

# A structure theorem for transformations in noncontextual models

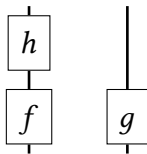
Matthew F. Pusey <m@physics.org>

joint work with John Selby, David Schmid and  
Rob Spekkens

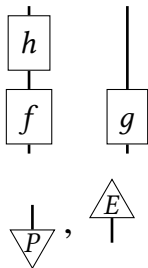
Part I

Diagrams

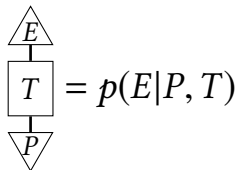
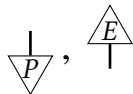
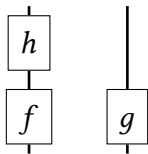
# Operational diagrams



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# Operational equivalence

$$\frac{\triangle E}{\triangle P} = \frac{\triangle E'}{\triangle P} \quad \forall P$$


# Operational equivalence

$$\frac{\triangle E}{\perp P} = \frac{\triangle E'}{\perp P} \quad \forall P$$




$$\triangle E = \triangle E'$$

# Ontological diagrams

The diagram symbol consists of a light blue square containing a white inverted triangle. Inside the triangle is the letter 'P'. A vertical line extends downwards from the top vertex of the triangle. To the right of the triangle is the letter 'M' in a subscript position.
$$\square \begin{array}{c} \downarrow \\ \triangle P \\ M \end{array} = p(\lambda|P)$$




# Ontological diagrams



The diagram consists of a light blue square containing an inverted white triangle with the letter 'P' inside. A vertical line extends upwards from the top vertex of the triangle. To the right of the square is the letter 'M'.


$$= p(\lambda|P)$$




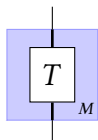
The diagram consists of a light blue square containing a white triangle with the letter 'E' inside. A vertical line extends downwards from the bottom vertex of the triangle. To the right of the square is the letter 'M'.

$$= p(E|\lambda)$$

# Ontological diagrams

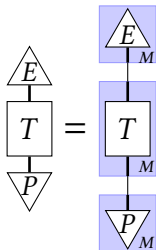
An ontological diagram consisting of a light blue square containing an inverted white triangle with the letter 'P' inside. A vertical line passes through the center of the triangle, extending above and below the square. The letter 'M' is positioned at the bottom right corner of the square.
$$P_M = p(\lambda|P)$$

An ontological diagram consisting of a light blue square containing a white triangle with the letter 'E' inside. A vertical line passes through the center of the triangle, extending above and below the square. The letter 'M' is positioned at the bottom right corner of the square.
$$E_M = p(E|\lambda)$$

An ontological diagram consisting of a light blue square containing a white square with the letter 'T' inside. A vertical line passes through the center of the square, extending above and below the square. The letter 'M' is positioned at the bottom right corner of the square.
$$T_M = p(\lambda'|\lambda, T)$$

# Ontological diagrams

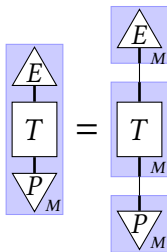
$$p(E|P, T) = \int \int p(E|\lambda')p(\lambda'|\lambda, T)p(\lambda|P)d\lambda'd\lambda$$



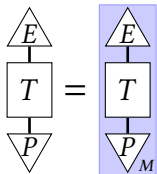
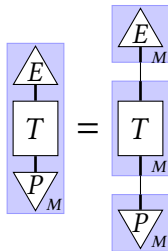
# Part II

## Statement of structure theorem

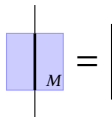
# Assumption 1: Diagram preservation (uncontroversial cases)



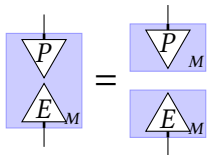
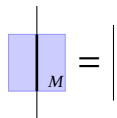
# Assumption 1: Diagram preservation (uncontroversial cases)



# Assumption 1: Diagram preservation (controversial cases?)



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$$\begin{array}{|c} \hline \\ \hline \end{array}_M = \begin{array}{|c} \hline \\ \hline \end{array} = \delta(\lambda' - \lambda)$$

$$\begin{array}{|c} \hline \triangle P \\ \hline \triangle E_M \\ \hline \end{array} = \begin{array}{|c} \hline \triangle P \\ \hline \triangle E_M \\ \hline \end{array}_M = p(\lambda'|P)p(E|\lambda)$$

## Assumption 2: Local tomography

$$\begin{array}{c} \triangle E \quad \triangle E' \\ | \quad | \\ \boxed{T} \\ | \quad | \\ \nabla P \quad \nabla P' \end{array} = \begin{array}{c} \triangle E \quad \triangle E' \\ | \quad | \\ \boxed{T'} \\ | \quad | \\ \nabla P \quad \nabla P' \end{array} \quad \forall E, E', P, P' \iff$$
  
$$\begin{array}{c} | \quad | \\ \boxed{T} \\ | \quad | \end{array} = \begin{array}{c} | \quad | \\ \boxed{T'} \\ | \quad | \end{array}$$

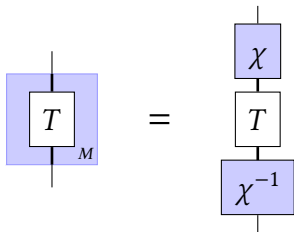
## Assumption 2: Local tomography (equivalent condition)

For any  $T$  there exist  $r_{ij}^T$  such that

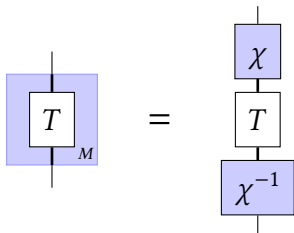
$$\begin{array}{c} | \\ \square \\ | \end{array} T = \sum_{ij} r_{ij}^T \begin{array}{c} | \\ \triangle \\ P_i \\ \triangle \\ E_j \\ | \end{array}$$

# Structure theorem

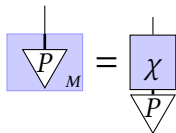
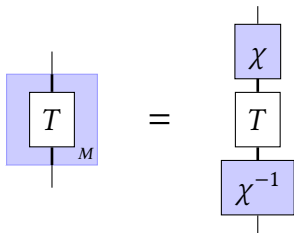
A diagram-preserving noncontextual model  $M$  of a locally tomographic theory can be written



What is  $\chi$ ?



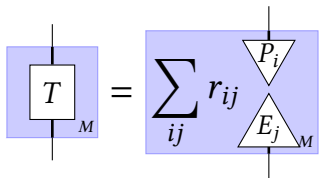
What is  $\chi$ ?



# Part III

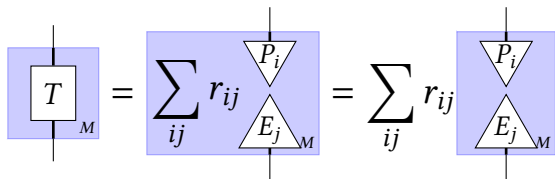
## Proof of structure theorem

# Step 1

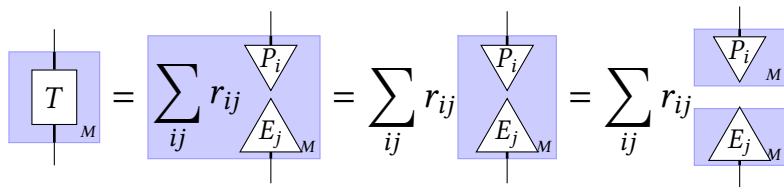




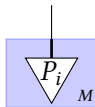
# Step 1



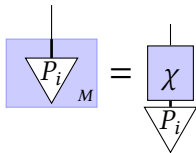
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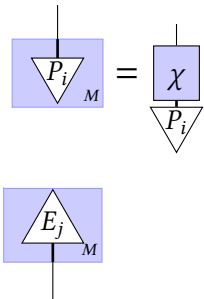
## Step 2



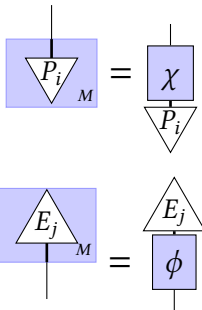
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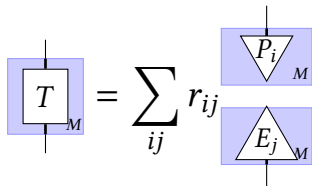
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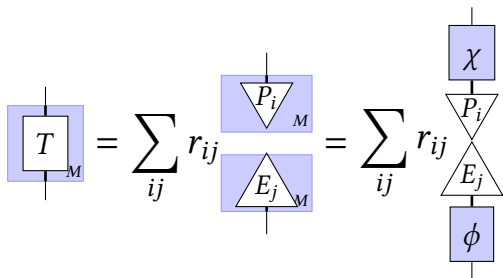
## Step 2



# Step 1 + Step 2

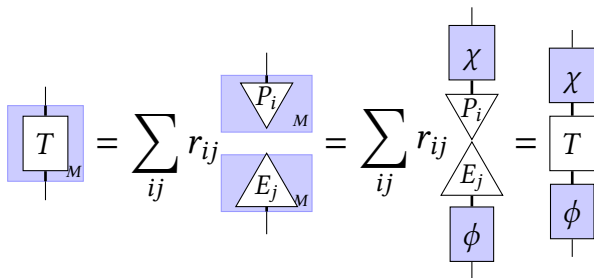


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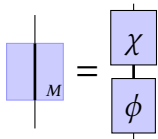




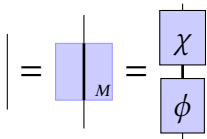
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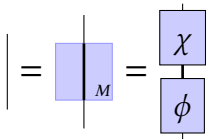
# Step 3



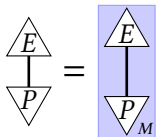
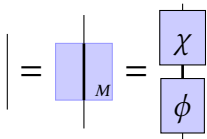
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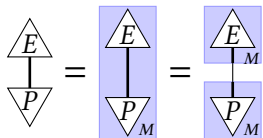
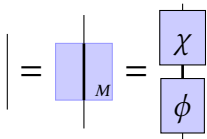
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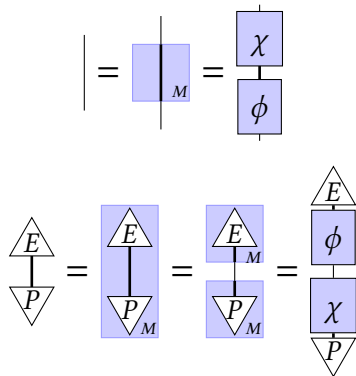
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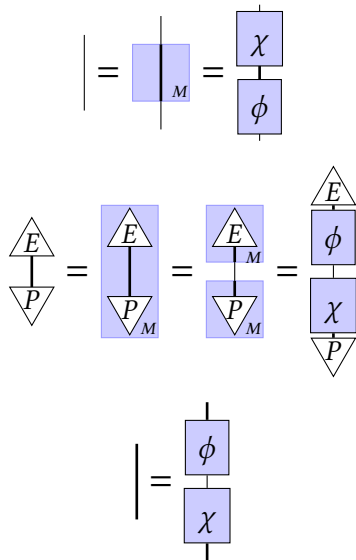
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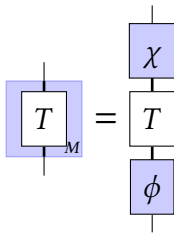


# Step 3

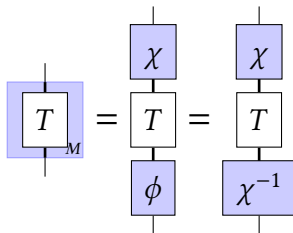




# QED



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# Part IV

## Consequences and other versions

# Number of ontic states

Number of ontic states = dimension of state space

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c.f. Hardy, Stud. Hist. Phil. Mod. Phys. **35** 267  
(2004)

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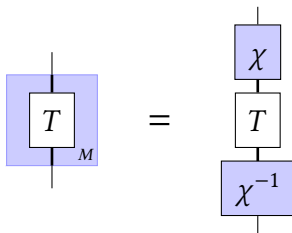
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c.f. Wallman & Bartlett, arXiv:1203.2652



# Quasiprobability representations

A diagram-preserving quasiprobability representation  $M$  of a locally tomographic theory can be written



# Stabilizer subtheory

The only noncontextual model of the stabilizer subtheory is

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We prove that there is a unique nonnegative and diagram-preserving quasiprobability representation of the stabilizer subtheory in all odd dimensions, namely Gross's discrete Wigner function. This representation is equivalent to Spekkens' epistemically restricted toy theory, and is consequently singled out as the unique noncontextual ontological model for the stabilizer subtheory. Strikingly, the principle of noncontextuality is powerful enough (at least in this setting) to rule out *one particular* classical realist interpretation. Our result explains the practical utility of the representation, e.g. why (in the setting of the stabilizer subtheory) negativity in this representation implies generalized contextuality, and hence sheds light on why negativity is a resource for quantum computational speedup. It also allows us to prove that generalized contextuality is a necessary resource for universal quantum computation in the state injection model. In all even dimensions, we prove that there does not exist any nonnegative and diagram-preserving quasiprobability representation of the stabilizer subtheory, and, conversely, the stabilizer subtheory is contextual in all even dimensions. Together, these results constitute a complete characterization of the (non)classicality of all stabilizer subtheories.

# Conclusions and outlook

- ▶ Noncontextual models (and quasiprobability representations) of locally tomographic theories have a very rigid structure
  - ▶ Fixed number of ontic states
  - ▶ Only freedom is representation of preparations
- ▶ Any model without this structure can immediately be declared contextual
- ▶ Computational searches for noncontextual models can leverage this structure
- ▶ First step towards unique characterization of noncontextual models of the Stabilizer subtheory