CZECH TECHNICAL
UNIVERSITY
IN PRAGUE

Generalised Kochen-Specker Theorem in Three Dimensions

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Kochen–Specker Theorem

Is it possible to assign zeros and ones to vectors of $\mathbb{R}^3$ such that for every three pairwise orthogonal vectors, **precisely one** of them is assigned 1?

Generalised Kochen–Specker Theorem

Is it possible to assign zeros and ones to vectors of $\mathbb{R}^3$ such that for every three pairwise orthogonal vectors, **an odd number** of them is assigned 1?

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Generalised Kochen–Specker Theorem

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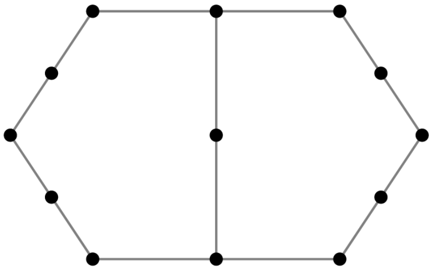
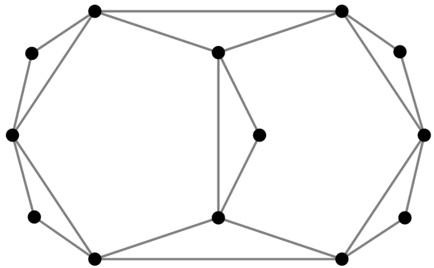
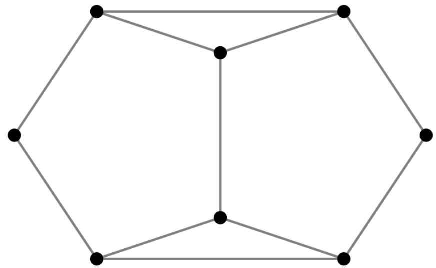
… and there is a vector assigned zero.

Orthogonality Diagrams


(i)


(ii)

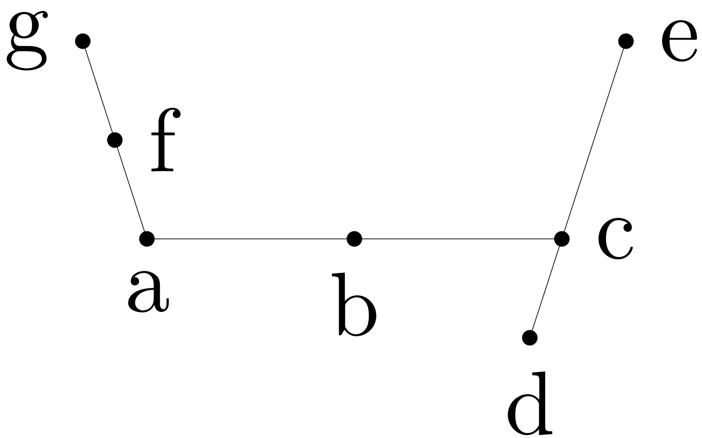

(iii)

Specker's bug

Vertices correspond to points of $\mathbb{R}^3$.
(i) Hypergraph representation: Edges are maximal straight line segments. They contain 3 vertices forming an orthogonal basis.
(ii), (iii) Graph representation: Edges denote orthogonality relations.

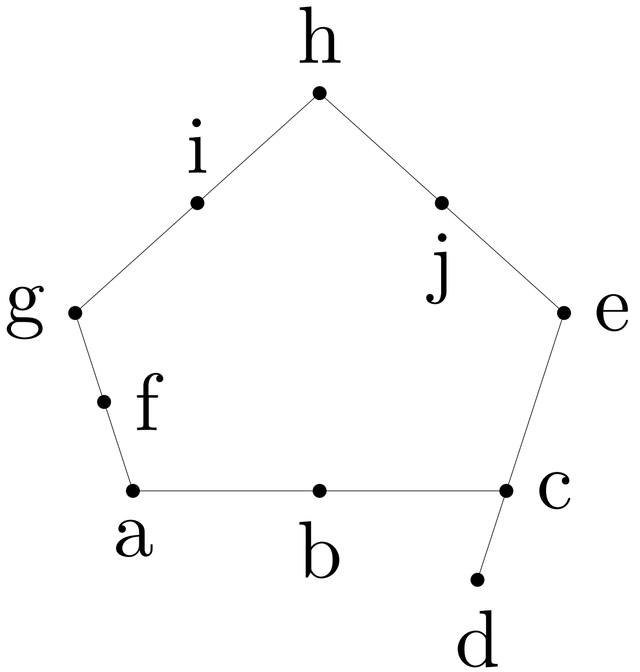
Representation (i) and (ii) are equivalent. Representation (iii) is a subgraph of (ii).

Construction



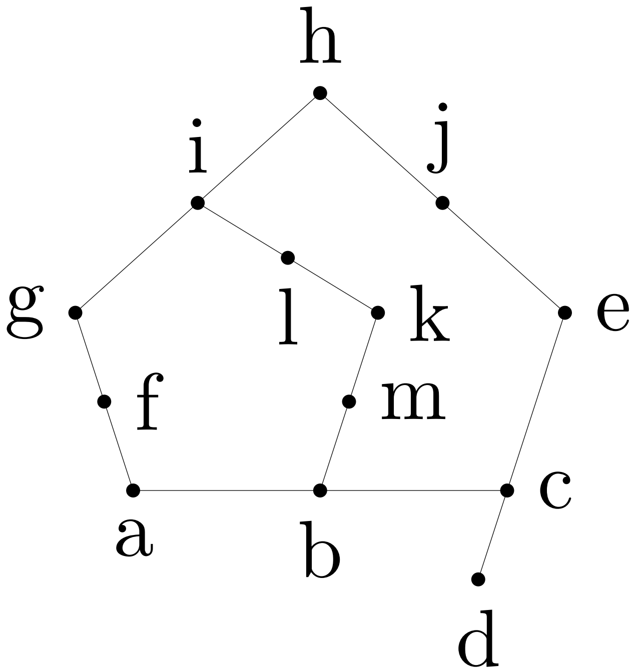
- $a = (1, 0, 0)$
- $b = (0, 1, 0)$
- $c = (0, 0, 1)$
- $d = (x, 1, 0)$
- $e = c \times d$
- $f = (0, 1, y)$
- $g = a \times f$

Construction

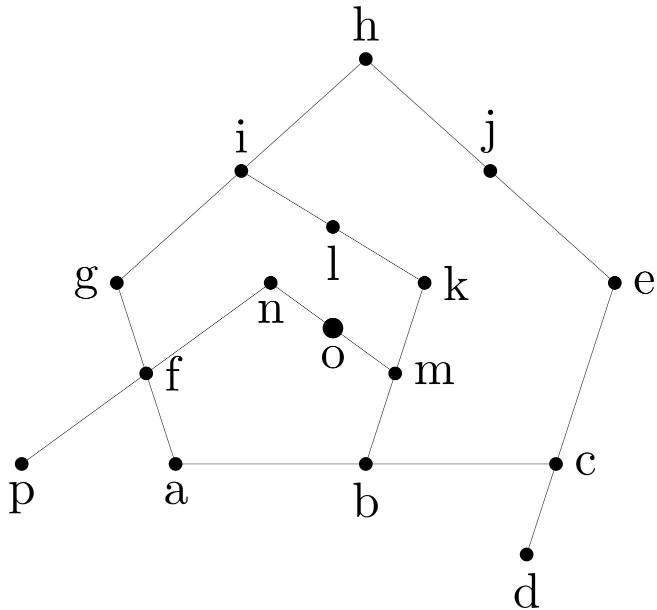


- $h = e \times g$
- $i = g \times h$
- $j = e \times h$

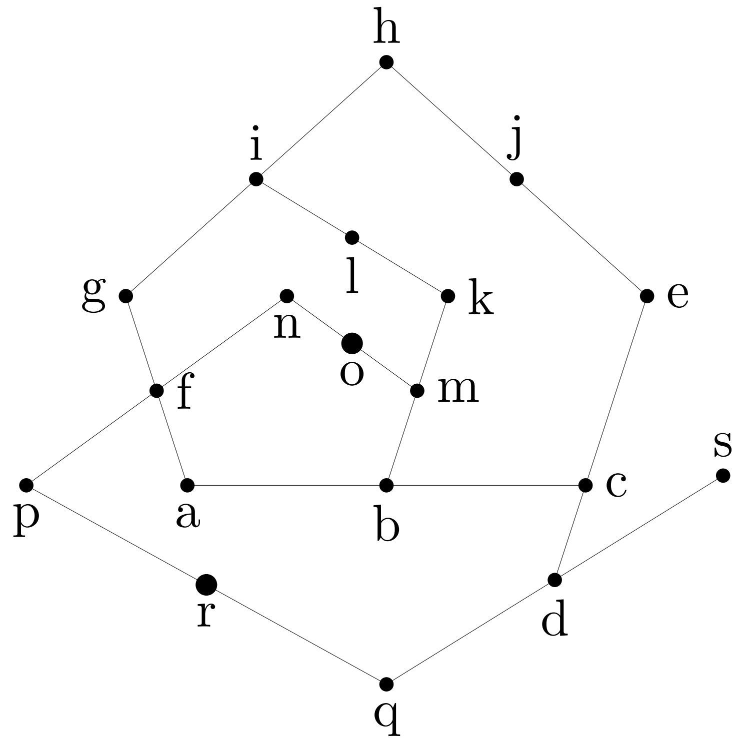
Construction



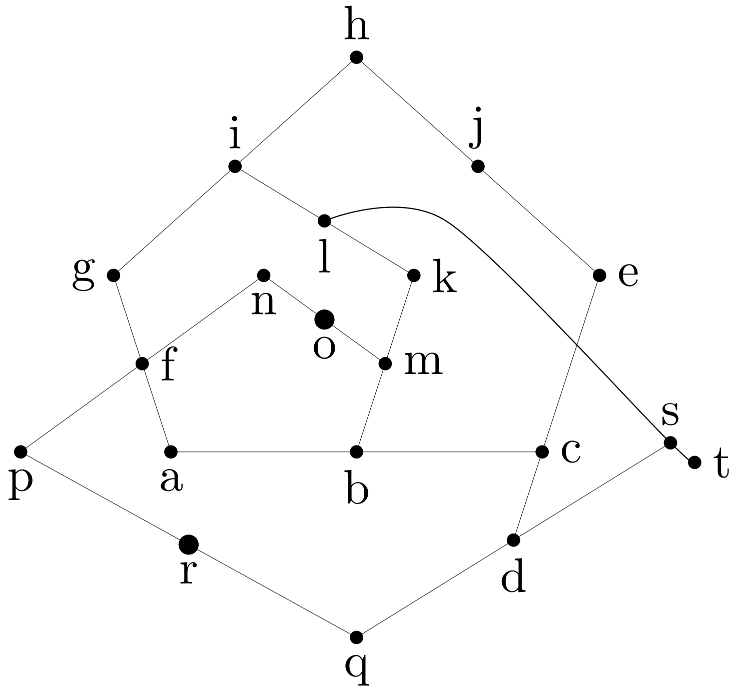
- $\mathsf{k} = \mathsf{b} \times \mathsf{i}$
- $\mathsf{l} = \mathsf{i} \times \mathsf{k}$
- $\mathsf{m} = \mathsf{b} \times \mathsf{k}$



- $\mathsf{n} = \mathsf{f} \times \mathsf{m}$
- $\mathsf{o} = \mathsf{m} \times \mathsf{n}$
- $\mathsf{p} = \mathsf{f} \times \mathsf{n}$

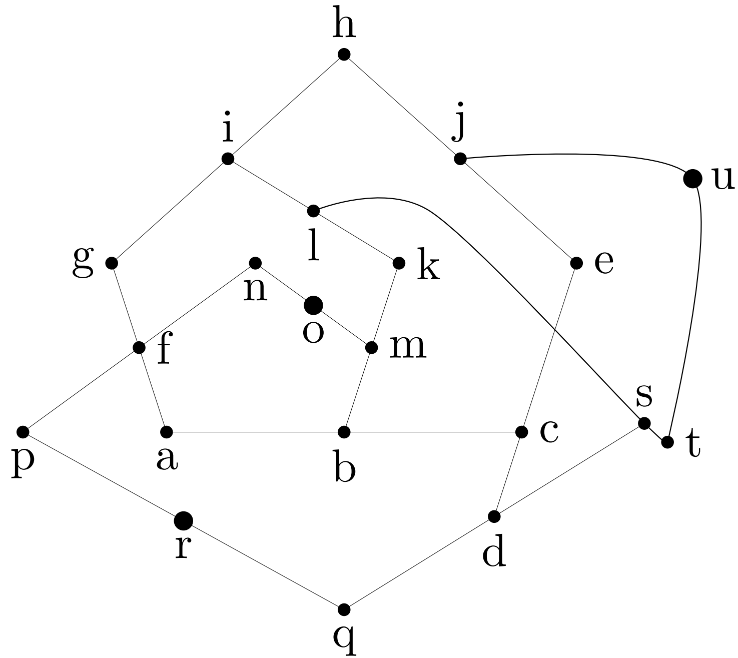


- $\mathbf{q} = \mathbf{d} \times \mathbf{p}$
- $\mathbf{r} = \mathbf{p} \times \mathbf{q}$
- $\mathbf{s} = \mathbf{d} \times \mathbf{q}$



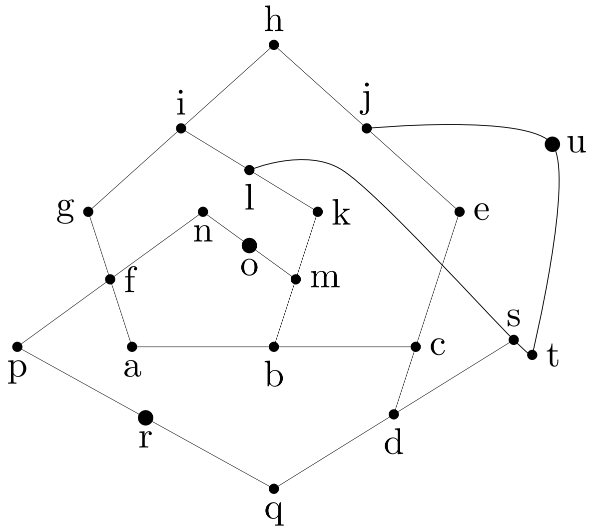
- $\mathbf{t} = \mathbf{s} \times \mathbf{l}$

$$\mathbf{l} \cdot \mathbf{s} = \mathbf{0}$$



- $\mathbf{u} = \mathbf{j} \times \mathbf{t}$

$$\mathbf{t} \cdot \mathbf{j} = \mathbf{0}$$



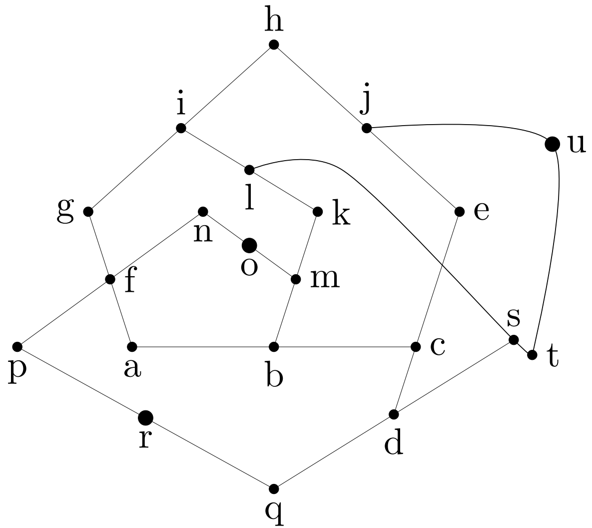
$\mathsf{l} \cdot \mathsf{s} = 0$

$\mathsf{t} \cdot \mathsf{j} = 0$

$$y = \frac{1}{3}\sqrt{1 + \sqrt[3]{163 - 9\sqrt{57}} + \sqrt[3]{163 + 9\sqrt{57}}} \doteq 1.14\,,$$

$$x = -\sqrt{\frac{y^2 + \sqrt{4y^8 + 16y^6 + 25y^4 + 16y^2 + 4}}{2y^2 + 2}} \doteq -1.61\,.$$

Application to the Coloring Problem



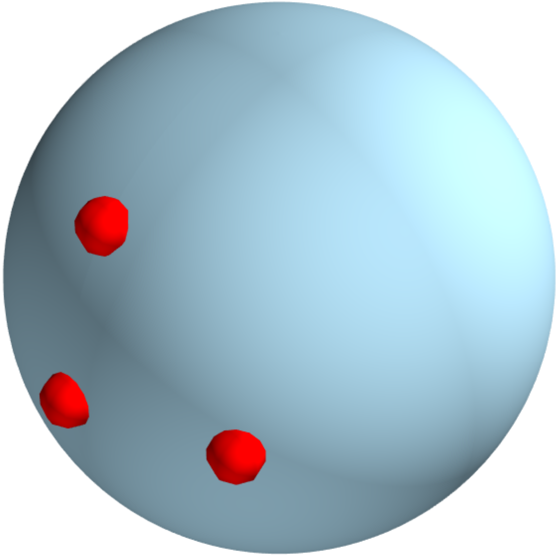
Suppose that there is a coloring m.

- There are 13 edges. Aside from u, r, o, every vertex is contained in 2 edges.

$$\overbrace{m(a) \oplus m(b) \oplus m(c)}^{=1} \oplus \overbrace{m(a) \oplus m(f) \oplus m(g)}^{=1} \oplus \ldots = m(r) \oplus m(u) \oplus m(o) = 13$$

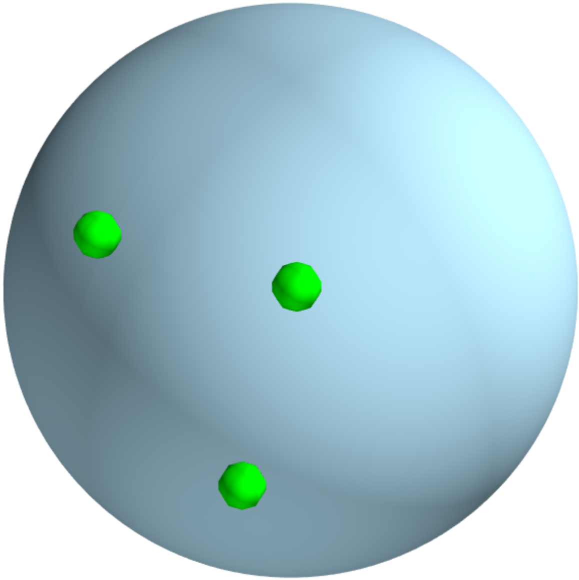
- $m(r) \oplus m(u) \oplus m(o) = 13 = 1 \pmod 2$

Application to the Coloring Problem



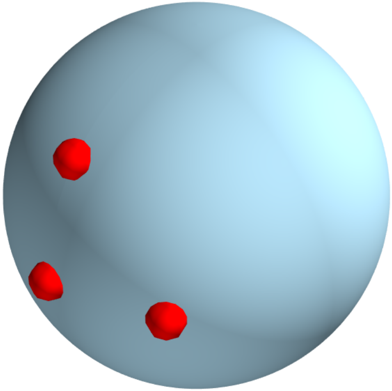
- $\boldsymbol{m}(\mathrm{a}) \oplus \boldsymbol{m}(\mathrm{b}) \oplus \boldsymbol{m}(\mathrm{c}) = 1$

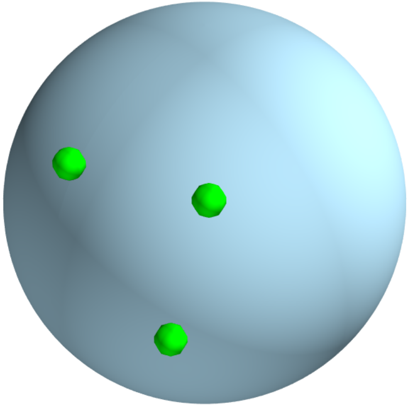
Application to the Coloring Problem

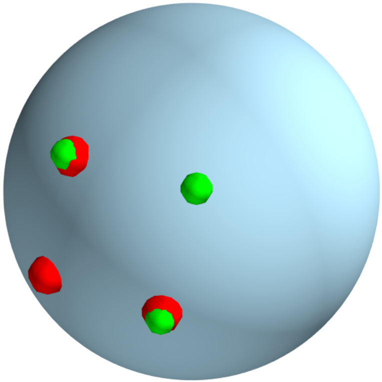


- $\boldsymbol{m}(a) \oplus \boldsymbol{m}(b) \oplus \boldsymbol{m}(c) = 1$

Application to the Coloring Problem



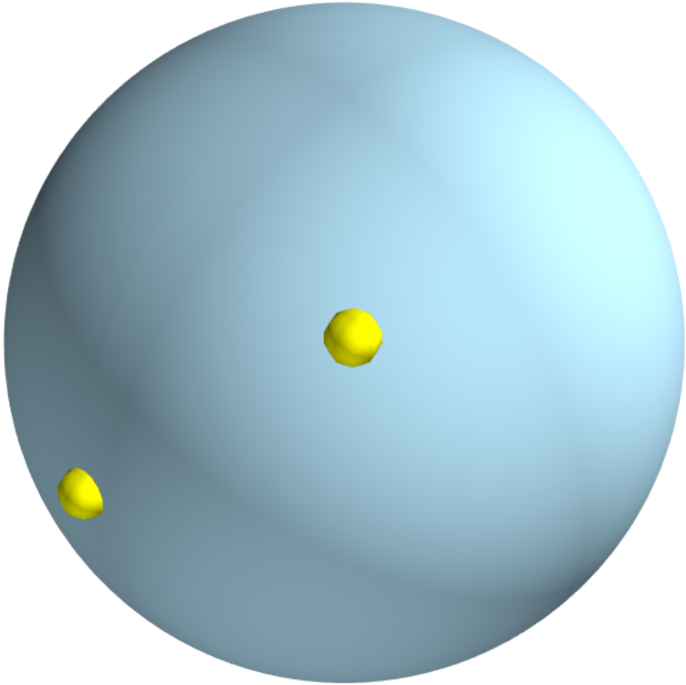




$\boldsymbol{m}(a) \oplus \boldsymbol{m}(b) \oplus \boldsymbol{m}(c) = 1$
$\boldsymbol{m}(a) \oplus \boldsymbol{m}(b) \oplus \boldsymbol{m}(c) = 1$

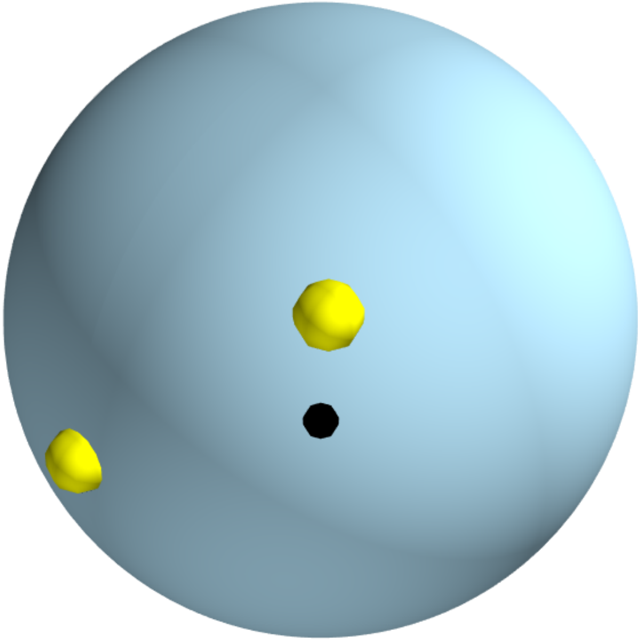
- $b = b$
- $c = c$
  $\boldsymbol{m}(a) \oplus \boldsymbol{m}(a) \oplus \boldsymbol{m}(b) \oplus \boldsymbol{m}(b) \oplus \boldsymbol{m}(c) \oplus \boldsymbol{m}(c) = 0$
- $\boldsymbol{m}(a) = \boldsymbol{m}(a)$

Application to the Coloring Problem



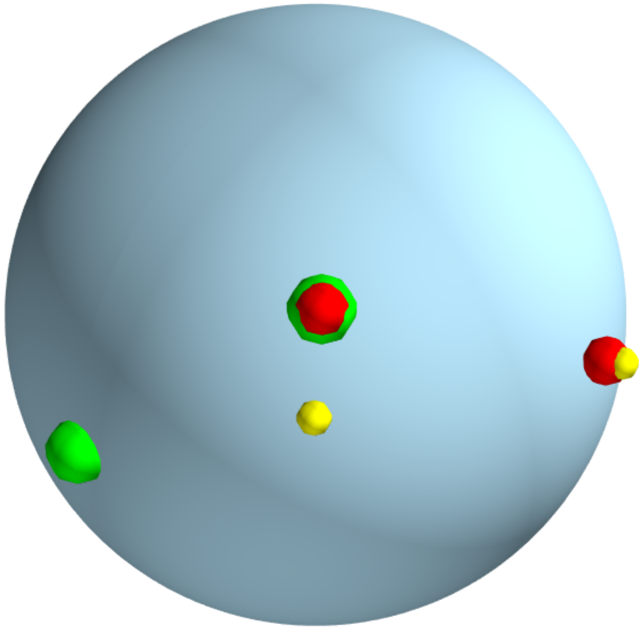
- $\boldsymbol{m}(\mathrm{a}) = \boldsymbol{m}(\mathrm{b})$

Application to the Coloring Problem



- Does $m(\mathrm{o})$ equal to $m(\mathrm{o})$?

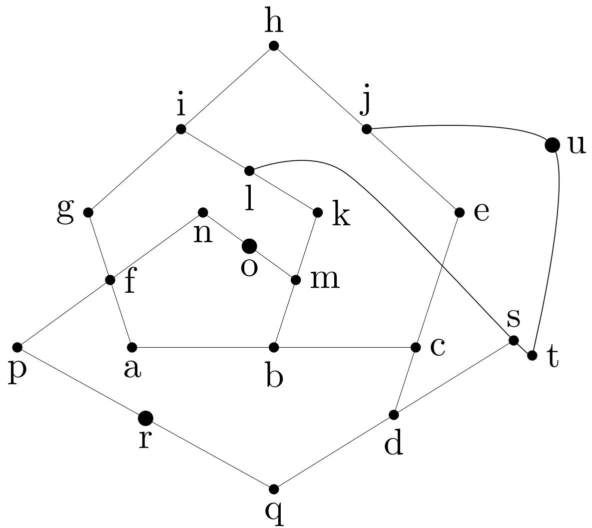
Application to the Coloring Problem



- $\boldsymbol{m}(\mathrm{o}) = \boldsymbol{m}(\mathrm{o})$
- $\boldsymbol{m}(\mathrm{o}) = \boldsymbol{m}(\mathrm{o})$
- $\boldsymbol{m}(\mathrm{o}) = \boldsymbol{m}(\mathrm{o})$

Result

The only possible assignment of zeros and ones to vectors of $\mathbb{R}^3$ such that for every three pairwise orthogonal vectors, an odd number of them is assigned 1 is the constant assignment.



Result

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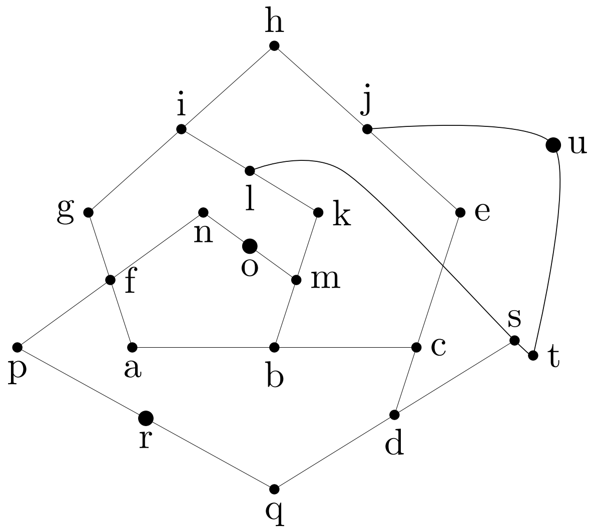

Thank you for the attention

Questions?