Agreement and Disagreement in a Non-Classical World

Adam Brandenburger, Patricia Contreras-Tejada, Pierfrancesco La Mura, Giannicola Scarpa, and Kai Steverson

New York University; Instituto de Ciencias Matemáticas, Madrid; Universidad Complutense de Madrid; HHL Leipzig Graduate School of Management; Universidad Politécnica de Madrid; DCI Solutions

Version 12/17/22

The Classical Agreement Theorem

Alice and Bob possess a common prior probability distribution on a state space

They each then receive different private information about the true state

They form their conditional (posterior) probabilities q_A and q_B of an underlying event of interest

Theorem (Aumann, 1976): *If these two values q_A and q_B are common knowledge between Alice and Bob, they must be equal*

Here, an event E is common knowledge between Alice and Bob if they both know it, both know they both know it, and so on indefinitely

R. Aumann, "Agreeing to Disagree," *Annals of Statistics*, 4, 1976, 1236-1239

Applications

The agreement theorem is considered a basic requirement in classical epistemics

It has been used to

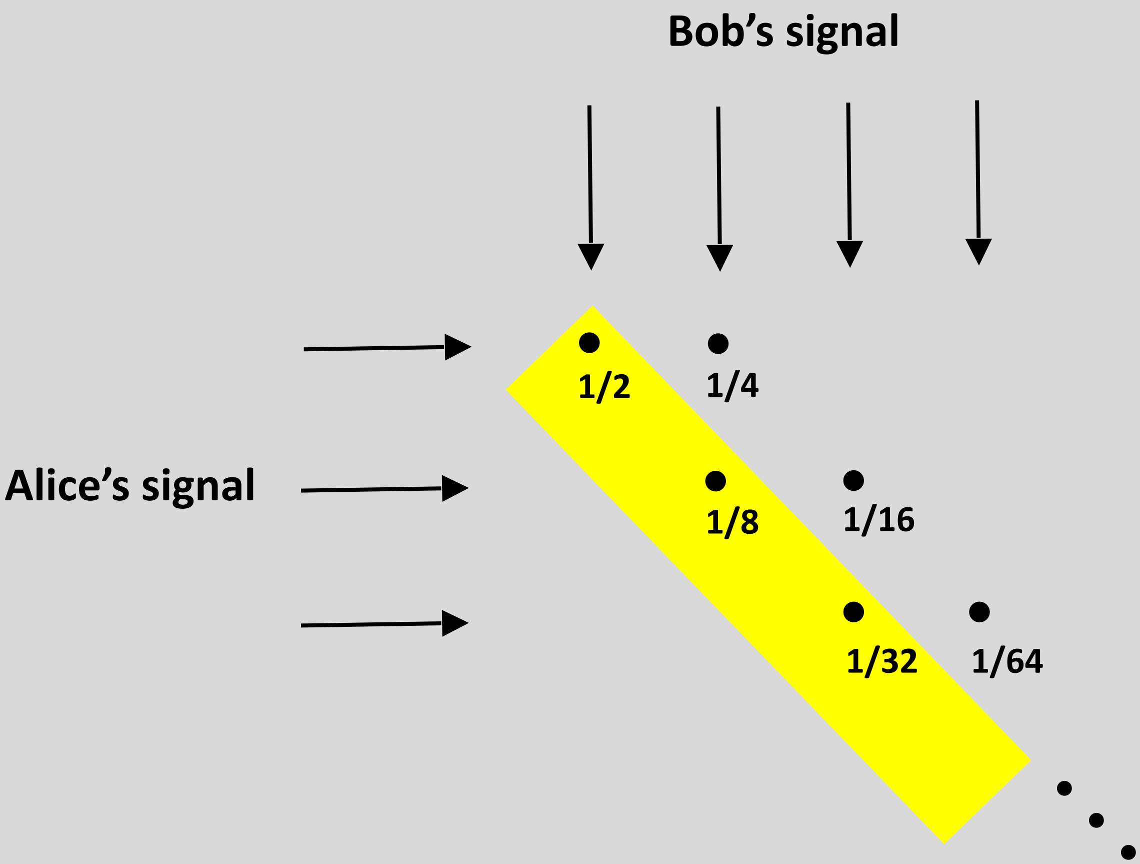
show that two risk-neutral agents, starting from a common prior, cannot agree to bet with each other
(Sebenius and Geanakoplos, 1983)

prove “no-trade” theorems for efficient markets
(Milgrom and Stokey, 1982)

establish epistemic conditions for Nash equilibrium
(Aumann and Brandenburger, 1995)

J. Sebenius and J. Geanakoplos. “Don’t Bet on It: Contingent Agreements with Asymmetric Information, *Journal of the American Statistical Association*, 78, 1983, 424-426; P. Milgrom and N. Stokey, “Information, Trade, and Common Knowledge,” *Journal of Economic Theory*, 26, 1982, 17–27; R. Aumann and A. Brandenburger, “Epistemic Conditions for Nash Equilibrium,” *Econometrica*, 63, 1995, 1161–1180

The Role of Common Knowledge: A "Discontinuity" at Infinity



J. Geanakoplos and H. Polemarchakis, "We Can't Disagree Forever," *Journal of Economic Theory*, 28, 1982, 192–200; this variant is due to John Geanakoplos (private communication)

Non-Classical Settings

What is the status of the Agreement Theorem when classical probability theory does not apply?

In the physical domain, the canonical case is quantum mechanics, where a fundamental result (Bell's Theorem, 1964) says that no local hidden-variable theory can model the results of all quantum experiments

This implies that the classical Bayesian model does not apply

In decision theory, there are proposals for non-classical alternatives with quantum features (e.g., La Mura, 2009; Busemeyer and Bruza, 2012; and Haven, Khrennikov, Ma, and Sozzo, 2018), which have been used to offer resolutions of anomalies in choice such as Ellsberg's Paradox (Ellsberg, 1961)

Perhaps, there is also an interesting non-classical decision theory with signed probabilities

J. Bell, "On the Einstein Podolsky Rosen Paradox," *Physics Physique Fizika*, 1, 1964, 195-200; P. La Mura, "Projective Expected Utility," *Journal of Mathematical Psychology*, 53, 2009, 408-414; J. Busemeyer and P. Bruza, *Quantum Models of Cognition and Decision*, Cambridge University Press, 2012; E. Haven, A. Khrennikov. C. Ma, and S. Sozzo, "Introduction to Quantum Probability Theory and its Economic Applications," *Journal of Mathematical Economics*, 78, 2018, 127-130; D. Ellsberg, "Risk, Ambiguity, and the Savage Axioms," *Quarterly Journal of Economics*, 75, 1961, 643-669

General Set-up

There is a finite abstract state space Ω

Alice and Bob have partitions $\mathscr{P}_A$ and $\mathscr{P}_B$ of Ω representing their private information

There is a common – possibly signed – prior probability measure p on Ω

Observability:

Assume throughout that all members of the partitions $\mathscr{P}_A$ and $\mathscr{P}_B$ receive probability in the interval $(0,1]$

Assume, too, that all events of interest receive probability in $(0,1]$

A Warm-Up Example

Alice's (Bob's) partition is red (blue)

The event of interest is

$$E = \{\omega_1, \omega_3, \omega_4\}$$

The true state is ω_1

At ω_1, Alice assigns (conditional) probability 1 to E

At ω_1, Bob assigns (conditional) probability 0 to E

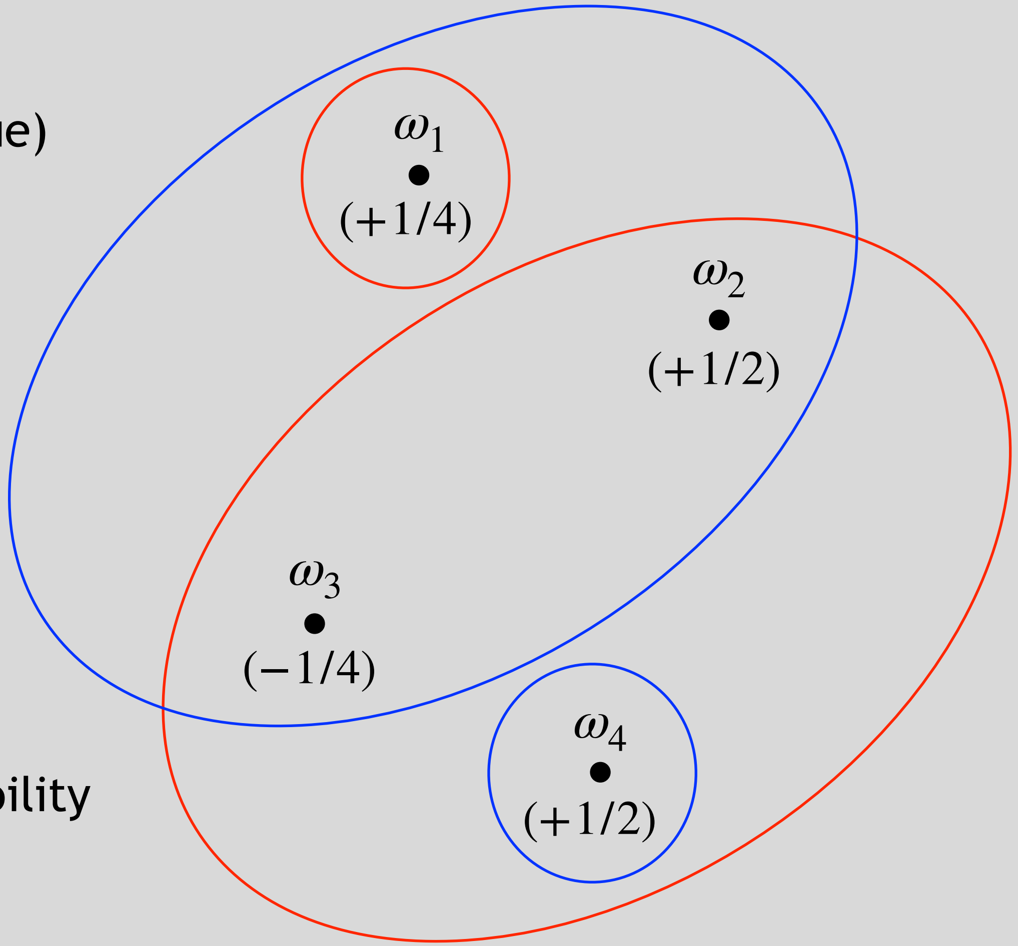
The event that Bob assigns probability 0 to E is

$$G = \{\omega_1, \omega_2, \omega_3\}$$

At ω_1, Alice assigns probability 1 to G

Call this **singular disagreement**

It is impossible classically!



Note: All partition cells (even events in the join) and E receive strictly positive probability and are therefore observable

From Knowledge to Certainty

Definition: Alice **knows** an event E at state ω if $\mathscr{P}_A(\omega) \subseteq E$

Definition: Alice is **certain** *of* an event E at a state ω if $p(E \,|\, \mathscr{P}_A(\omega)) = 1$

Fix an event E and probabilities q_A and q_B, and let

$$A_0 = \left\{\omega \in \Omega : p(E \mid \mathscr{P}_A(\omega)) = q_A\right\}$$

$$B_0 = \left\{\omega \in \Omega : p(E \mid \mathscr{P}_B(\omega)) = q_B\right\}$$

$$A_{n+1} = A_n \cap \left\{\omega \in \Omega : p(B_n \mid \mathscr{P}_A(\omega)) = 1\right\}$$

$$B_{n+1} = B_n \cap \left\{\omega \in \Omega : p(A_n \mid \mathscr{P}_B(\omega)) = 1\right\}$$

for all $n \geq 0$

Definition: It is **common certainty** at a state ω^* that Alice assigns probability q_A to E and Bob assigns probability q_B to E if $\omega^* \in \cap_{n=0}^{\infty} A_n \cap \cap_{n=0}^{\infty} B_n$

Relationship Between Knowledge and Certainty

If Alice knows an event E at state ω, then she is certain of E at ω

It is also true that common knowledge of E implies common certainty of E

(Proof: If Alice knows Bob knows E, then she knows Bob is certain of E, since knowledge is monotonic. From this, Alice is certain Bob is certain of E. The argument continues to higher levels.)

Arguably, the distinction between these modalities is "small" in the classical domain (arguably, not!)

Also, in the classical domain, there is an Agreement Theorem for common certainty

Theorem (classical): Fix a (non-negative) common prior and an event E. Suppose at a state ω^* it is common certainty that Alice's probability of E is q_A and Bob's probability of E is q_B. Then $q_A = q_B$.

But what happens in the non-classical world?

Non-Classical Agreement with Knowledge

Even without our observability conditions, we get a non-classical analog to the classical Agreement Theorem

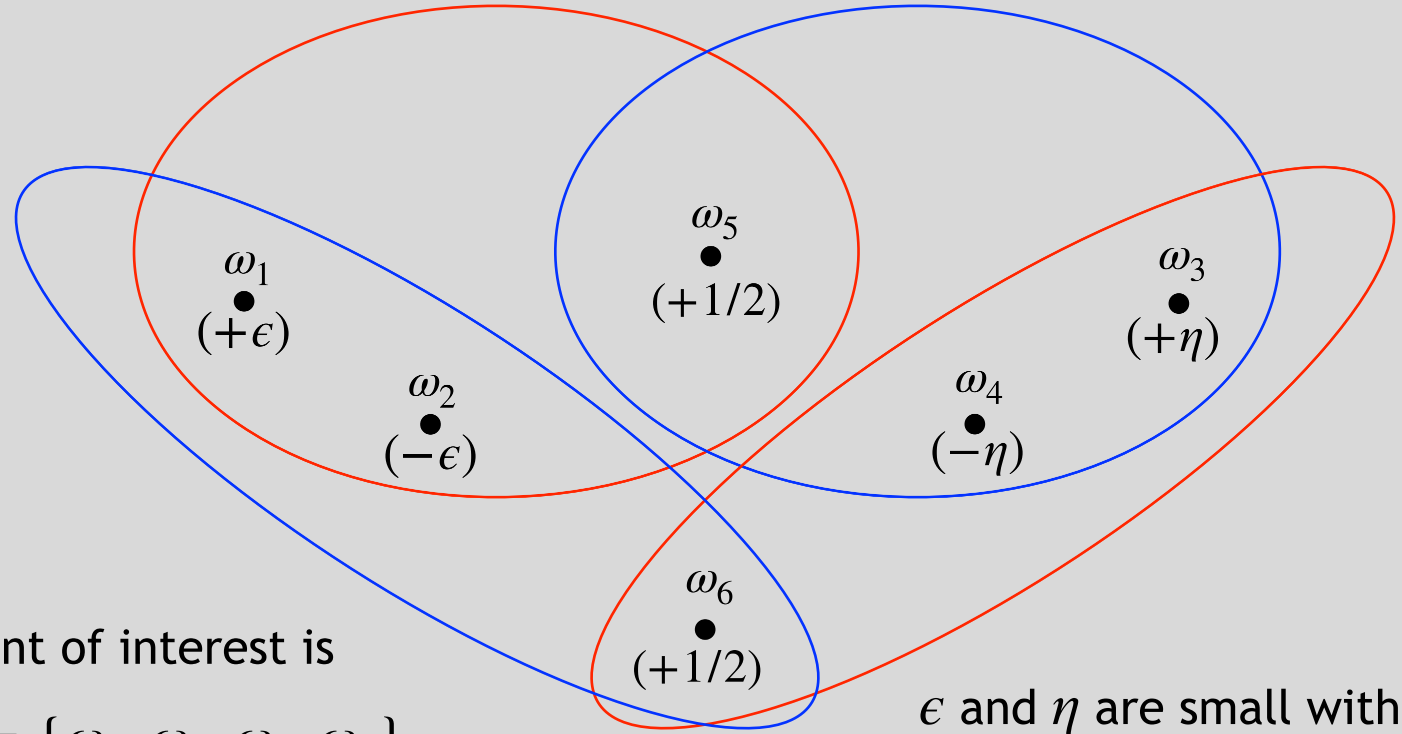
Theorem (non-classical): *Fix a signed common prior and an event* E. *Suppose at a state* ω^* *it is common knowledge that Alice's probability of* E *is* q_A *and Bob's probability of* E *is* q_B. *Then* $q_A = q_B$.

Proof: Follow closely the classical argument. Consider the (equal) conditional probabilities q_A for Alice, calculated for cells of her partition that are contained in the cell $(\mathscr{P}_A \wedge \mathscr{P}_B)(\omega^*)$ of the meet. This time, we take an affine rather than convex combination of this constant probability to get $p(E\,|\,(\mathscr{P}_A \wedge \mathscr{P}_B)(\omega^*)) = q_A$. Then run the same argument for Bob.

But let's see what happens with common certainty …

https://www.xixilogic.org/events/wp-content/uploads/2022/05/lfdsn2019-proceedings.pdf; see also M. Leifer and C. Duarte, "Generalizing Aumann's Agreement Theorem," 2022, at https://arxiv.org/abs/2202.02156

Common Certainty of Disagreement



The event of interest is

$$E = \{\omega_2, \omega_4, \omega_5, \omega_6\}$$

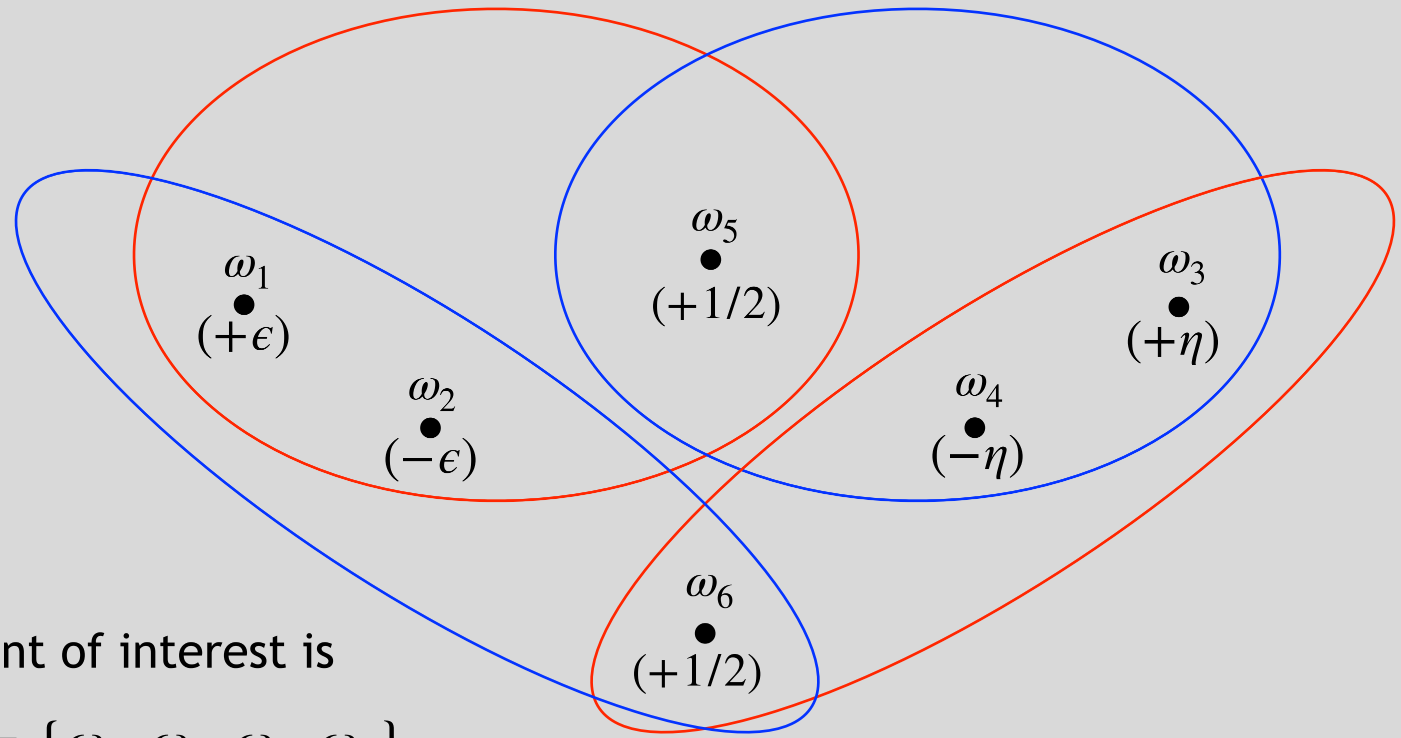
ϵ and η are small with $\epsilon \neq \eta$

The true state is ω_5

At ω_5, it is common certainty that Alice assigns probability $1 - 2\epsilon$ to E while Bob assigns probability $1 - 2\eta$ to E

That is, there is common certainty of disagreement!

Communication



The event of interest is

$$E = \{\omega_2, \omega_4, \omega_5, \omega_6\}$$

The true state is ω_1

Alice communicates her probability $1 - 2\epsilon$ to Bob (cf. Geanakoplos and Polemarchakis, 1982), which tells him she has information $\{\omega_1, \omega_2, \omega_5\}$

Bob's information is then $\{\omega_1, \omega_2\}$, so he forms a (new) probability of $-\epsilon/0$, which is not well-defined!

Communication-Enabled Structures

Define a sequence of partitions for Alice, corresponding to announcements she could make about her probability of E, her certainty of Bob's probability, etc., and likewise for Bob

$$\mathscr{M}_A^{(n)} = \{A_n, A_n^c\}$$

$$\mathscr{M}_B^{(n)} = \{B_n, B_n^c\}$$

For any $\pi, E \subseteq \Omega$, say π is **regular with respect to** E if $p(\pi) \geq 0$ and $0 \leq p(\pi \cap E) \leq p(\pi)$

A structure $(\Omega, p, \mathscr{P}_A, \mathscr{P}_B)$ is **communication-enabled with respect to** E if for each $n \geq 0$, each $\pi \in \mathscr{P}_A \vee \mathscr{M}_B^{(n)}$ and each $\pi \in \mathscr{P}_B \vee \mathscr{M}_A^{(n)}$ is regular with respect to E

Note: This property fails in the previous example

A New Agreement Theorem

Theorem: *Fix a structure that is communication-enabled with respect to* E *and suppose at a state* ω^* *it is common certainty that Alice's probability of* E *is* q_A *and Bob's probability of* E *is* q_B*. Then* $q_A = q_B$.

Notice that Alice's potential announcements are made relative to her (initial) partition $\mathscr{P}_A$; and likewise for Bob

This is different from Geanakopolos and Polemarchakis (1982) where the (actual) announcements are made relative to updated partitions

Alternatively put, the mere ability to "confirm" the epistemic state (here, the state is common certainty of the posteriors) is enough to rule out disagreement — the confirmation need not actually be carried out

Realizability of Common Certainty of Disagreement?

In the physical domain, it can be shown that common certainty of disagreement (CCD) is impossible when observing quantum systems but possible for superquantum (no-signaling) systems

The impossibility of CCD can therefore be proposed as a physical axiom

In decision theory, if we equip agents with signed probability measures, it seems we can get highly non-classical behavior, such as betting between risk-neutral agents

Or, should the impossibility of CCD be elevated to an (epistemic) decision-theoretic principle?

If yes, what non-classical behavior is then allowed? This appears to be an open direction …

P. Contreras-Tejada, G. Scarpa, A. Kubicki, A. Brandenburger, and P. La Mura, "Observers of Quantum Systems Cannot Agree to Disagree," *Nature Communications*, 12, 2021, at https://doi.org/10.1038/ s41467-021-27134-6

Two Alternative Models

1. Khrennikov and Basieva (2014) and Khrennikov (2015) consider quantum-like observers of a quantum system who employ either the knowledge or certainty modality

   This approach allows CCD even for quantum systems

2. We could strengthen* the belief modality to say:

   Alice is **fully certain** of E if all events in the complement of E receive probability 0

   We could investigate this route by developing a preference-based definition of full certainty (analogous to defining Savage-null events) from a decision theory with signed probabilities

   This appears to be an interesting open direction …

*Our thanks to Miklós Pintér for this suggestion; A. Khrennikov and I. Basieva, "Possibility to Agree on Disagree from Quantum Information and Decision Making," *Journal of Mathematical Psychology*, 62, 2014, 1-5; A. Khrennikov, "Quantum Version of Aumann's Approach to Common Knowledge: Sufficient Conditions of Impossibility to Agree on Disagree," *Journal of Mathematical Economics*, 60, 2015, 89-104


Thank You