# On global states of collections of random variables

Federico Holik

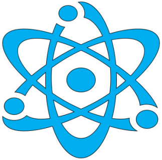
I F L P
CONICET
U N L P

4th Workshop
Quantum Contextuality in Quantum Mechanics and Beyond
21-05-2021

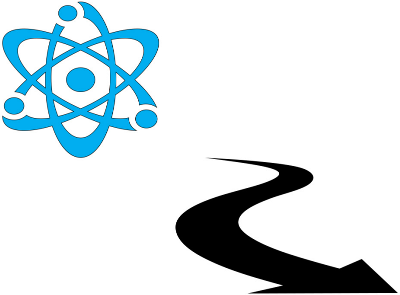
# Outline

# Outline

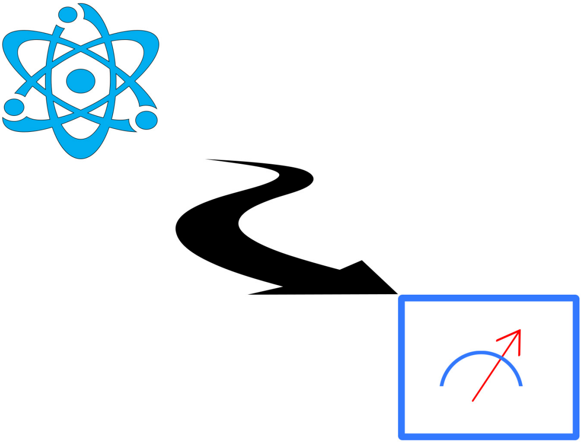
# A (probabilistic) system under study



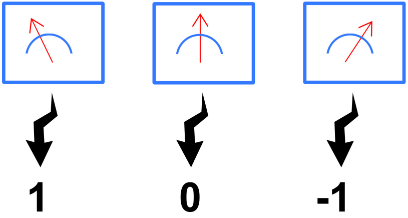
# A (probabilistic) system under study



# A (probabilistic) system under study: an experiment



# A (probabilistic) system under study: an experiment




$$\Omega = \{1, 0, -1\}$$

$$\Sigma = \{\emptyset, \{1\}, \{0\}, \{-1\}, \{1, 0\}, \{1, -1\}, \{0, -1\}, \{1, 0, -1\}\}$$

$$\mu : \Sigma \rightarrow \mathbb{R}$$

# Kolmogorov's axioms

## Probability measures

$$\mu : \Sigma \to [0, 1] \tag{1}$$

such that:

1. $\mu(\Omega) = 1$
2. For any denumerable family of pairwise disjoint sets $\{A_i\}_{i \in I}$

$$\mu(\bigcup_{i \in I} A_i) = \sum_i \mu(A_i)$$

## Measurable space

The triad $(\Omega, \Sigma, \mu)$ is a measurable space. $\Sigma$ is a $\sigma$-algebra (but the important thing for us, is that it is also a Boolean algebra).

# Random variables

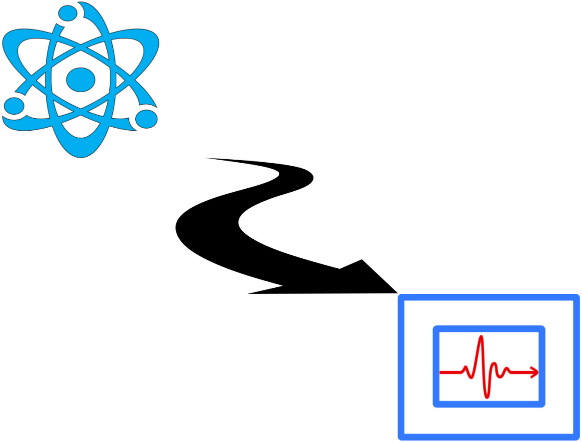
## Random variable

$$f : \Omega \to \mathbb{R} \tag{2}$$

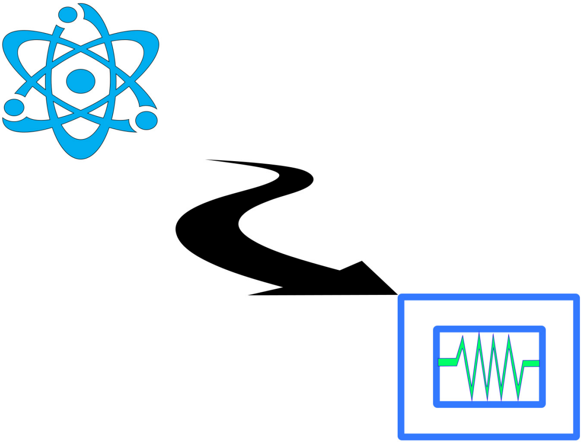
such that for every Borel set $\Delta \subseteq \mathbb{R}$,

$$f^{-1}(\Delta) \in \Sigma \tag{3}$$

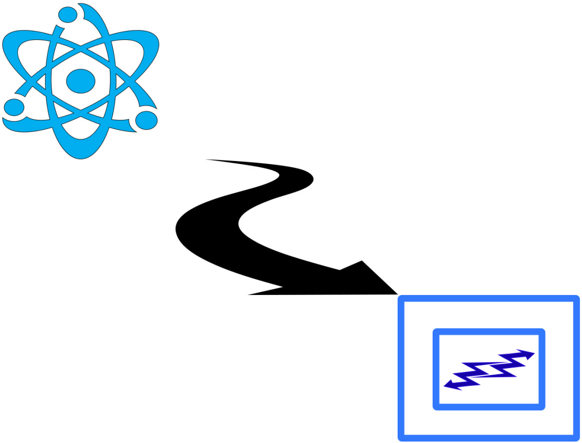
# A (probabilistic) system under study: a different experiment



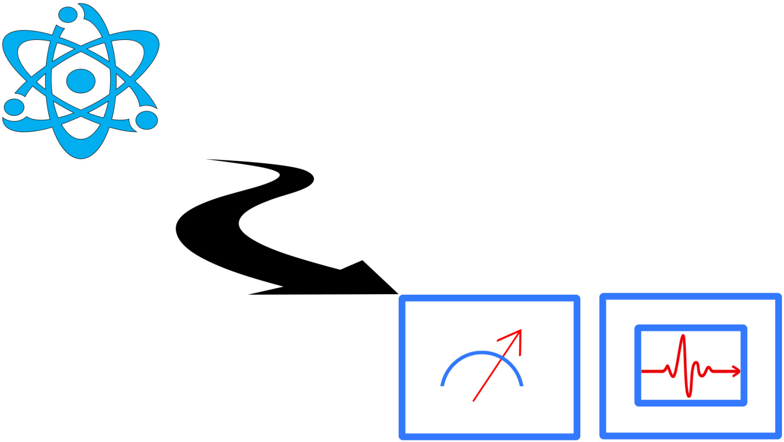
# A (probabilistic) system under study: a different experiment



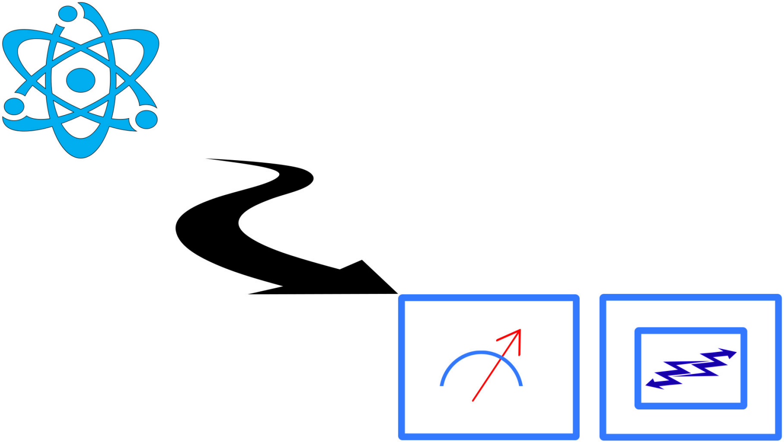
# A (probabilistic) system under study: a different experiment



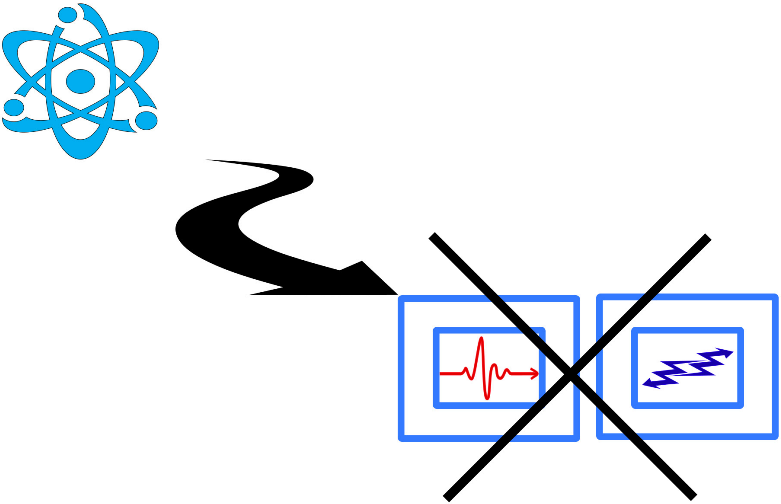
# Different, but jointly measurable experiments



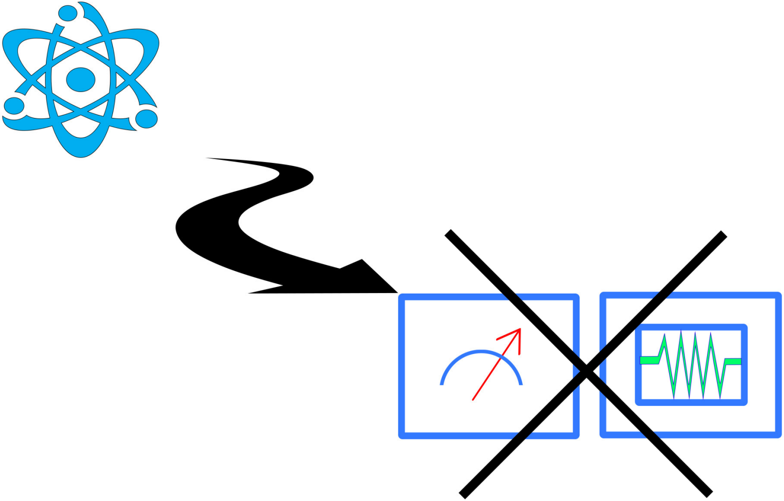
# Different, but jointly measurable



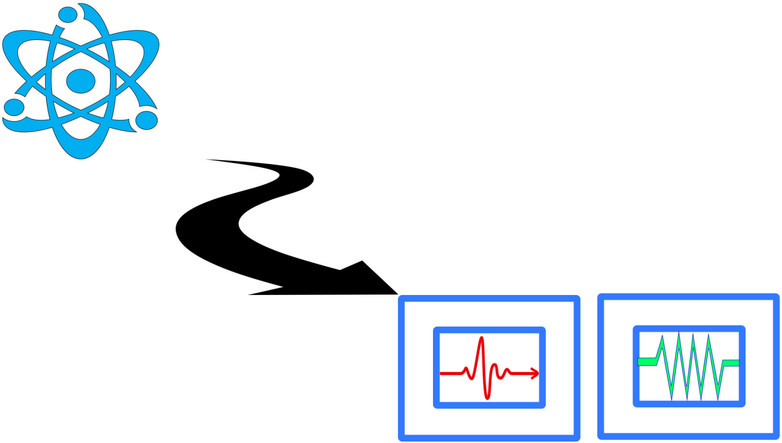
# Not jointly measurable



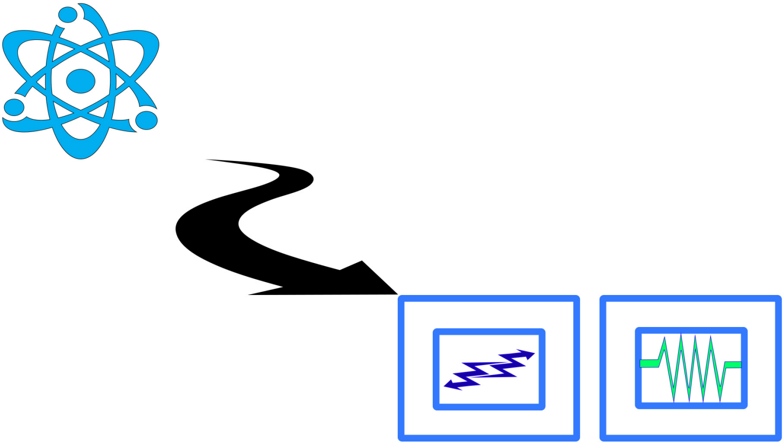
# Not jointly measurable



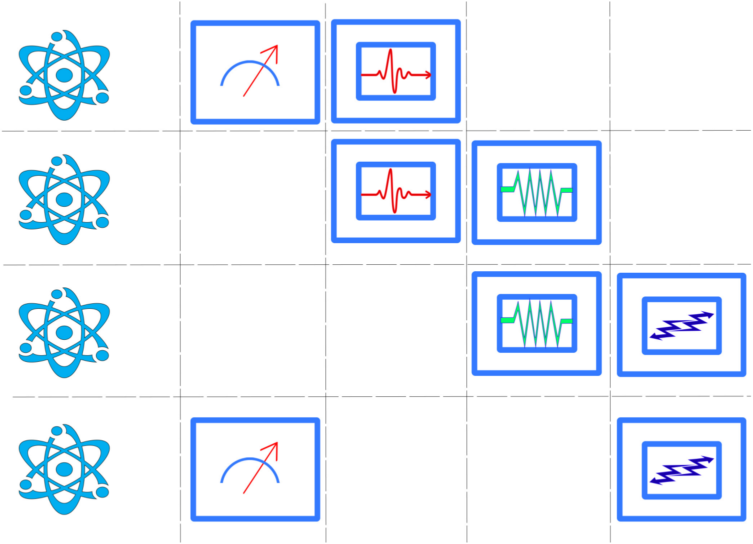
# Joint probability distribution



# Joint probability distribution



# One ends up in something like a matrix



In every row: existence of joint probability distribution.

# Using Ehtibar's notation

## Problem

Consider an arbitrary family $\{f_{i,j}\}$ of random variables.

| | | | |
|---|---|---|---|
| $f_{1,1}$ | $f_{1,2}$ | $\emptyset$ | $\emptyset$ |
| $\emptyset$ | $f_{2,2}$ | $f_{2,3}$ | $\emptyset$ |
| $\emptyset$ | $\emptyset$ | $f_{3,3}$ | $f_{3,4}$ |
| $f_{1,4}$ | $\emptyset$ | $\emptyset$ | $f_{4,4}$ |

- For each $f_{i,j}$, we have a probability space $(\Omega_{ij}, \Sigma_{ij}, \mu_{ij})$.
- For each row $i$ of the matrix, we have $(\Omega_i, \Sigma_i, \mu_i)$.
- $f_{1,1}$ and $f_{1,4}$ have the same content, but they should not be a priori identified.

# Using Ehtibar's notation

## Problem

Consider an arbitrary family $\{f_{i,j}\}$ of random variables.

| | | | |
|---|---|---|---|
| $f_{1,1}$ | $f_{1,2}$ | $\emptyset$ | $\emptyset$ |
| $\emptyset$ | $f_{2,2}$ | $f_{2,3}$ | $\emptyset$ |
| $\emptyset$ | $\emptyset$ | $f_{3,3}$ | $f_{3,4}$ |
| $f_{1,4}$ | $\emptyset$ | $\emptyset$ | $f_{4,4}$ |

- For each $f_{i,j}$, we have a probability space $(\Omega_{ij}, \Sigma_{ij}, \mu_{ij})$.
- For each row $i$ of the matrix, we have $(\Omega_i, \Sigma_i, \mu_i)$.
- $f_{1,1}$ and $f_{1,4}$ have the same content, but they should not be a priori identified.

# Using Ehtibar's notation

## Problem

Consider an arbitrary family $\{f_{i,j}\}$ of random variables.

| | | | |
|---|---|---|---|
| $f_{1,1}$ | $f_{1,2}$ | $\emptyset$ | $\emptyset$ |
| $\emptyset$ | $f_{2,2}$ | $f_{2,3}$ | $\emptyset$ |
| $\emptyset$ | $\emptyset$ | $f_{3,3}$ | $f_{3,4}$ |
| $f_{1,4}$ | $\emptyset$ | $\emptyset$ | $f_{4,4}$ |

- For each $f_{i,j}$, we have a probability space $(\Omega_{ij}, \Sigma_{ij}, \mu_{ij})$.
- For each row $i$ of the matrix, we have $(\Omega_i, \Sigma_i, \mu_i)$.
- $f_{1,1}$ and $f_{1,4}$ have the same content, but they should not be a priori identified.

# Using Ehtibar's notation

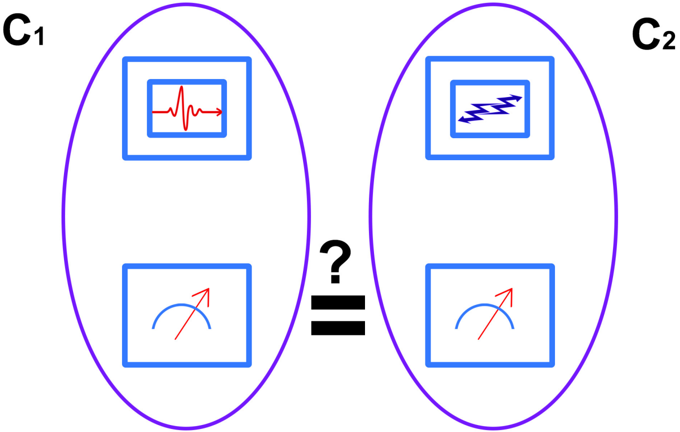
## Problem

Consider an arbitrary family $\{f_{i,j}\}$ of random variables.

| | | | |
|---|---|---|---|
| $f_{1,1}$ | $f_{1,2}$ | $\emptyset$ | $\emptyset$ |
| $\emptyset$ | $f_{2,2}$ | $f_{2,3}$ | $\emptyset$ |
| $\emptyset$ | $\emptyset$ | $f_{3,3}$ | $f_{3,4}$ |
| $f_{1,4}$ | $\emptyset$ | $\emptyset$ | $f_{4,4}$ |

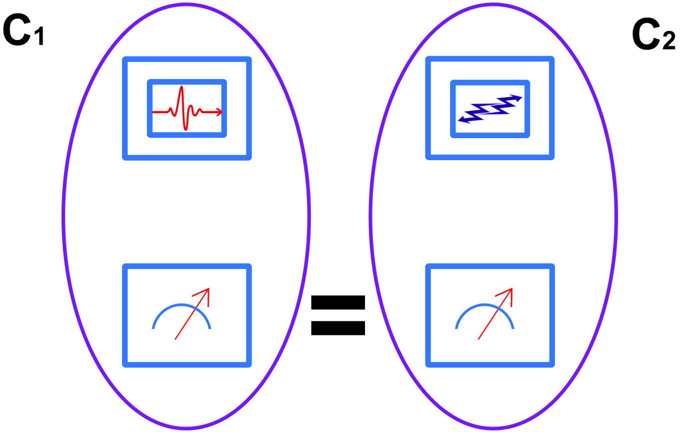
- For each $f_{i,j}$, we have a probability space $(\Omega_{ij}, \Sigma_{ij}, \mu_{ij})$.
- For each row $i$ of the matrix, we have $(\Omega_i, \Sigma_i, \mu_i)$.
- $f_{1,1}$ and $f_{1,4}$ have the same content, but they should not be a priori identified.

# Are they the same?



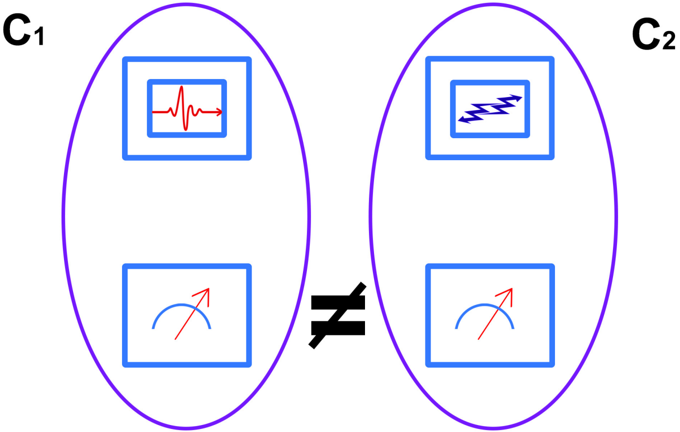

Should we identify those random variables?

# Option 1: they are *equal*



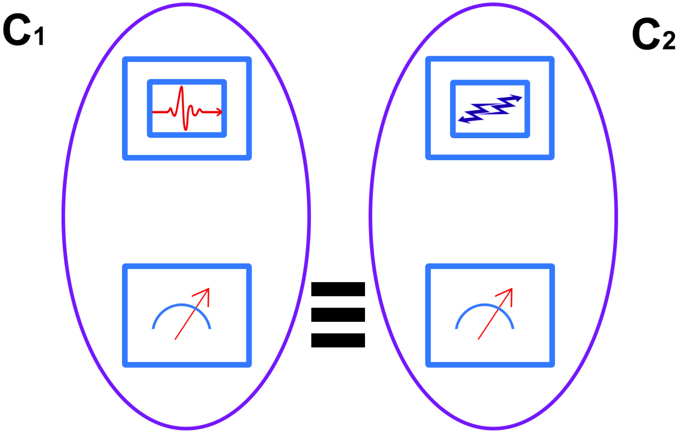

They are identified, even if they belong to different contexts.

# Option 2: they are *different*



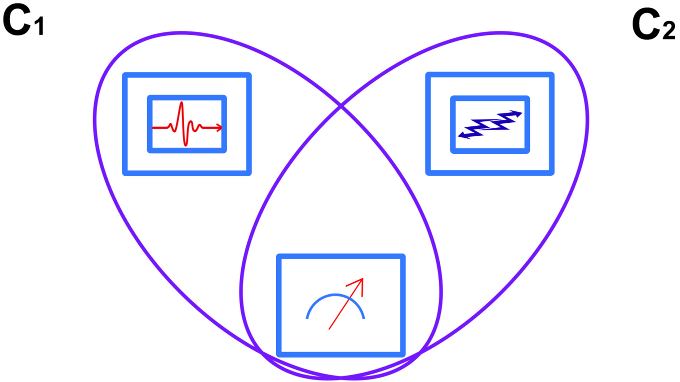

The probabilities might be different.

# Option 3: they are *indistinguishable*




[J.A. de Barros, F. Holik, D. Krause, *Phil. Trans. R. Soc. A.*, **377** (2157),

# Implies a mathematical identification




An *identification* usually takes place...

# If quantum probabilities were not known...

## Problem

- A state gives us a collection of classical probabilities: one for each context.
- But we know that things are intertwined...

$$\begin{tikzcd} \Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"] & \Sigma_i \arrow[dr, "\mu_i"] & \\ & & [0,1] \end{tikzcd}$$

# If quantum probabilities were not known...

## Problem

- A state gives us a collection of classical probabilities: one for each context.
- But we know that things are intertwined...

$$
\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[dr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[d, "\mu_i"] \\
& [0,1]
\end{tikzcd}
$$

# If quantum probabilities were not known...

## Problem

- A state gives us a collection of classical probabilities: one for each context.
- But we know that things are intertwined...

$$\begin{tikzcd} \Sigma_{ij} \arrow[r, "\iota"] \arrow[ddrr, bend right, "\mu_{ij}"] & \Sigma_i \arrow[ddr, "\mu_i"] & \\ & & \\ & & {[0,1]} \end{tikzcd}$$

# The diagram closes in QM

## Global Objects

- In QM we have a global object (and get a commutative diagram below).
- It contains information about all contexts and correlations. Very useful in practice.

$$
\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[dr, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu_\rho"] \\
& & [0,1]
\end{tikzcd}
$$

# The diagram closes in QM

## Global Objects

- In QM we have a global object (and get a commutative diagram below).
- It contains information about all contexts and correlations. Very useful in practice.

$$
\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[dr, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu_\rho"] \\
 & & {[0,1]}
\end{tikzcd}
$$

# The diagram closes in QM

## Global Objects

- In QM we have a global object (and get a commutative diagram below).
- It contains information about all contexts and correlations. Very useful in practice.

$$\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[ddrr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[ddr, "\mu_i"'] & \mathcal{L} \arrow[dd, dotted, "\mu_\rho"] \\
& & \\
& & {[0,1]}
\end{tikzcd}$$

# The diagram closes in QM

## Global Objects

- Global object = non-distributive orthomodular lattice (= Closed subspaces of the Hilbert space).
- Global state = Density Matrix.

$$
\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[dr, "\mu_i"'] & \mathcal{L} \arrow[r, "\iota"] \arrow[d, dotted, "\mu_\rho"] & \mathbb{A} \arrow[dl, dotted, "\exists!\rho"] \\
 & & [0,1] &
\end{tikzcd}
$$

# Different paths to follow

## Problem

There are –at least– two ways of describing global states.

- Alternative 1: keep using Boolean algebras, but negative probabilities.
- Alternative 2: paste Boolean algebras and end up in a non-Boolean structure (and define states as usual in quantum logical approaches).

$$\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[dr, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu!"] \\
& & {[0,1]}
\end{tikzcd}$$

# Different paths to follow

### Problem

There are –at least– two ways of describing global states.

- Alternative 1: keep using Boolean algebras, but negative probabilities.
- Alternative 2: paste Boolean algebras and end up in a non-Boolean structure (and define states as usual in quantum logical approaches).

$$\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[rrd, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[rd, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu!"] \\
 & & {[0,1]}
\end{tikzcd}$$

# Different paths to follow

## Problem

There are –at least– two ways of describing global states.

- Alternative 1: keep using Boolean algebras, but negative probabilities.
- Alternative 2: paste Boolean algebras and end up in a non-Boolean structure (and define states as usual in quantum logical approaches).

$$\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[dr, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu!"] \\
 & & {[0,1]}
\end{tikzcd}$$

# Different paths to follow

## Problem

There are –at least– two ways of describing global states.

- Alternative 1: keep using Boolean algebras, but negative probabilities.
- Alternative 2: paste Boolean algebras and end up in a non-Boolean structure (and define states as usual in quantum logical approaches).

$$\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[dr, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu!"] \\
 & & [0,1]
\end{tikzcd}$$

# Outline

# Contexts in signed measurable spaces

## Joint work

This part is based on a joint work with José Acacio de Barros (San Francisco State University).

## Works

- J. A. de Barros and F. Holik, *Entropy* 2020, 22(8), 829.
- Connections between indistinguishability and negative probabilities?
- General definition of *negative probability*? Based on measure theory (as in Kolmogorov's axioms)? Independent of any Hilbert space structure? How to incorporate the notion of context from the very beginning?

# Contexts in signed measurable spaces

**Joint work**

This part is based on a joint work with José Acacio de Barros (San Francisco State University).

**Works**

- J. A. de Barros and F. Holik, *Entropy* 2020, **22**(8), 829.
- Connections between indistinguishability and negative probabilities?
- General definition of *negative probability*? Based on measure theory (as in Kolmogorov's axioms)? Independent of any Hilbert space structure? How to incorporate the notion of context *from the very beginning*?

# Contexts in signed measurable spaces

### Joint work

This part is based on a joint work with José Acacio de Barros (San Francisco State University).

### Works

- J. A. de Barros and F. Holik, *Entropy* 2020, **22**(8), 829.
- Connections between indistinguishability and negative probabilities?
- General definition of *negative probability*? Based on measure theory (as in Kolmogorov's axioms)? Independent of any Hilbert space structure? How to incorporate the notion of context *from the very beginning*?

# Contexts in signed measurable spaces

## Joint work

This part is based on a joint work with José Acacio de Barros (San Francisco State University).

## Works

- J. A. de Barros and F. Holik, *Entropy* 2020, **22**(8), 829.
- Connections between indistinguishability and negative probabilities?
- General definition of *negative probability*? Based on measure theory (as in Kolmogorov's axioms)? Independent of any Hilbert space structure? How to incorporate the notion of context *from the very beginning*?

# Contexts in signed measurable spaces

### Joint work

This part is based on a joint work with José Acacio de Barros (San Francisco State University).

### Works

- J. A. de Barros and F. Holik, *Entropy* 2020, **22**(8), 829.
- Connections between indistinguishability and negative probabilities?
- General definition of *negative probability*? Based on measure theory (as in Kolmogorov's axioms)? Independent of any Hilbert space structure? How to incorporate the notion of context *from the very beginning*?

# Signed measurable spaces

## Definition

Let $\Omega$ be a sample space and $\Sigma$ a $\sigma$-algebra over $\Omega$. A *signed measure* is a function $\mu : \Sigma \to \mathbb{R}$ such that

$$\mu(\emptyset) = 0 \tag{4}$$

and for every denumerable and disjoint family $\{A_i\}_{i\in\mathbb{N}}$

$$\mu(\bigcup_i A_i) = \sum_i \mu(A_i) \tag{5}$$

The triple $(\Omega, \Sigma, \mu)$ is called a *signed measure space*.

# Contexts in signed measurable spaces

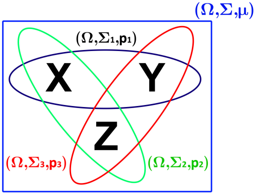
**Definition**

Let $(\Omega, \Sigma, \mu)$ be a signed measure space, and let $(M, \mathcal{M})$ be a Borel space with elements of $M$ being real numbers, i.e. $\mathcal{M}$ is a $\sigma$-algebra over $M$. A (real-valued) *extended random variable* $R$ is a measurable function $R : \Omega \to M$, i.e. for all $\Delta \in \mathcal{M}$, $R^{-1}(\Delta) \in \Sigma$.

# Contexts in signed measurable spaces

## Definition

Consider a family of signed probability models $\mathcal{S}_{(\Omega,\Sigma)}$. Let $\{R_i\}$, $i = 1, \ldots, n$, be a collection of extended random variables defined on $\mathcal{S}_{(\Omega,\Sigma)}$. A *general context* is a subset $C_j = \{R_k\}_{k\in N_j}$, $N_j \subset \{1, \ldots, n\}$ of those extended random variables, for which there exists a sub-$\sigma$-algebra $\Sigma_j$ of $\Sigma$ satisfying that, for all $\mu \in \mathcal{S}$, by defining $p_j^{\mu}(F) := \mu(F)$ for all $F \in \Sigma_j$, the triad $(\Omega, \Sigma_j, p_j^{\mu})$ becomes a probability space, and $R_{i_k}$ is a random variable with respect to it, for all $k \in \{1, ..., n_j\}$.



# An **Identification Principle** is operating here...

Random variables with same content from different contexts are considered **Indistinguishable**.

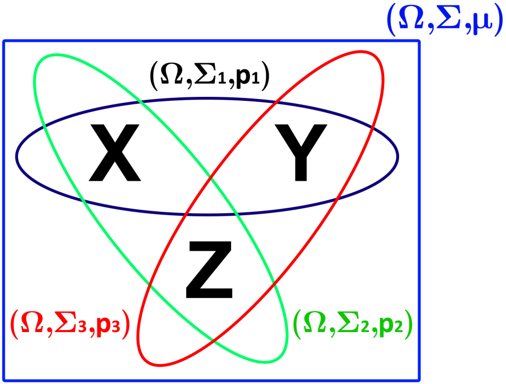
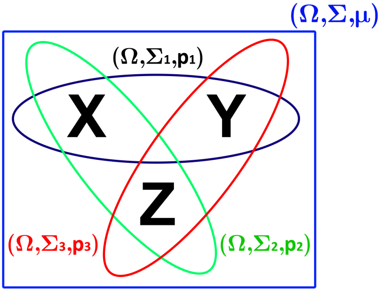



Figure: The sets $\Sigma_1$, $\Sigma_2$ and $\Sigma_3$ are sub-$\sigma$-algebras of $\Sigma$. When $\mu$ is restricted to them, we have $\mu|_{\Sigma_j}(F) = p_j(F)$, for all $F \in \Sigma_j$.

# Signed probability space

## Definition

A *signed probability space*, also called here *negative probability space*, is a signed measure space $(\Omega, \Sigma, \mu)$ endowed with a non-empty set of contexts $C = \{C_j^\mu\}$ (in the sense of the previous Definition), such that $\mu(\Omega) = 1$. The measure $\mu$ in this space is a *signed probability* or *negative probability*.



# Features

- It can be used to define contextuality measures (how negative is your state?)
- It includes previous examples (of no-signal models) as particular cases of the above definition.
- Negative probabilities never appear in experiments (only concrete contexts represent what actually happens).
- It is possible to define an *entropic measure* in the usual way: $s(\mu) = \inf_{\{C_i\}}(-\sum_j p_j^i \ln(p_j^i))$ (*measurement entropy*).

$$
\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[dr, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu"] \\
 & & {[0,1]}
\end{tikzcd}
$$

# Features

- It can be used to define contextuality measures (how negative is your state?)
- It includes previous examples (of no-signal models) as particular cases of the above definition.
- Negative probabilities never appear in experiments (only concrete contexts represent what actually happens).
- It is possible to define an *entropic measure* in the usual way: $s(\mu) = \inf_{\{C_i\}}(-\sum_j p_j^i \ln(p_j^i))$ (*measurement entropy*).

$$
\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[rrd, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[rd, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu"] \\
 & & [0,1]
\end{tikzcd}
$$

# Features

- It can be used to define contextuality measures (how negative is your state?)
- It includes previous examples (of no-signal models) as particular cases of the above definition.
- Negative probabilities never appear in experiments (only concrete contexts represent what actually happens).
- It is possible to define an *entropic measure* in the usual way: $s(\mu) = \inf_{\{C_i\}}(-\sum_j p^i_j \ln(p^i_j))$ (*measurement entropy*).

$$
\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[rrd, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[rd, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu"] \\
 & & {[0,1]}
\end{tikzcd}
$$

# Features

- It can be used to define contextuality measures (how negative is your state?)
- It includes previous examples (of no-signal models) as particular cases of the above definition.
- Negative probabilities never appear in experiments (only concrete contexts represent what actually happens).
- It is possible to define an *entropic measure* in the usual way: $s(\mu) = \inf_{\{C_i\}}(-\sum_j p_j^i \ln(p_j^i))$ (*measurement entropy*).

$$\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[dr, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu"] \\
 & & {[0,1]}
\end{tikzcd}$$

# Outline

# Remember

| | | | | |
|---|---|---|---|---|
| $f_{1,1}$ | $\emptyset$ | $f_{1,3}$ | $f_{1,4}$ | $\emptyset$ |
| $\emptyset$ | $f_{2,2}$ | $f_{2,3}$ | $f_{2,4}$ | $f_{2,5}$ |
| $f_{3,1}$ | $\emptyset$ | $\emptyset$ | $f_{3,4}$ | $\emptyset$ |
| $\emptyset$ | $f_{4,2}$ | $f_{4,3}$ | $\emptyset$ | $f_{4,4}$ |

## Rows and Columns

- Random variables in a row $\Longrightarrow$ jointly measurable$\Longrightarrow$ $\Longrightarrow$ There exists a Boolean algebra $\Sigma_i$ for each row (each $\Sigma_{i,j}$ is a sub-algebra of $\Sigma_i$).
- Should we identify $\Sigma_{i,j}$ with $\Sigma_{i',j}$ (when $i \neq i'$)?

# Remember

| | | | | |
|---|---|---|---|---|
| $f_{1,1}$ | $\emptyset$ | $f_{1,3}$ | $f_{1,4}$ | $\emptyset$ |
| $\emptyset$ | $f_{2,2}$ | $f_{2,3}$ | $f_{2,4}$ | $f_{2,5}$ |
| $f_{3,1}$ | $\emptyset$ | $\emptyset$ | $f_{3,4}$ | $\emptyset$ |
| $\emptyset$ | $f_{4,2}$ | $f_{4,3}$ | $\emptyset$ | $f_{4,4}$ |

## Rows and Columns

- Random variables in a row $\Longrightarrow$ jointly measurable$\Longrightarrow$ $\Longrightarrow$ There exists a Boolean algebra $\Sigma_i$ for each row (each $\Sigma_{i,j}$ is a sub-algebra of $\Sigma_i$).
- Should we identify $\Sigma_{i,j}$ with $\Sigma_{i',j}$ (when $i \neq i'$)?

# Remember

| | | | | |
|---|---|---|---|---|
| $f_{1,1}$ | $\emptyset$ | $f_{1,3}$ | $f_{1,4}$ | $\emptyset$ |
| $\emptyset$ | $f_{2,2}$ | $f_{2,3}$ | $f_{2,4}$ | $f_{2,5}$ |
| $f_{3,1}$ | $\emptyset$ | $\emptyset$ | $f_{3,4}$ | $\emptyset$ |
| $\emptyset$ | $f_{4,2}$ | $f_{4,3}$ | $\emptyset$ | $f_{4,4}$ |

## Rows and Columns

- Random variables in a row $\Longrightarrow$ jointly measurable$\Longrightarrow$ $\Longrightarrow$ There exists a Boolean algebra $\Sigma_i$ for each row (each $\Sigma_{i,j}$ is a sub-algebra of $\Sigma_i$).
- Should we identify $\Sigma_{i,j}$ with $\Sigma_{i',j}$ (when $i \neq i'$)?

# If we don't identify...

## Horizontal pasting

Given a family $\{\Sigma_i\}_{i\in I}$ of Boolean algebras whose intersections are given by $\Sigma_i \cap \Sigma_j = \{\mathbf{0}, \mathbf{1}\}$ $(i \neq j)$, define $\leq$ as the union of the orders, and the complement as the local complement $(a' := a'^{i})$. Then, we obtain an orthomodular lattice.

$$\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[dr, "\mu_i"] & \\
 & & {[0,1]}
\end{tikzcd}$$

# If we don't identify...

## Horizontal pasting

Given a family $\{\Sigma_i\}_{i\in I}$ of Boolean algebras whose intersections are given by $\Sigma_i \cap \Sigma_j = \{\mathbf{0},\mathbf{1}\}$ $(i \neq j)$, define $\leq$ as the union of the orders, and the complement as the local complement $(a' := a'^{i})$. Then, we obtain an orthomodular lattice.

$$\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[rrd, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[rd, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu!"] \\
 & & {[0,1]}
\end{tikzcd}$$

# Identifying things among different contexts

## Main feature

A global object is defined by intertwining contexts.

## Reasons

- In *physics*, the no-signal condition suggests that we *should identify* the random variables of the columns, because they have the same content and the same probability distribution.
- Indistinguishability Principle $\longrightarrow$ Identification Rule for intertwining Boolean algebras.
- The lattice of projection operators of the Hilbert space can be described as a pasted family of its maximal Boolean subalgebras: $\mathcal{L} = \bigvee \Sigma_i$

# Identifying things among different contexts

## Main feature

A global object is defined by intertwining contexts.

## Reasons

- In *physics*, the no-signal condition suggests that we *should identify* the random variables of the columns, because they have the same content and the same probability distribution.
- **Indistinguishability Principle** $\Longrightarrow$ **Identification Rule** for intertwining Boolean algebras.
- The lattice of projection operators of the Hilbert space can be described as a pasted family of its maximal Boolean subalgebras: $\mathcal{L} = \bigvee \Sigma_i$.

# Identifying things among different contexts

## Main feature

A global object is defined by intertwining contexts.

## Reasons

- In *physics*, the no-signal condition suggests that we *should identify* the random variables of the columns, because they have the same content and the same probability distribution.
- **Indistinguishability Principle** $\Longrightarrow$ **Identification Rule** for intertwining Boolean algebras.
- The lattice of projection operators of the Hilbert space can be described as a pasted family of its maximal Boolean subalgebras: $\mathcal{L} = \bigvee \Sigma_i$.

# Identifying things among different contexts

## Main feature

A global object is defined by intertwining contexts.

## Reasons

- In *physics*, the no-signal condition suggests that we *should identify* the random variables of the columns, because they have the same content and the same probability distribution.
- **Indistinguishability Principle** $\Longrightarrow$ **Identification Rule** for intertwining Boolean algebras.
- The lattice of projection operators of the Hilbert space can be described as a pasted family of its maximal Boolean subalgebras: $\mathcal{L} = \bigvee \Sigma_i$.

# Identifying things among different contexts

## Main feature

A global object is defined by intertwining contexts.

## Reasons

- In *physics*, the no-signal condition suggests that we *should identify* the random variables of the columns, because they have the same content and the same probability distribution.
- **Indistinguishability Principle** $\Longrightarrow$ **Identification Rule** for intertwining Boolean algebras.
- The lattice of projection operators of the Hilbert space can be described as a pasted family of its maximal Boolean subalgebras: $\mathcal{L} = \bigvee \Sigma_i$.

# Identifying things among different contexts

Posing the problem in this way opens interesting questions.

## Several Options

- Try to use standard results of algebra.
- Impose group theoretical conditions: in quantum mechanics, all maximal Boolean subalgebras are isomorphic (everything is generated by the action of a group).

# Identifying things among different contexts

Posing the problem in this way opens interesting questions.

## Several Options

- Try to use standard results of algebra.
- Impose group theoretical conditions: in quantum mechanics, all maximal Boolean subalgebras are isomorphic (everything is generated by the action of a group).

# Identifying things among different contexts

Posing the problem in this way opens interesting questions.

## Several Options

- Try to use standard results of algebra.
- Impose group theoretical conditions: in quantum mechanics, all maximal Boolean subalgebras are isomorphic (everything is generated by the action of a group).

# Identifying things among different contexts

Posing the problem in this way opens interesting questions.

## Several Options

- Try to use standard results of algebra.
- Impose group theoretical conditions: in quantum mechanics, all maximal Boolean subalgebras are isomorphic (everything is generated by the action of a group).

# Some References:

Navara, M.; Rogalewicz, V. The pasting constructions for orthomodular posets. *Math. Nachr.* 1991, **154**, 157-168
Kalmbach, G. Orthomodular Lattices; Academic Press: Waltham, MA, USA, 1983; Volume 18.

# Joint work

## Joint work

Joint work with Cesar Massri (UBA-CONICET).

## Works

- C. Massri and F. Holik, "On the representation of measures over bounded lattices", to appear in *Algebra Universalis*, (2021). arXiv:1705.11051 [math.AC]
- C. Massri, F. Holik and A. Plastino, "States In Generalized Probabilisic Models: An Approach Based In Algebraic Geometry", *Math. Slovaca* **69** No. **1**, 53-70 (2019).

# Joint work

## Joint work

Joint work with Cesar Massri (UBA-CONICET).

## Works

- C. Massri and F. Holik, "On the representation of measures over bounded lattices", to appear in *Algebra Universalis*, (2021). arXiv:1705.11051 [math.AC]
- C. Massri, F. Holik and A. Plastino, "States In Generalized Probabilisic Models: An Approach Based In Algebraic Geometry", *Math. Slovaca* **69** No. **1**, 53-70 (2019).

# Why Hilbert spaces?

## Problem

“One imposes certain physical requirements on these probabilities, which are suggested by earlier experience and developments, and the satisfaction of which calls for certain relations between the probabilities. Then, secondly, one searches for a simple analytical apparatus in which quantities occur that satisfy these relations exactly” [Hilbert, von Neumann, and Nordheim].

The “simple analytical apparatus” is the *Hilbert space*.

A. Duncan and M. Janssen, *Eur. Phys. J. H* **38**, 175-259 (2013).

# Why Hilbert spaces?

## Problem

"One imposes certain physical requirements on these probabilities, which are suggested by earlier experience and developments, and the satisfaction of which calls for certain relations between the probabilities. Then, secondly, one searches for a simple analytical apparatus in which quantities occur that satisfy these relations exactly" [Hilbert, von Neumann, and Nordheim].

The "simple analytical apparatus" is the *Hilbert space*.

A. Duncan and M. Janssen, *Eur. Phys. J. H* **38**, 175-259 (2013).

# Most strategies focus on looking for axioms to justify Hilbert spaces

## Alternative

It is also interesting to pose the problem in a different way.

## Our Approach

- We start by assuming that systems are described by a global algebraic structre.
- These structures might not be Boolean (as the previous discussion suggests).
- Try to understand how a *(possibly non Abelian) ring* emerges out of this minimal operational structure.

# Most strategies focus on looking for axioms to justify Hilbert spaces

## Alternative

It is also interesting to pose the problem in a different way.

## Our Approach

- We start by assuming that systems are described by a global algebraic structre.
- These structures might not be Boolean (as the previous discussion suggests).
- Try to understand how a *(possibly, non Abelian) ring* emerges out of this minimal operational structure.

# Most strategies focus on looking for axioms to justify Hilbert spaces

## Alternative

It is also interesting to pose the problem in a different way.

## Our Approach

- We start by assuming that systems are described by a global algebraic structre.
- These structures might not be Boolean (as the previous discussion suggests).
- Try to understand how a *(possibly, non Abelian) ring* emerges out of this minimal operational structure.

# Most strategies focus on looking for axioms to justify Hilbert spaces

## Alternative

It is also interesting to pose the problem in a different way.

## Our Approach

- We start by assuming that systems are described by a global algebraic structre.
- These structures might not be Boolean (as the previous discussion suggests).
- Try to understand how a *(possibly, non Abelian) ring* emerges out of this minimal operational structure.

# Most strategies focus on looking for axioms to justify Hilbert spaces

## Alternative

It is also interesting to pose the problem in a different way.

## Our Approach

- We start by assuming that systems are described by a global algebraic structre.
- These structures might not be Boolean (as the previous discussion suggests).
- Try to understand how a *(possibly, non Abelian) ring* emerges out of this minimal operational structure.

# Looking for a ring.

Now we want to see how a ring might appear (as the rings of operators that are used in quantum mechanics).

$$\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[dr, "\mu_i"'] & \mathcal{L} \arrow[d, dotted, "\mu"] \\
& & {[0,1]}
\end{tikzcd}$$

# Looking for a ring.

Now we want to see how a ring might appear (as the rings of operators that are used in quantum mechanics).

$$\begin{tikzcd}
\Sigma_{ij} \arrow[r, "\iota"] \arrow[drr, bend right, "\mu_{ij}"'] & \Sigma_i \arrow[r, "\iota"] \arrow[dr, "\mu_i"'] & \mathcal{L} \arrow[r, "\iota"] \arrow[d, dotted, "\mu"] & \mathbb{A} \arrow[dl, dotted, "\exists!\rho"] \\
 & & {[0,1]} &
\end{tikzcd}$$

# Compatible propositions and Free algebras

## Definitions

- For elements of an ortholattice, define $xCy$ ("$x$ commutes with $y$") if $x = (x \wedge y) \vee (x \wedge y')$.
- $\mathbb{C}\langle\mathcal{L}\rangle$ is the algebra of all possible expressions of the form $\alpha x + \beta y$, $xy$, $yx$, $x \wedge y + z$, etc.

# Compatible propositions and Free algebras

## Definitions

- For elements of an ortholattice, define $xCy$ ("$x$ commutes with $y$") if $x = (x \wedge y) \vee (x \wedge y')$.
- $\mathbb{C}\langle\mathcal{L}\rangle$ is the algebra of all possible expressions of the form $\alpha x + \beta y$, $xy$, $yx$, $x \wedge y + z$, etc.

Let $\mathbb{C}\langle\mathcal{L}\rangle$ be the free noncommutative algebra over $\mathbb{C}$ generated by the elements of $\mathcal{L}$. Let $I$ be the bilateral ideal generated by the relations,

$$I = \langle 0_{\mathcal{L}}, \quad 1 - 1_{\mathcal{L}}, \quad x \wedge y - xy \text{ if } xCy, \quad x \vee y - x - y - x \wedge y \text{ if } xCy \rangle.$$

### Definition

Let $\mathcal{L}$ be an orthomodular lattice. We define the *universal noncommutative algebra* associated to $\mathcal{L}$ as

$$U(\mathcal{L}) := \mathbb{C}\langle\mathcal{L}\rangle / I.$$

Clearly, we have a map $\iota : \mathcal{L} \to U(\mathcal{L})$ sending $x \in \mathcal{L}$ to its class $\iota(x)$. The map $\iota$ is not necessarily injective.

# Properties

Thus, we have created a copy of $\mathcal{L}$ in an algebraic structure $U(\mathcal{L})$ in such a way that:

- Let $x \in \mathcal{L}$. Given that $xCx$, we get $\iota(x)^2 = \iota(x)$, hence any $\iota(x)$ is idempotent.
- Let $x, y \in \mathcal{L}$ be such that $x \perp y$. Then $xCy$ and we get $\iota(x \vee y) = \iota(x) + \iota(y)$ and $\iota(x)\iota(y) = 0$ .
- If $xCy$, we get $\iota(x)\iota(y) = \iota(x \wedge y) = \iota(y \wedge x) = \iota(y)\iota(x)$.
- For $A = \sum_{i=1}^{N} \alpha_i \iota(x_i)$ and $B \in U(\mathcal{L})$, we set $AB = \sum_{i=1}^{N} \alpha_i \iota(x_i) B$ and $BA = \sum_{i=1}^{N} \alpha_i B \iota(x_i)$.

# Properties

Thus, we have created a copy of $\mathcal{L}$ in an algebraic structure $U(\mathcal{L})$ in such a way that:

- Let $x \in \mathcal{L}$. Given that $xCx$, we get $\iota(x)^2 = \iota(x)$, hence any $\iota(x)$ is idempotent.
- Let $x, y \in \mathcal{L}$ be such that $x \perp y$. Then $xCy$ and we get $\iota(x \vee y) = \iota(x) + \iota(y)$ and $\iota(x)\iota(y) = 0$ .
- If $xCy$, we get $\iota(x)\iota(y) = \iota(x \wedge y) = \iota(y \wedge x) = \iota(y)\iota(x)$.
- For $A = \sum_{i=1}^{N} \alpha_i \iota(x_i)$ and $B \in U(\mathcal{L})$, we set $AB = \sum_{i=1}^{N} \alpha_i \iota(x_i) B$ and $BA = \sum_{i=1}^{N} \alpha_i B \iota(x_i)$.

# Properties

Thus, we have created a copy of $\mathcal{L}$ in an algebraic structure $U(\mathcal{L})$ in such a way that:

- Let $x \in \mathcal{L}$. Given that $xCx$, we get $\iota(x)^2 = \iota(x)$, hence any $\iota(x)$ is idempotent.
- Let $x, y \in \mathcal{L}$ be such that $x \perp y$. Then $xCy$ and we get $\iota(x \vee y) = \iota(x) + \iota(y)$ and $\iota(x)\iota(y) = 0$ .
- If $xCy$, we get $\iota(x)\iota(y) = \iota(x \wedge y) = \iota(y \wedge x) = \iota(y)\iota(x)$.
- For $A = \sum_{i=1}^{N} \alpha_i \iota(x_i)$ and $B \in U(\mathcal{L})$, we set $AB = \sum_{i=1}^{N} \alpha_i \iota(x_i) B$ and $BA = \sum_{i=1}^{N} \alpha_i B \iota(x_i)$.

# Properties

Thus, we have created a copy of $\mathcal{L}$ in an algebraic structure $U(\mathcal{L})$ in such a way that:

- Let $x \in \mathcal{L}$. Given that $xCx$, we get $\iota(x)^2 = \iota(x)$, hence any $\iota(x)$ is idempotent.
- Let $x, y \in \mathcal{L}$ be such that $x \perp y$. Then $xCy$ and we get $\iota(x \vee y) = \iota(x) + \iota(y)$ and $\iota(x)\iota(y) = 0$ .
- If $xCy$, we get $\iota(x)\iota(y) = \iota(x \wedge y) = \iota(y \wedge x) = \iota(y)\iota(x)$.
- For $A = \sum_{i=1}^{N} \alpha_i \iota(x_i)$ and $B \in U(\mathcal{L})$, we set $AB = \sum_{i=1}^{N} \alpha_i \iota(x_i) B$ and $BA = \sum_{i=1}^{N} \alpha_i B \iota(x_i)$.

# Operator algebras might not be so strange after all...

**Theorem (Universality)**

*Let $\mathcal{L}$ be an orthomodular lattice and let $U$ be its universal noncommutative algebra. Let $f$ be a function from $\mathcal{L}$ to a noncommutative algebra $A$ over $\mathbb{C}$ such that,*

- $f(x \vee y) = f(x) + f(y) - f(x \wedge y)$ *if* $xCy$, $x, y \in \mathcal{L}$.
- $f(x \wedge y) = f(x)f(y)$ *if* $xCy$, $x, y \in \mathcal{L}$.
- $f(0) = 0$ *and* $f(1) = 1$.

*Then, there exists a unique $\mathbb{C}$-algebra map $\hat{f} : U \to A$ such that $f = \hat{f} \circ \iota$.*

$$
\begin{tikzcd}
\mathcal{L} \arrow[r, "\mu"] \arrow[d, "\iota_1"'] & A \\
\mathbb{C}\langle\mathcal{L}\rangle \arrow[r, "\iota_2"] \arrow[ur, dashed, "\exists!\mu'"] & \mathbb{C}\langle\mathcal{L}\rangle / I \arrow[u, dashed, "\exists!\hat{\mu}"']
\end{tikzcd}
$$

## Discussion

- Understand contextuality $\Longleftrightarrow$ Understand how contexts are intertwined.
- How global states and event algebras emerge? The *identification* of random variables among different contexts seems to play a key role in their genesis $\Longrightarrow$ **Indistinguishability Principle**.
- We have discussed the general features of two alternatives:
  *Boolean Algebras + Negative Probabilities*
  *Non-Boolean Algebras + Generalized States.*

# Conclusions

## Discussion

- Understand contextuality $\Longleftrightarrow$ Understand how contexts are intertwined.
- How global states and event algebras emerge? The *identification* of random variables among different contexts seems to play a key role in their genesis $\Longrightarrow$ **Indistinguishability Principle**.
- We have discussed the general features of two alternatives: *Boolean Algebras + Negative Probabilities* *Non-Boolean Algebras + Generalized States.*

# Conclusions

## Discussion

- Understand contextuality $\Longleftrightarrow$ Understand how contexts are intertwined.
- How global states and event algebras emerge? The *identification* of random variables among different contexts seems to play a key role in their genesis $\Longrightarrow$ **Indistinguishability Principle**.
- We have discussed the general features of two alternatives:
  *Boolean Algebras* + *Negative Probabilities*
  *Non-Boolean Algebras* + *Generalized States*.

# References


- C. Massri, F. Holik and A. Plastino, "States In Generalized Probabilisic Models: An Approach Based In Algebraic Geometry", *Math. Slovaca* **69** No. **1**, 53-70 (2019).
- C. Massri and F. Holik, "On the representation of measures over bounded lattices", to appear in *Algebra Universalis*, (2021). arXiv:1705.11051 [math.AC]
- J.A. de Barros, F. Holik. "Indistinguishability and Negative Probabilities", *Entropy*, Vol **22**, N. **8**, pp. 829, (2020).
- J.A. de Barros, F. Holik, D. Krause. "Indistinguishability and the origins of contextuality in physics", *Philosophical Transactions of the Royal Society A*, **377** (2157), 20190150, (2019).
- J.A. De Barros, F. Holik, D. Krause, "Contextuality and indistinguishability", *Entropy* **19** (**9**), 435, (2017).