- Contexts: joint measurements of (A_x, B_y) (locality assumption).
- Identification of same measurement in different context: same local "black-box".
- "Noncontextuality assumption": the choice of measurement on B does not influence the outcome of A.

Local hidden variable theory

$$p(ab|xy) = \sum_{\lambda} p(\lambda) p(a|x,\lambda) p(b|y,\lambda)$$

Experimental tests of contextuality (OP)



Generalization?

Analogous expression in terms of probabilities

$$p(abc|xyz) = \sum_{\lambda} p(\lambda)p(a|x,\lambda)p(b|y,\lambda)p(c|z,\lambda)$$

Can we interpret measurements as black-boxes? What are the physical assumptions?

Experimental tests of contextuality (OP)

Identification of measurements and contexts

Intuition (ideal case):

- **Repeatable measurements:** we are measuring a "property" of the system (not a random signal/outcome)
- **Nondisturbing measurements:** measurements do not influence each other.

QM perspective: we recover the ideal case of commuting projective measurements (sharp, repeatable, nondisturbing).

A simple example: Peres-Mermin square

Abstract formulation

$$\begin{bmatrix} A & B & C \\ a & b & c \\ \alpha & \beta & \gamma \end{bmatrix}$$

- Compatible observables along rows and columns
- Possible to measure them jointly (contexts are "given")
- Assuming a noncontextual value (± 1) for $A, B, \dots, \gamma$ [5]

$$\langle \mathrm{PM} \rangle \equiv \langle ABC \rangle + \langle abc \rangle + \langle \alpha\beta\gamma \rangle + \langle Aa\alpha \rangle + \langle Bb\beta \rangle - \langle Cc\gamma \rangle \le 4.$$

[5] A. Cabello, PRL **101** (2008)

A simple example: Peres-Mermin square

Quantum realization

$$\begin{bmatrix} A & B & C \\ a & b & c \\ \alpha & \beta & \gamma \end{bmatrix} = \begin{bmatrix} \sigma_z \otimes \mathbb{1} & \mathbb{1} \otimes \sigma_z & \sigma_z \otimes \sigma_z \\ \mathbb{1} \otimes \sigma_x & \sigma_x \otimes \mathbb{1} & \sigma_x \otimes \sigma_x \\ \sigma_z \otimes \sigma_x & \sigma_x \otimes \sigma_z & \sigma_y \otimes \sigma_y \end{bmatrix}$$

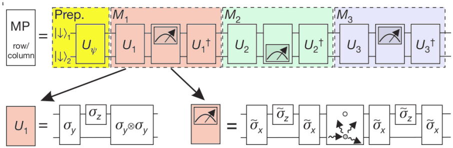
For this specific realization: $ABC = abc = \ldots = -Cc\gamma = \mathbb{1}$

$$\langle \mathrm{PM} \rangle = \langle ABC \rangle + \langle abc \rangle + \langle \alpha\beta\gamma \rangle + \langle Aa\alpha \rangle + \langle Bb\beta \rangle - \langle Cc\gamma \rangle = 6 > 4,$$

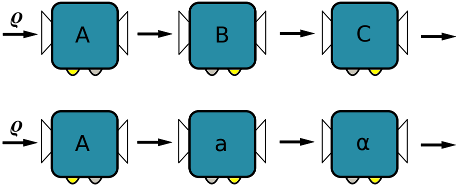
independently of the initial state.

A simple example: Peres-Mermin square

Experimental realization



or "more abstractly"



to measure $\langle ABC \rangle, \langle Aa\alpha \rangle, \ldots, \langle Cc\gamma \rangle$.

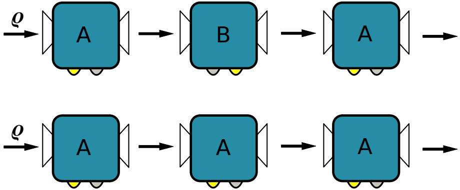
[5]Picture from Kirchmair et al. Nature 460 (2009)

A simple example: Peres-Mermin square

How do we know that $A, B, C, \ldots, \gamma$ **are the right measurements?**
(without assuming QM)

Properties:

- Nondisturbance (outcome of A confirmed by measurement after B)
- Repeatability (outcome of A confirmed by later measurements)



and so on for longer sequences and permutations.

We still need a (more detailed) classical model to interpret this.

A simple example: Peres-Mermin square

Common misconceptions

- Assumptions from QM are needed to define NCHV model:
  - The hidden variable description assumes $ABC = +1$, etc.
  - The product AB (in QM $\sigma_z \otimes \mathbb{1} \cdot \mathbb{1} \otimes \sigma_z$) is equal to C (in QM: $\sigma_z \otimes \sigma_z$)
  - I substitute the ABC measurement with just a measurement of $\mathbb{1}$.
- The measurement, e.g., $\sigma_x \otimes \sigma_x$ is given by two single-qubit measurements with four outcomes $(\pm 1, \pm 1)$.
- I need to assume determinism.

A simple example: Peres-Mermin square

Common misconceptions

- Assumptions from QM are needed to define NCHV model:
  - The hidden variable description assumes $ABC = +1$, etc. (**no need**)
  - The product AB (in QM $\sigma_z \otimes \mathbb{1} \cdot \mathbb{1} \otimes \sigma_z$) is equal to C (in QM: $\sigma_z \otimes \sigma_z$) (**no need**)
  - I substitute, e.g., ABC measurement with just a measurement of $\mathbb{1}$ (**no contradiction**).
- The measurement, e.g., $\sigma_x \otimes \sigma_x$ is given by two single-qubit measurements with four outcomes $(\pm 1, \pm 1)$ (**the PVMs are noncommuting**).
- I need to assume determinism (**no statistical identification**).

ALL THE ABOVE ARE WRONG!

Dealing with experimental noise

No general recipe

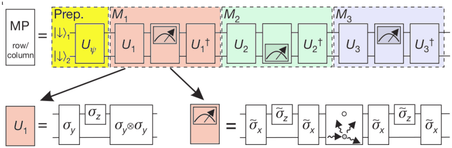
Several approaches, with different physical assumptions, e.g.,

- O. Gühne *et al.* Phys. Rev. A, 81(2), 022121, (2010) [disturbance]
- J. Szangolies *et al.* Phys. Rev. A, 87(5), 050101, (2013) [noncontextual evolution]
- J. V. Kujala *et al.* Phys. Rev. Lett. 115, 150401 (2015) [disturbance]
- Liang *et al.* Phys. Rep. 2011 [sharpness, but very different framework]
- Shane's talk? [disturbance and sharpness]

Noise assumptions

Examples of physical assumptions

- Noise quantification under assumption of **cumulative noise** (correction terms to the classical bound of the form $p^{\mathrm{err}}[BAB]$) [6].



[6]Kirchmair et al. Nature 460 (2009). Gühne et al. Phys. Rev. A, 81, (2010)

Summary of procedure (OP)

Experimental test in observable perspective (OP):

S.1 **Define experimental measurement procedures** and associate to each one a classical random variable with same values as possible outcomes.

S.2 **Identify contexts** in terms of outcome-repeatable and statistical-nondisturbing measurements.

S.3 **Perform measurements** in different sequences, according to the defined contexts. For each measurement the same procedure is repeated in different contexts.

S.4 **Compare the observed statistics** for contexts (sequences) with the one predicted by the NCHV for the corresponding classical variables.

Summary of procedure (OP)

What if measurements are not ideal?

They must introduce some disturbance, we can try to quantify it

S.5 **Quantify deviations from ideal** (outcome-repeatable and nondisturbing) measurements, performing additional experimental runs, and compare with the classical models accordingly.

Conclusions and outlook

Summary

- Difficult to move from logical contradiction/ideal case to experiments
- Different approaches possibles (Effects and Observables)
- In many (theor.) approaches context are "given" (by QM?). How to identify them in experiments?
- Conceptual role of disturbance and sharpness
- No general solution (specialized models for each setup)

Open problems

- Role of sharpness? (Quantitative)
- Most general noise model?
- Possible to make a (reasonable) loophole-free experiment?