Converting contextuality into nonlocality

(From nonlocality to contextuality and back)

Adán Cabello

UNIVERSIDAD Ð SEVILLA 1505

INSTITUTO CARLOS I DE FÍSICA TEÓRICA Y COMPUTACIONAL

QUANTERA

GOBIERNO DE ESPAÑA MINISTERIO DE CIENCIA E INNOVACIÓN

SECRET SECuRe quantum communication based on Energy-Time/time-bin entanglement

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UNIÓN EUROPEA Fondo Europeo de Desarrollo Regional

QDISC Quantum Device-Independent Secure Communication

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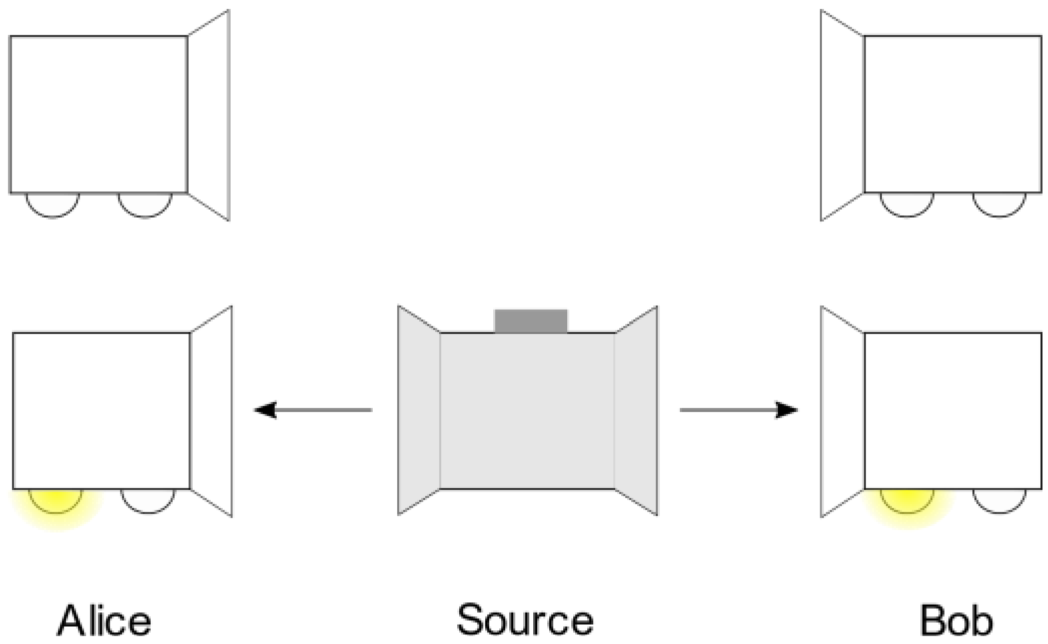
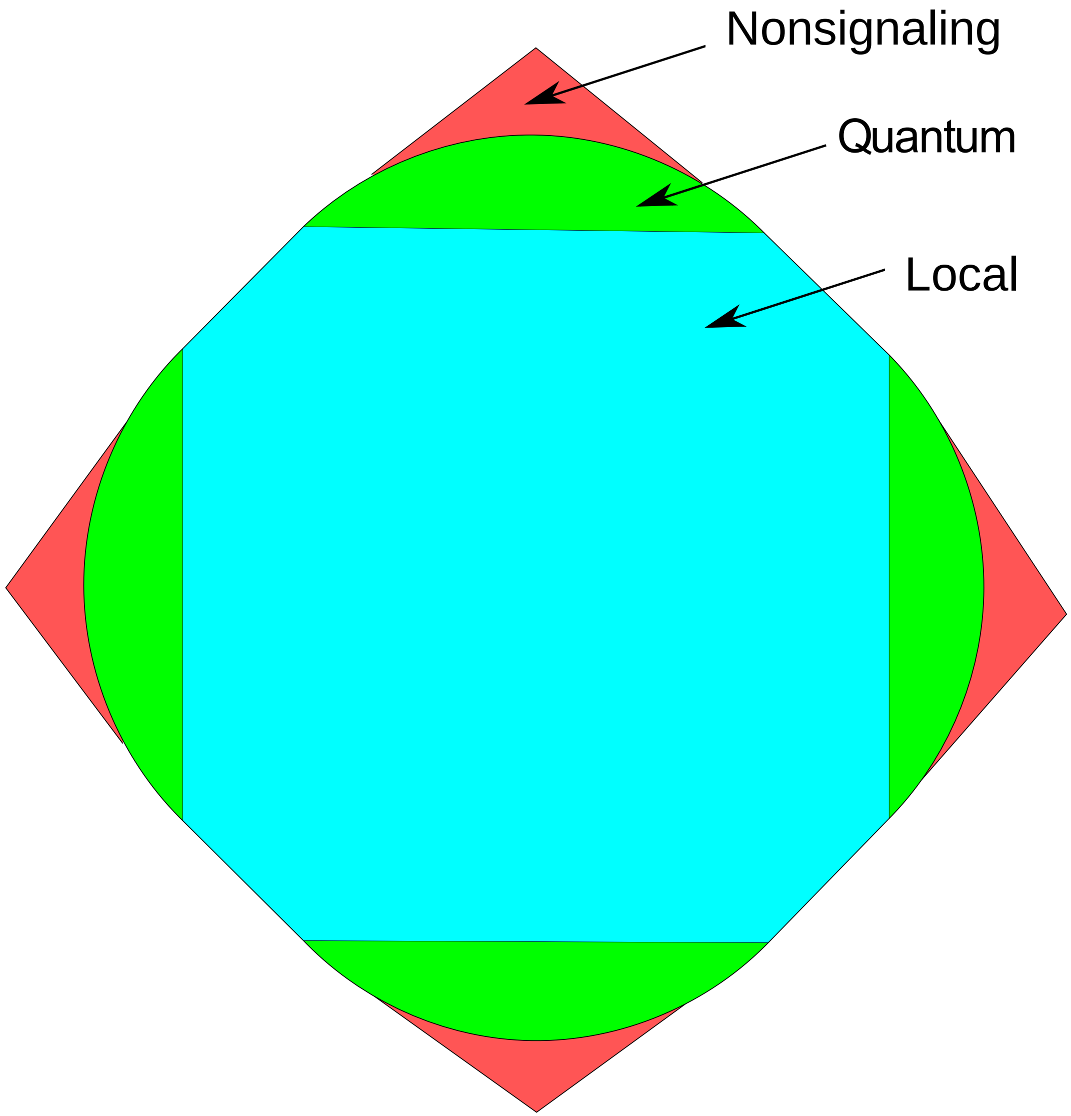
Bell nonlocality experiment

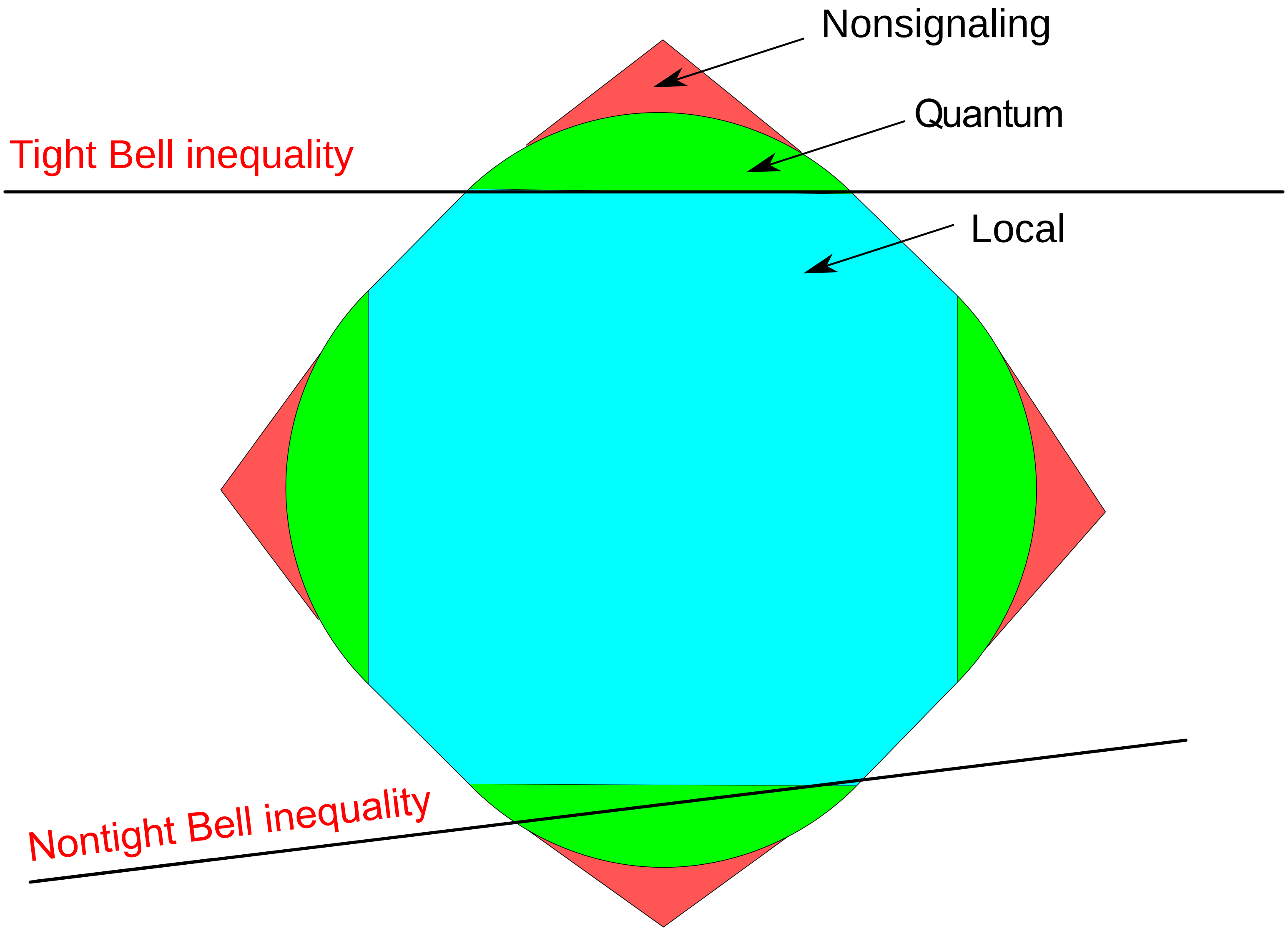


- Composite systems needed
- Spacelike separation needed
- Entanglement needed
- Measurements can be destructive

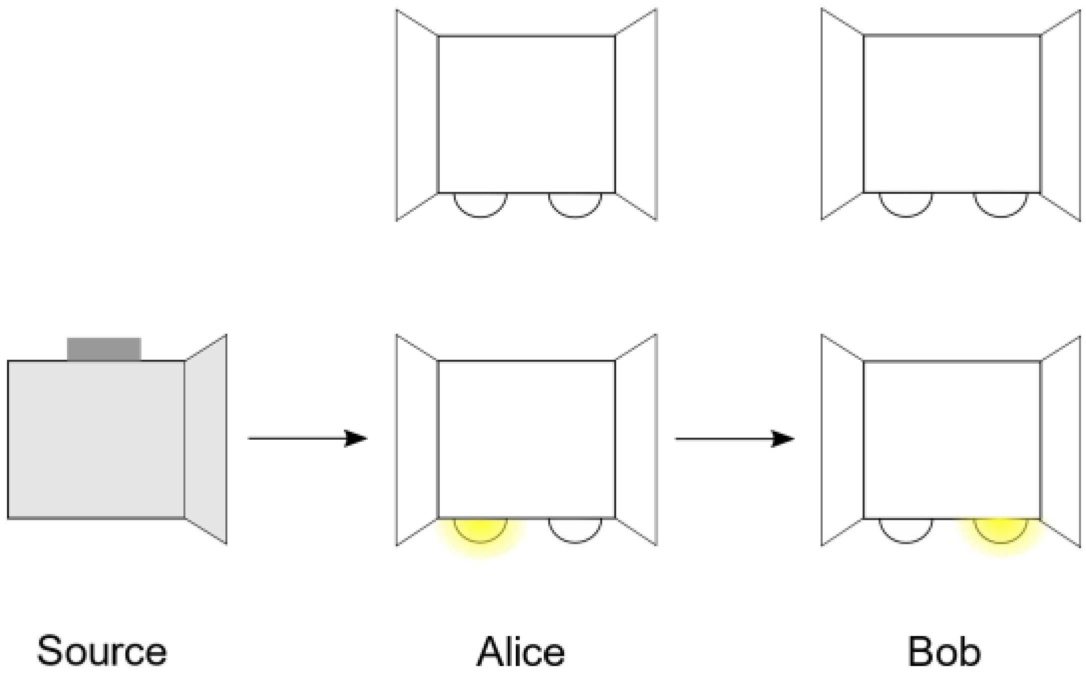
Geometry of the sets of behaviors



Bell inequalities

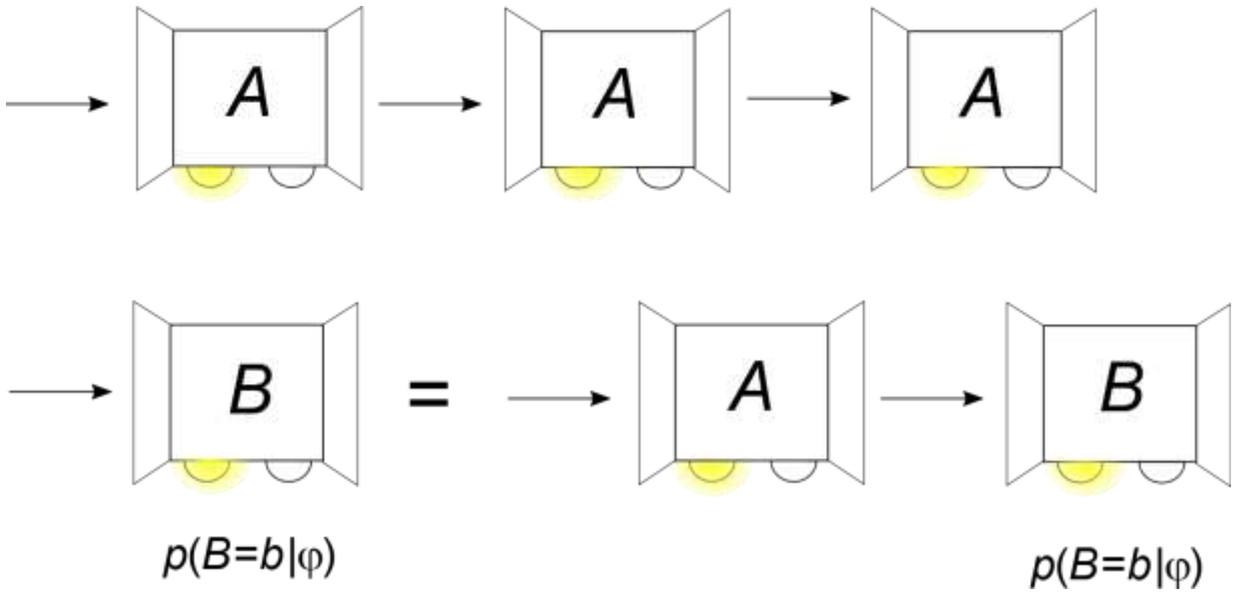


Kochen-Specker contextuality experiment



- Composite systems not needed
- Spacelike separation not needed
- Entanglement not needed
- Sharp measurements needed

Sharp (aka ideal) measurements



- Yield the same result when repeated
- Do not disturb compatible observables

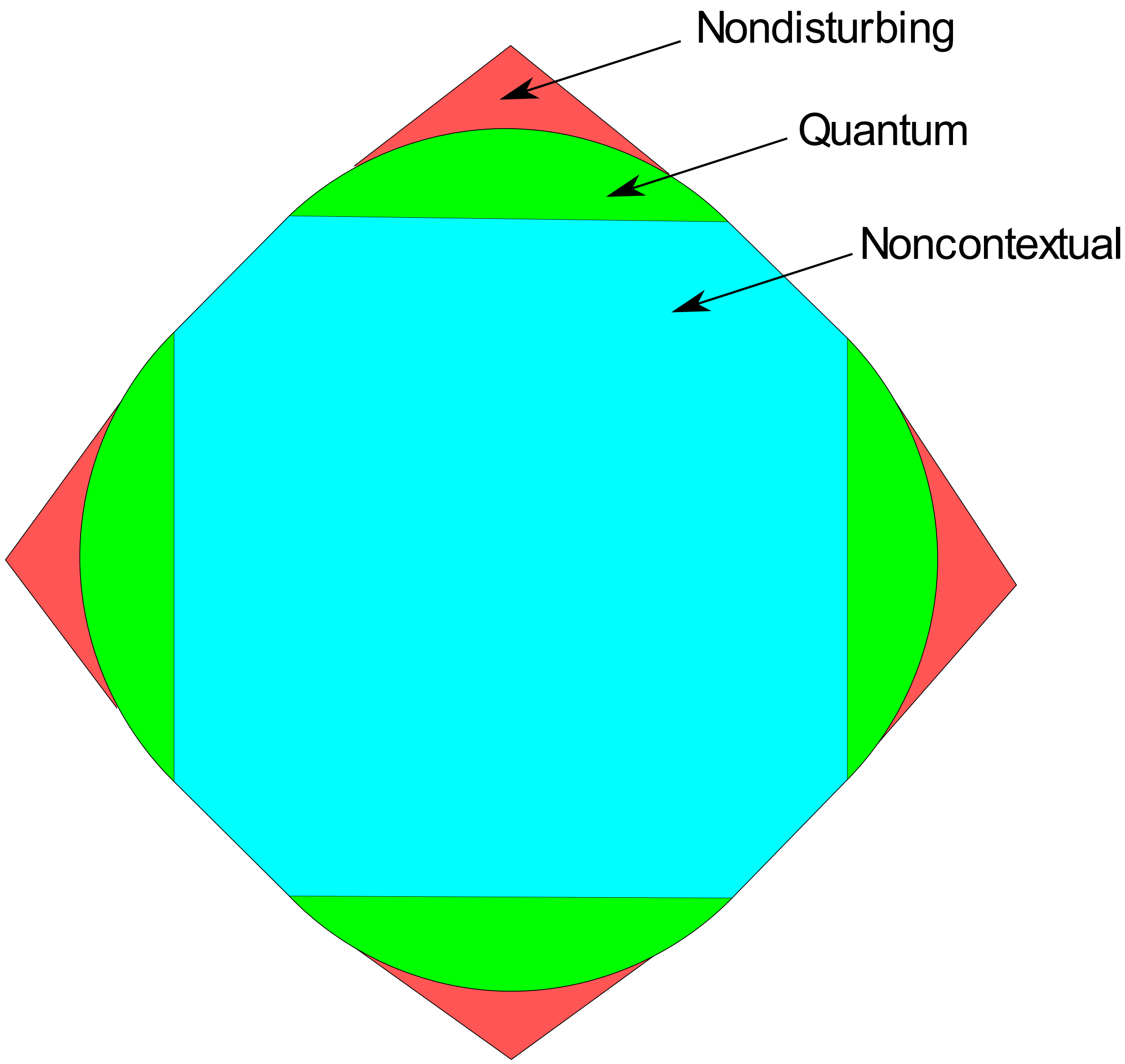
- Compatible (aka joint measurable) = having a common refinement

Why measurements should be sharp?

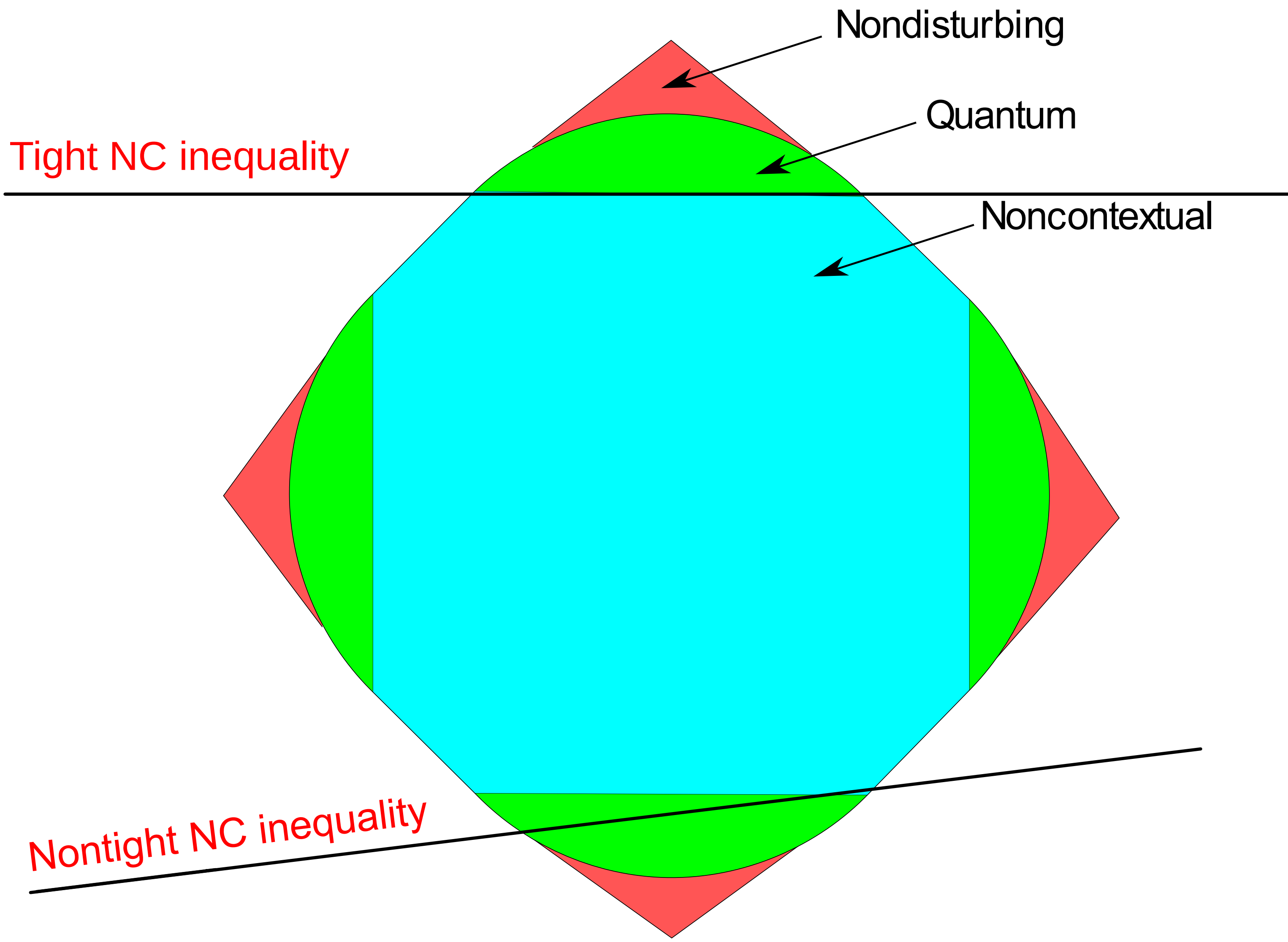
- Because otherwise the assumption of outcome noncontextuality for contexts made of compatible observables is not justified appealing to classical physics

- R. W. Spekkens, Phys. Rev. A **71**, 052108 (2005); Found. Phys. **44**, 1125 (2014)
- T. Fritz, Rev. Math. Phys. **24**, 1250012 (2012)
- J. Henson and A. B. Sainz, Phys. Rev. A **91**, 042114 (2015)
- R. Kunjwal, Quantum **4**, 219 (2020)
- C. Budroni *et al.*, arXiv: 2102.13036

Geometry of the sets of behaviors



Noncontextuality inequalities



Iff measurements are sharp

- Then, for <u>any</u> scenario whose graph of compatibility is completely *n*-partite and each part has at least two incompatible observables (e.g., the scenario whose graph of compatibility is a square)
  - *Local set = noncontextual set*
  - *Quantum Bell nonlocal = quantum KS contextual*
  - *Tight Bell inequality = tight noncontextuality inequality*

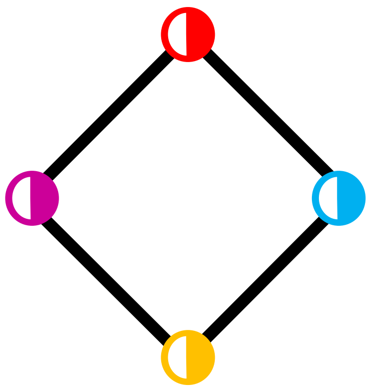
Vorob'yev's theorem

Theorem 1. *[7] The only contextuality scenarios that admit contextual behaviors are those in which the graph of compatibility is nonchordal. That is, it contains, as induced subgraph, at least one cycle of four or more vertices (i.e., squares, pentagons, hexagons, etc.).*

Definition 8. *An* induced subgraph *of a graph G is a graph formed from a subset of the vertices of G and all of the edges connecting pairs of vertices in that subset.*

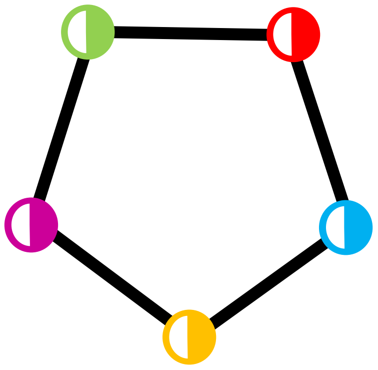
KS contextuality scenarios

Colloraly 1. *The simplest scenario in which contextuality with ideal measurements is possible is the one consisting of four dichotomic observables whose graph of compatibility is a square. The* Clauser-Horne-Shimony-Holt [8] Bell scenario *has this graph of orthogonality.*



Theorem 3. *[10, 11] In quantum theory, contextuality for ideal measurements requires quantum systems of dimension three or higher.*

Theorem 4. *[12] The simplest scenario in which contextuality with ideal measurements on qutrits is possible is the one consisting of five dichotomic observables whose graph of compatibility is a pentagon. This is the* Klyachko-Binicioğlu-Can-Shumovsky (KCBS) scenario *[12].*



Why focusing on KS scenarios rather than in Bell's?

- To understand why quantum theory
- Quantum theory produces contextuality in <u>all</u> scenarios in which Vorob'yev's theorem allows for contextuality
- <u>Most</u> scenarios for which Vorob'yev's theorem allows for contextuality are not Bell scenarios
- If we understand which principle singles out quantum contextuality we will understand much more than Bell nonlocality (without assuming quantum theory)

Z.-P. Xu and A. Cabello, Necessary and sufficient condition for contextuality from incompatibility, Phys. Rev. A **99**, 020103 (2019).

How to attract Bell nonlocality fans?

- Many people that love Bell nonlocality have never been interested in KS contextuality
  - They may argue that sharp measurements are “unphysical” (and therefore that quantum theory is a theory about unphysical measurements)
  - They may argue that compatibility cannot be granted for sequential measurements
  - They may argue that KS contextuality for single systems can be classically simulated
- How to attract Bell nonlocality fans to KS contextuality?
- Answer: By converting <u>any</u> example of quantum KS contextuality into a <u>related</u> example of quantum Bell nonlocality

Question

- Given <u>any</u> set S of sharp measurements producing a violation Q of a noncontextuality inequality with noncontextual bound k

- Can two parties, each of them having S, produce a violation Q of a Bell inequality (formally identical to the previous noncontextuality inequality) with local bound k?

- <u>Answer</u>: No

Result

- Given <u>any</u> S that violates a noncontextuality inequality
- There is always S' such that $S \cup S'$ produce a state-independent violation Q of a noncontextuality inequality with noncontextual bound k
- and two parties, one of them having $S \cup S'$ and the other the transpose projectors, can produce a violation Q of a Bell inequality (formally identical to the previous noncontextuality inequality) with bound k
- The state-independent violation and the Bell violation disappear if we remove any element of $S \cup S'$

Contextuality and graphs

Theorem 5. *[13, 14] Every quantum contextual behavior produced by a set of projectors $S = \{\Pi_i, \ldots, \Pi_n\}$ violates a noncontextuality (NC) inequality of the form*

$$\sum_{i \in V(\mathcal{G})} w_i P(\Pi_i = 1) - \sum_{(i,j) \in E(\mathcal{G})} \max(w_i, w_j) P(\Pi_i = 1, \Pi_j = 1) \overset{\text{NCHV}}{\leq} \alpha(\mathcal{G}, w)$$

where G is the graph of orthogonality of S, $E(G)$ is the set of edges of G, $w = (w_1, \ldots, w_n)$ are nonnegative numbers, and $\alpha(G, w)$ is the independence number of the weighted graph (G, w) (i.e., of the graph G in which vertex i has associated weight w_i).

[13] A. Cabello, S. Severini, and A. Winter, Graph-theoretic approach to quantum correlations, Phys. Rev. Lett. **112**, 040401 (2014).

[14] A. Cabello, Simple method for experimentally testing any form of quantum contextuality, Phys. Rev. A **93**, 032102 (2016).

Take out message

- Any example of quantum KS contextuality can be converted into a violation of a noncontextuality inequality whose bound is the independence number of the graph of orthogonality of a set *S* of projectors

State-independent contextuality sets

Definition 14. *A* state-independent contextuality (SI-C) set *is a set of projectors that produces contextual behaviors for any initial state.*

Remark 10. *To every SI-C set of rank-n projectors one can associate a SI-C set of rank-one projectors.*

Remark 11. *[31, 39] Every proof of state-independent contextuality made of self-adjoint operators which are not rank-n projectors (e.g., [40, 41]) has associated a SI-C set of rank-one projectors.*

Theorem 7. *[17] A set of projectors $S = \{\Pi_i, \ldots, \Pi_n\}$ is a SI-C set if and only if there are nonnegative numbers $w = (w_1, \ldots, w_n)$ and a number $0 \leq y < 1$ such that $\sum_{j \in \mathcal{I}} w_j \leq y$ for all $\mathcal{I}$, where $\mathcal{I}$ is any independent set of the graph of orthogonality of S, and $\sum_i w_i \Pi_i \geq \mathbb{1}$.*

[17] A. Cabello, M. Kleinmann, and C. Budroni, Necessary and Sufficient Condition for Quantum State-Independent Contextuality, Phys. Rev. Lett. **114**, 250402 (2015).

SI-C inequalities

In particular, w gives rise to a NC inequality violated by any quantum state, inequality that can be written as

$$\sum_{i \in V(\mathcal{G})} w_i P(\Pi_i = 1) - \sum_{(i,j) \in E(\mathcal{G})} \max(w_i, w_j) P(\Pi_i = 1, \Pi_j = 1) \overset{\text{NCHV}}{\leq} \alpha(\mathcal{G}, w)$$

Observation

Definition 18. *A* Kochen-Specker (KS) assignment *to a set of rank-one projectors is an assignment of* 0 *or* 1 *satisfying that: (I) two orthogonal projectors cannot both have assigned* 1*, (II) for every set of mutually orthogonal projectors summing the identity, one of them must be assigned* 1.

Definition 19. *A* KS set *is a set of rank-one projectors which does not admit a KS assignment.*

- All KS sets are SI-C sets
- Most SI-C sets are not KS sets

Critical SI-C sets

Definition 20. *A KS set S is* critical *if by removing any element of S the resulting set is not a KS set.*

Remark 7. *[28] The original KS set [11] is critical.*

Definition 21. *A SI-C set S is* critical *if by removing any element of S the resulting set is not a SI-C set.*

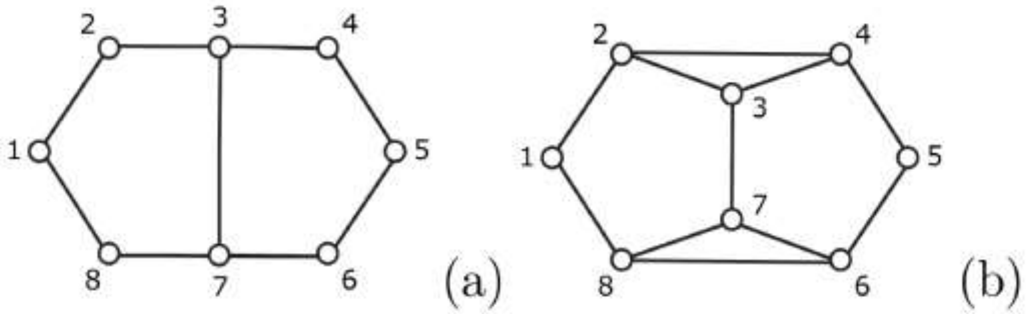
J. Zimba and R. Penrose, On Bell non-locality without probabilities: More curious geometry, Stud. Hist. Philos. Sci. A **24**, 697 (1993).

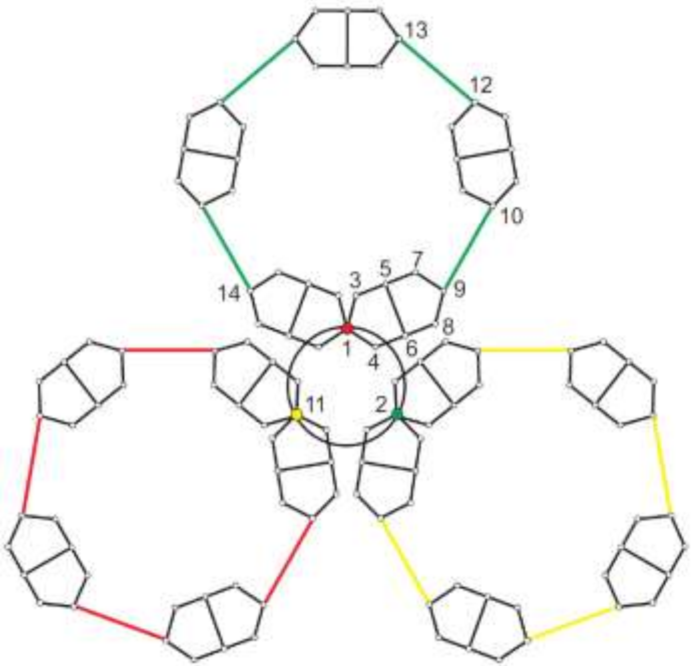
True-implies-false sets

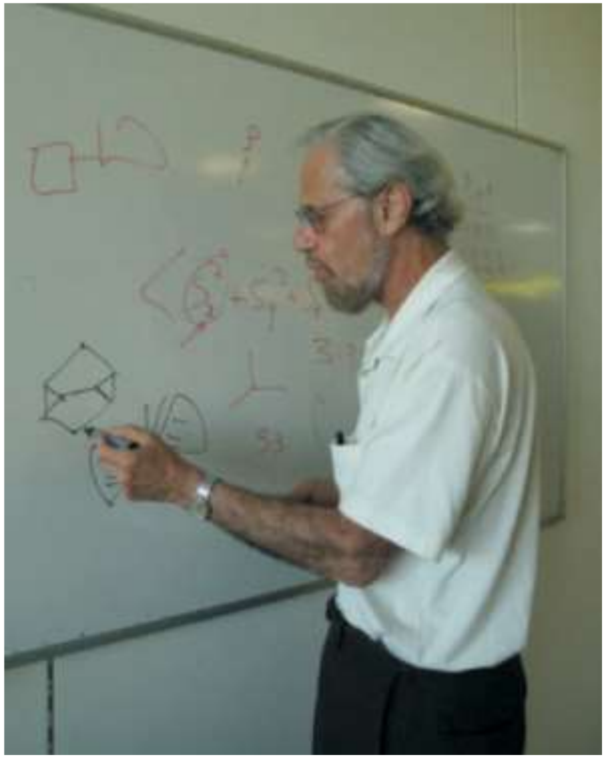
Definition 22. *A set of projectors* $S = \{\Pi_A = |\psi_A\rangle\langle\psi_A|\} \cup \{\Pi_i, \ldots, \Pi_n\} \cup \{\Pi_B = |\psi_B\rangle\langle\psi_B|\}$ *in dimension* $d \geq 3$, *where* Π_A *and* Π_B *are nonorthogonal and* $\{\Pi_i, \ldots, \Pi_n\}$ *are not necessarily rank-one projectors, is a* true-implies-false set (TIFS) *if, for any KS assignment* f, $f(\Pi_A) + f(\Pi_B) \leq 1$. *Therefore,* $f(\Pi_A) = 1$ *implies* $f(\Pi_B) = 0$, *and* $f(\Pi_B) = 1$ *implies* $f(\Pi_A) = 0$.

TIFSs made of rank-one projectors are called *definite prediction sets* in [43], 01-*gadgets* in [44], and *Hardy-like proofs* in [19] (after

True-implies-false sets







From state-dependent contextuality to critical SI-C

Theorem 27. *Any SD-C set S of projectors can be extended to a critical SI-C set containing all the projectors of S.*

From critical SI-C to Bell nonlocality

- Physical implementation (the same as in Stairs 1983)

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |kk\rangle$$

- Alice has *S*

- Bob has the transpose projectors

The Bell inequality

$$\sum_{i\in V(\mathcal{G})} w_i P(\Pi_i^A = 1, \Pi_i^B = 1) - \sum_{(i,j)\in E(\mathcal{G})} \frac{\max(w_i, w_j)}{2} \left[P(\Pi_i^A = 1, \Pi_j^B = 1) + P(\Pi_j^A = 1, \Pi_i^B = 1)\right] \overset{\text{LHV}}{\leq} \alpha(\mathcal{G}, w)$$

- Formally identical to the SI-C inequality

$$\sum_{i\in V(\mathcal{G})} w_i P(\Pi_i = 1) - \sum_{(i,j)\in E(\mathcal{G})} \max(w_i, w_j) P(\Pi_i = 1, \Pi_j = 1) \overset{\text{NCHV}}{\leq} \alpha(\mathcal{G}, w)$$

Examples of SI-C producing Bell nonlocality

A. Stairs, Quantum logic, realism, and value definiteness, Philos. Sci. **50**, 578 (1983).

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Examples of SI-C producing Bell nonlocality

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S. Abramsky, R. Soares Barbosa, N. de Silva, and O. Zapata, The quantum monad on relational structures, in *42nd International Symposium on Mathematical Foundations of Computer Science (MFCS 2017)*, edited by K. G. Larsen, H. L. Bodlaender, and J.-F. Raskin (Schloss Dagstuhl–Leibniz-Zentrum für Informatik, Saarbrücken, 2017) Leibniz International Proceedings in Informatics **83**, p. 35:1.

Examples of SI-C producing Bell nonlocality

- Convert some SI-C sets into nonlocal games
- Do not use SI-C inequalities
- Do not preserve *k* and *Q*
- Some add constraints (in addition to the assumption of outcome noncontextuality)

Example of how the method works

- We are given a set *S* of measurements which (for the right states) violate a noncontextuality inequality

- Example:

$$\Pi_i = |v_i\rangle\langle v_i|$$

$$\begin{aligned}
\langle v_1| &= (1,0,0)\,, \\
\langle v_2| &= \tfrac{1}{\sqrt{2}}\,(0,1,1)\,, \\
\langle v_3| &= \tfrac{1}{\sqrt{3}}\,(1,-1,1)\,, \\
\langle v_4| &= \tfrac{1}{\sqrt{2}}\,(1,1,0)\,, \\
\langle v_5| &= (0,0,1)\,,
\end{aligned}$$

$$\sum_{i=1}^{5} P(\Pi_i = 1) - \sum_{i=1}^{5} P(\Pi_i = 1, \Pi_{i+1} = 1) \leq 2.$$

Example of how the method works

- We are given a set *S* of measurements which (for the right states) violate a noncontextuality inequality

- Example:

$$\Pi_i = |v_i\rangle\langle v_i|$$

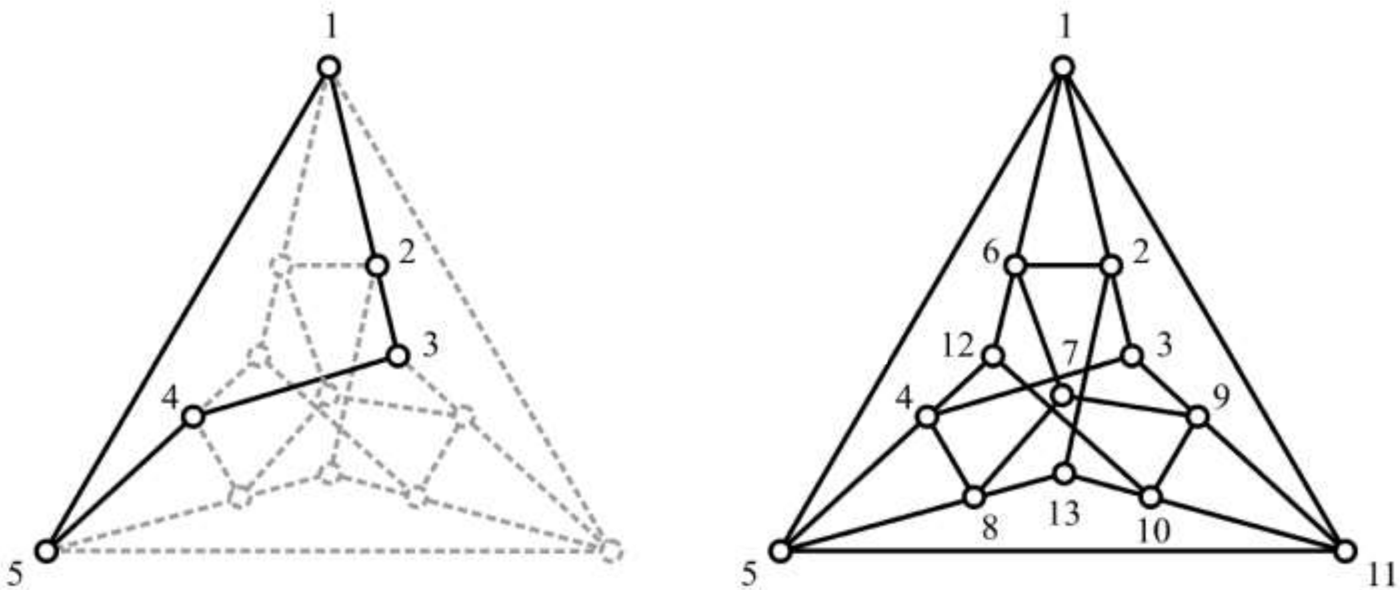
$$\langle\psi| = \frac{1}{\sqrt{3}}(1,1,1)$$

$$\langle v_1| = (1,0,0)\,,$$
$$\langle v_2| = \frac{1}{\sqrt{2}}(0,1,1)\,,$$
$$\langle v_3| = \frac{1}{\sqrt{3}}(1,-1,1)\,,$$
$$\langle v_4| = \frac{1}{\sqrt{2}}(1,1,0)\,,$$
$$\langle v_5| = (0,0,1)\,,$$

$$\sum_{i=1}^{5} P(\Pi_i = 1) - \sum_{i=1}^{5} P(\Pi_i = 1, \Pi_{i+1} = 1) \leq 2.$$

Step 1

- Extend *S* to a minimal critical S-IC set



(a) (b)

$$
\begin{aligned}
\langle v_6| &= \tfrac{1}{\sqrt{2}}\,(0,1,-1)\,,\\
\langle v_7| &= \tfrac{1}{\sqrt{3}}\,(1,1,1)\,,\\
\langle v_8| &= \tfrac{1}{\sqrt{2}}\,(1,-1,0)\,,\\
\langle v_9| &= \tfrac{1}{\sqrt{2}}\,(1,0,-1)\,,\\
\langle v_{10}| &= \tfrac{1}{\sqrt{2}}\,(1,0,1)\,,\\
\langle v_{11}| &= (0,1,0)\,,\\
\langle v_{12}| &= \tfrac{1}{\sqrt{3}}\,(-1,1,1)\,,\\
\langle v_{13}| &= \tfrac{1}{\sqrt{3}}\,(1,1,-1)\,.
\end{aligned}
$$

(c)

It is the Yu-Oh set

Yu, S., and C. H. Oh (2012), Phys. Rev. Lett. **108** (3), 030402.

The Yu-Oh set is the minimal SI-C set

Yu, S., and C. H. Oh (2012), Phys. Rev. Lett. **108** (3), 030402.

Cabello, A., M. Kleinmann, and J. R. Portillo (2016a), J. Phys. A: Math. Theor. **49**, 38LT01.

Step 2

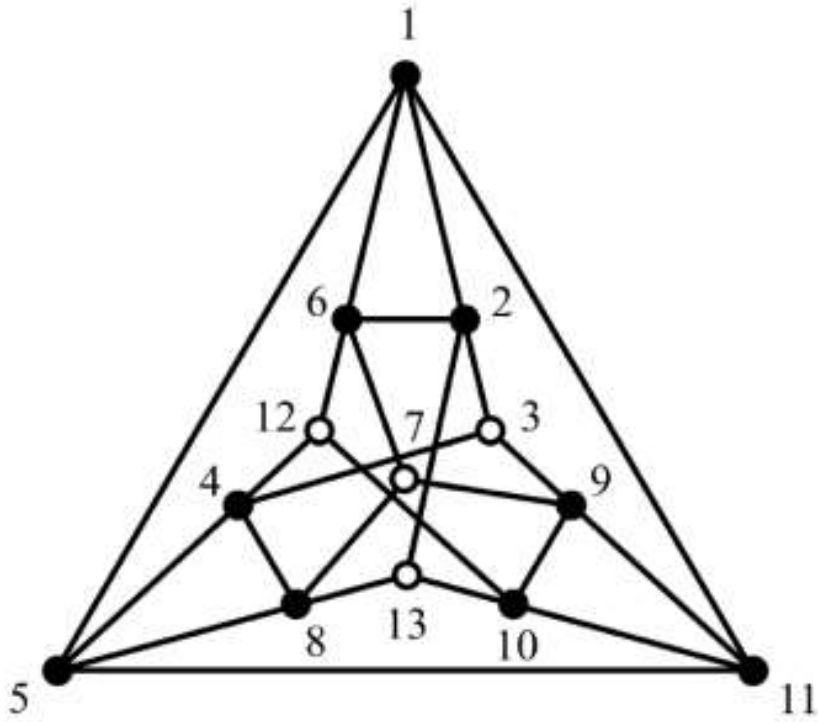
- Find the weights leading to the optimal state-independent violation of

$$\sum_{i\in V(\mathcal{G})} w_i P(\Pi_i = 1) - \sum_{(i,j)\in E(\mathcal{G})} \max(w_i, w_j) P(\Pi_i = 1, \Pi_j = 1) \overset{\text{NCHV}}{\leq} \alpha(\mathcal{G}, w)$$

$$\alpha(\mathcal{G}, w) < Q(\mathcal{G}, w) = \operatorname{tr}\left(\frac{\mathbb{I}}{d} \sum_{i\in V(\mathcal{G})} w_i \Pi_i\right) = \frac{1}{d} \sum_{i\in V(\mathcal{G})} w_i$$

Step 2

- Example $w_i = 2$ for $i = 3, 7, 12, 13$, and $w_i = 3$ otherwi



$$\alpha(\mathcal{G}, w) = 11$$

$$Q(\mathcal{G}, w) = 11 + \frac{2}{3}$$

Step 3

- Inject

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |kk\rangle$$

- Allow Alice to perform the "first" measurement and Bob the (transpose of the) "second" measurement

Step 3

- Then, the correlations between Alice and Bob violate the Bell inequality

$$\sum_{i\in V(\mathcal{G})} w_i P(\Pi_i^A = 1, \Pi_i^B = 1) - \sum_{(i,j)\in E(\mathcal{G})} \frac{\max(w_i, w_j)}{2} \left[P(\Pi_i^A = 1, \Pi_j^B = 1) + P(\Pi_j^A = 1, \Pi_i^B = 1)\right] \overset{\text{LHV}}{\leq} \alpha(\mathcal{G}, w)$$

- Which has as local bound the bound of the original noncontextuality inequality

- As quantum value the original quantum value

Virtues of the method

- Works for <u>**any**</u> quantum contextual correlations

- Convert quantum violation of noncontextuality inequalities that might only be testable by performing <u>**sequential measurements**</u> on single systems into Bell inequalities that can be tested with <u>**local measurements**</u> on spatially separated systems

- The <u>**compatibility/sharpness loophole**</u> in contextuality experiments with sequential measurements <u>**disappears**</u> in the Bell test, as there, measurements are not need to be ideal and measurements on different locations are automatically compatible

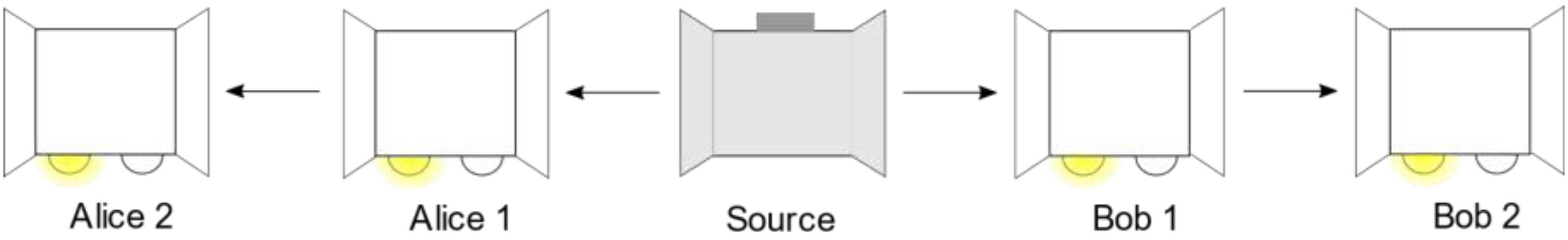
Virtues of the method

- The **quantum/local gap** for the violation of the Bell inequality is the **same as the quantum/noncontextual gap** of the S-IC inequality. And both are produced using the same measurements

- Nonlocality vanishes whenever we remove any element of *S*

Virtues of the method

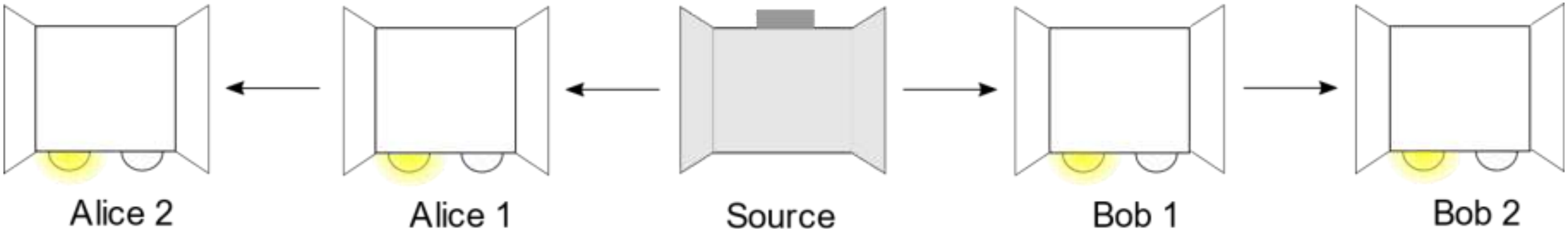
- Allows for tests of the SI-C inequality and the Bell inequality



- SI-C between Alice 1 and Alice 2
- SI-C between Bob 1 and Bob 2
- Bell nonlocality between Alice 1 and Bob 1
- Bell nonlocality between Alice 1 and Bob 2
- Bell nonlocality between Alice 2 and Bob 1
- Bell nonlocality between Alice 2 and Bob 2
- In all cases the same classical bounds and the same quantum values

Virtues of the method

- Allows for simultaneous tests of the SI-C inequality and the Bell inequality



- There is **no "contextuality-nonlocality tradeoff"**. The quantum violations of the SIC inequality and the Bell inequality can be tested simultaneously in the same experiment

P. Kurzyński, A. Cabello, and D. Kaszlikowski, Fundamental Monogamy Relation between Contextuality and Nonlocality, Phys. Rev. Lett. **112**, 100401 (2014).

Limitations of the method

- Except for the case of S-IC sets, the nonlocal correlations resulting from the application of the method do not have the same quantum/local gap than the quantum/noncontextual gap of the original state-dependent contextual correlations

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