Causal reappraisal of the quantum three box paradox

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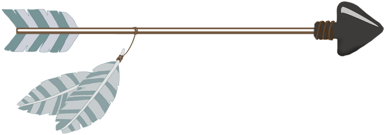
Causal framework:



- **structural causal model,**
  functional relations:

e.g. $X := f(Z, U, N_X)$,
$Y := g(X, Z, U, N_Y)$,
$Z := N_Z,\ U := N_U$,
f, g- arbitrary functions,
N_i- independent error terms,

- **causal DAGs,**
  quantitative relations:

$$\begin{tikzcd} & & U \arrow[dl] \arrow[dr] & \\ Z \arrow[r] \arrow[rrr, bend right] & X \arrow[rr] & & Y \end{tikzcd}$$

- **generated statistics,**
  joint probability distribution:
  $P(x, y, z, u) =$
  $P(x|z, u)P(y|x, z, u)P(z)P(u)$

3 box paradox in a nutshell

State space: $|1\rangle, |2\rangle, |3\rangle \in \mathbb{C}^3$

pre-selection:

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle),$$

post-selection:

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle - |3\rangle).$$

Initial state: $\rho_0 = |\psi_0\rangle\langle\psi_0|$.

Measurement M_2 **:**

$$\begin{cases} \mathbb{P}_1^{post} = |\psi_2\rangle\langle\psi_2|, & M_2 = 1, \\ \mathbb{P}_0^{post} = \mathbb{1} - |\psi_2\rangle\langle\psi_2|, & M_2 = 0. \end{cases}$$

E.g. check box 1:

$$\begin{cases} \mathbb{P}_1^{[1]} = |1\rangle\langle 1|, & M_1 = 1, \\ \mathbb{P}_0^{[1]} = |2\rangle\langle 2| + |3\rangle\langle 3|, & M_1 = 0, \end{cases}$$

or check box 2:

$$\begin{cases} \mathbb{P}_1^{[2]} = |2\rangle\langle 2|, & M_1 = 1, \\ \mathbb{P}_0^{[2]} = |1\rangle\langle 1| + |3\rangle\langle 3|, & M_1 = 0. \end{cases}$$

$C = 1, 2$

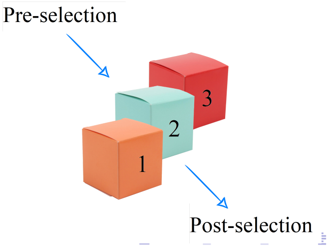
$P(M_1, M_2 \mid C)$	$M_2 = 0$	$M_2 = 1$
$M_1 = 0$	$2/3$	0
$M_1 = 1$	$2/9$	$1/9$

$C = 3$

$P(M_1, M_2 \mid C)$	$M_2 = 0$	$M_2 = 1$
$M_1 = 0$	$2/9$	$4/9$
$M_1 = 1$	$2/9$	$1/9$

Joint probability:

$P(M_\mathbf{2} = j, M_\mathbf{1} = i \mid C = k) = Tr[\mathbb{P}_j^{post}\mathbb{P}_i^{[k]}\rho_\mathbf{0}\mathbb{P}_i^{[k]}]$



Assumptions:

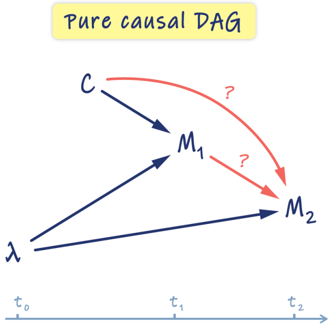
- no backward in time causation,
- free choice of a box measurement.

Variables:

- C- choice of a box,
- M_1- result of a box measurement,
- M_2- result of a post-selection,
- λ- hidden (latent) variable.

What is the role of a measurement disturbance?

- $M_1 \rightarrow M_2$ outcome dependence?
- $C \rightarrow M_2$ parameter dependence?



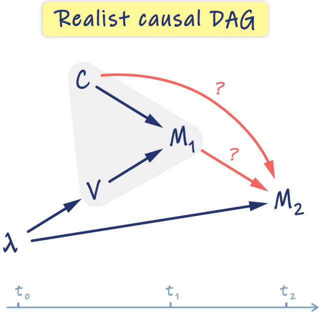
Assumptions:

- no backward in time causation,
- free choice of a box measurement,
- realism: the measurement reveals pre-existing properties of a particle (e.g. a position of a particle).

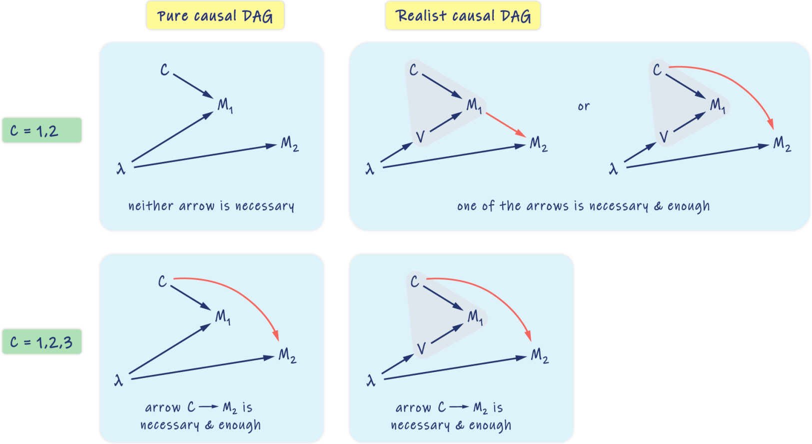
Variables:

- C- choice of a box,
- M_1- result of a box measurement,
- M_2- result of a post-selection,
- λ- hidden (latent) variable,
- V- position of a particle.

$M_1 := M_1(C, V) = \delta_{C,V}$.



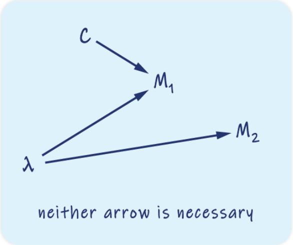
Results: which arrows are *necessary & enough*?



Results: Pure causal DAG, $C = 1, 2$

Pure causal DAG

C = 1,2



Structural causal model:

$\lambda \sim Ber(1/3)$ i.e.

$P(\lambda = 0) = 2/3$ & $P(\lambda = 1) = 1/3$

$M_1 := M_1(\lambda, C) = \lambda$

$M_2 := M_2(\lambda) = N \cdot \lambda$

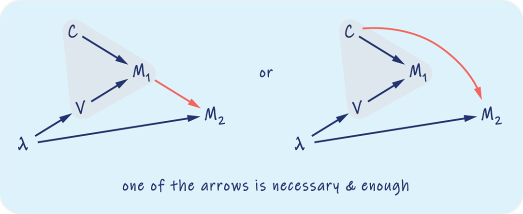
$N \sim Ber(1/3)$- noise variable

$C = 1, 2$

$P(M_1, M_2 \vert C)$	$M_2 = 0$	$M_2 = 1$
$M_1 = 0$	$2/3$	0
$M_1 = 1$	$2/9$	$1/9$

Realist causal DAG

$C = 1,2$



Structural causal model:
$\lambda = Uni(1, 3)$ i.e.
$P(\lambda = i) = 1/3$ for $i = 1, 2, 3$,
$M_1 := M_1(V, C) = \delta_{V,C}$,
$V := V(\lambda) = \lambda$,
$M_2 := M_2(M_1, \lambda) = M_1 \cdot N$
$N \sim Ber(1/3)$- noise variable
(DAG on the left-hand side)

$C = 1, 2$

$P(M_1, M_2 \mid C)$	$M_2 = 0$	$M_2 = 1$
$M_1 = 0$	$2/3$	0
$M_1 = 1$	$2/9$	$1/9$

Why do we need additional arrows?

Without both arrows $M_1 \to M_2$ and $C \to M_2$ all paths between V & C are **blocked**, since M_1 is **a collider**, i.e. we necessarily have $V \perp\!\!\!\perp C \mid M_2$, but by realism assumption: $M_1 = \delta_{C,V}$, we get $V = C$ for $M_2 = 1 \implies V \not\perp\!\!\!\perp C \mid M_2 \implies$ additional arrows are needed to explain the experiment's statistics.

Results: Pure causal DAG, $C = 1, 2, 3$

$C = 1, 2$

$P(M_1, M_2 \vert C)$	$M_2 = 0$	$M_2 = 1$
$M_1 = 0$	$2/3$	0
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$C = 3$

$P(M_1, M_2 \vert C)$	$M_2 = 0$	$M_2 = 1$
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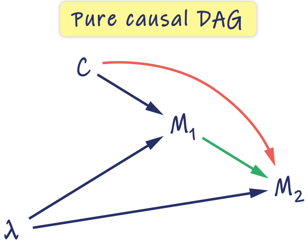
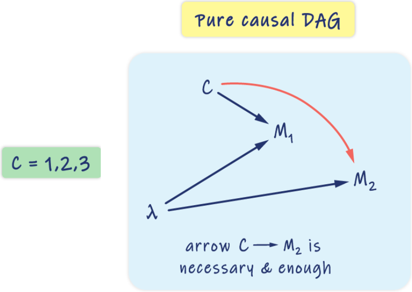


Can we explain the statistics with an arrow $M_1 \to M_2$?

Instrumental inequalities state that without an arrow $C \to M_2$,
we would need to have:

$$P(M_1 = 0, M_2 = 0 | C = k) + P(M_1 = 0, M_2 = 1 | C = l) \leq 1,$$
$$P(M_1 = 1, M_2 = 0 | C = k) + P(M_1 = 1, M_2 = 1 | C = l) \leq 1,$$
$$P(M_1 = 0, M_2 = 1 | C = k) + P(M_1 = 0, M_2 = 0 | C = l) \leq 1,$$
$$P(M_1 = 1, M_2 = 1 | C = k) + P(M_1 = 1, M_2 = 0 | C = l) \leq 1,$$

for each $kl = 12, 13, 23$, but e.g. for $kl = 23$, we get $2/3 + 4/9 = 10/9 > 1$,
$\Longrightarrow$ an arrow $M_1 \to M_2$ is not enough, $\Longrightarrow$ we need an arrow $C \to M_2$.



Is an arrow $C \to M_2$ necessary & enough? Yes, it is.

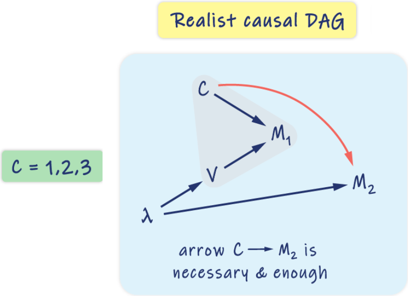
Structural causal model:

$\lambda = Uni(1, 3)$,
$M_1 := M_1(\lambda, C) = \delta_{\lambda,C}$,

$$M_2 := M_2(\lambda, C) = \begin{cases} \delta_{\lambda,C} \cdot N, & C = 1, 2, \\ (1 - \delta_{\lambda,C}) \cdot (1 - N) + \delta_{\lambda,C} \cdot N, & C = 3, \end{cases}$$

$N \sim Ber(1/3)$- noise variable.

Results: Realist causal DAG, $C = 1, 2, 3$



Structural causal model:

$\lambda = Uni(1, 3)$,

$M_1 := M_1(V, C) = \delta_{V,C}$,

$V := V(\lambda) = \lambda$,

$M_2 := M_2(\lambda, C) =$

$$\begin{cases} \delta_{\lambda,C} \cdot N, & C = 1, 2, \\ (1 - \delta_{\lambda,C}) \cdot (1 - N) + \delta_{\lambda,C} \cdot N, & C = 3, \end{cases}$$

$N \sim Ber(1/3)$- noise variable.

$C = 1, 2$

$P(M_1, M_2 \vert C)$	$M_2 = 0$	$M_2 = 1$
$M_1 = 0$	$2/3$	0
$M_1 = 1$	$2/9$	$1/9$

$C = 3$

$P(M_1, M_2 \vert C)$	$M_2 = 0$	$M_2 = 1$
$M_1 = 0$	$2/9$	$4/9$
$M_1 = 1$	$2/9$	$1/9$

Summary

- We have presented two possible models of causal DAGs that reproduce the statistics of three box experiment.
- There is a difference whether the full statistics is considered $C = \{1, 2, 3\}$ or just the famous *paradoxical* part related to the choice of measurements $C = \{1, 2\}$.
- The choice of a structure (*pure causal vs. realist causal*) impacts which measurement disturbance needs to be taken into account in description of the experiment.
- The famous *paradoxical* part related to the choice of measurements $C = \{1, 2\}$, does not require any additional arrows in the pure causal structure. The statistics can be explained by a hidden/latent variable.
- Whatever structure is taken into account (*pure causal or realist causal*), when we analyze the full statistics $C = \{1, 2, 3\}$, the same additional arrows are needed.

Thank you for your attention!