# Which is the simplest bipartite AVN, FN, PT and SC in nature?

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# AVN, FN, PT, SC

- AVN = all versus nothing (nonlocality)
- FN = full nonlocality (aka maximal nonlocality)
- PT = pseudo telepathy
- SC = strong contextuality

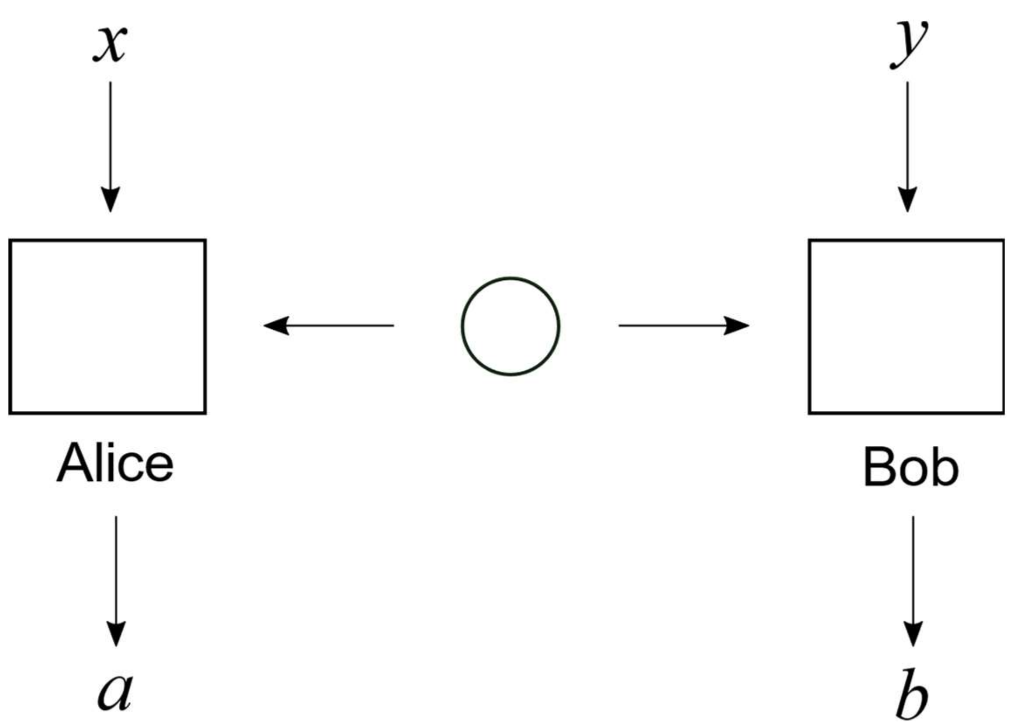
# The Nobel Prize in Physics 2022

# Bell scenario

- A Bell scenario is characterized by:
  - a number of parties,
  - the number of observables each party has,
  - their numbers of outcomes,
  - any observable of one party can be jointly measured with any observable of a different party.

# The simplest Bell scenario: (2,2,2)

- (2,2,2) = 2 parties, 2 observables per party, 2 outcomes per observable
- *x*, *y* in {0,1}; *a*, *b* in {0, 1}



# Quantum behaviors (aka correlations)

- In QM (algebraic quantum field theory), the set of behaviors (correlations, probability models, empirical models) for (2,2,2) is

$$\mathbf{p} = \{\langle\psi|M_{a|x}M_{b|y}|\psi\rangle\} = \begin{bmatrix} \langle\psi|M_{0|1}M_{0|1}|\psi\rangle & \langle\psi|M_{0|1}M_{1|1}|\psi\rangle & \langle\psi|M_{1|1}M_{0|1}|\psi\rangle & \langle\psi|M_{1|1}M_{1|1}|\psi\rangle \\ \langle\psi|M_{0|1}M_{0|2}|\psi\rangle & \langle\psi|M_{0|1}M_{1|2}|\psi\rangle & \langle\psi|M_{1|1}M_{0|2}|\psi\rangle & \langle\psi|M_{1|1}M_{1|2}|\psi\rangle \\ \langle\psi|M_{0|2}M_{0|1}|\psi\rangle & \langle\psi|M_{0|2}M_{1|1}|\psi\rangle & \langle\psi|M_{1|2}M_{0|1}|\psi\rangle & \langle\psi|M_{1|2}M_{1|1}|\psi\rangle \\ \langle\psi|M_{0|2}M_{0|2}|\psi\rangle & \langle\psi|M_{0|2}M_{1|2}|\psi\rangle & \langle\psi|M_{1|2}M_{0|2}|\psi\rangle & \langle\psi|M_{1|2}M_{1|2}|\psi\rangle \end{bmatrix}$$

$|\psi\rangle$ is vector of unit norm in a complex Hilbert space $\mathcal{H}$
$\langle\psi|$ is its Hermitian conjugate
$M_{a|x}$ and $M_{b|y}$ are orthogonal projectors on $\mathcal{H}$ satisfying

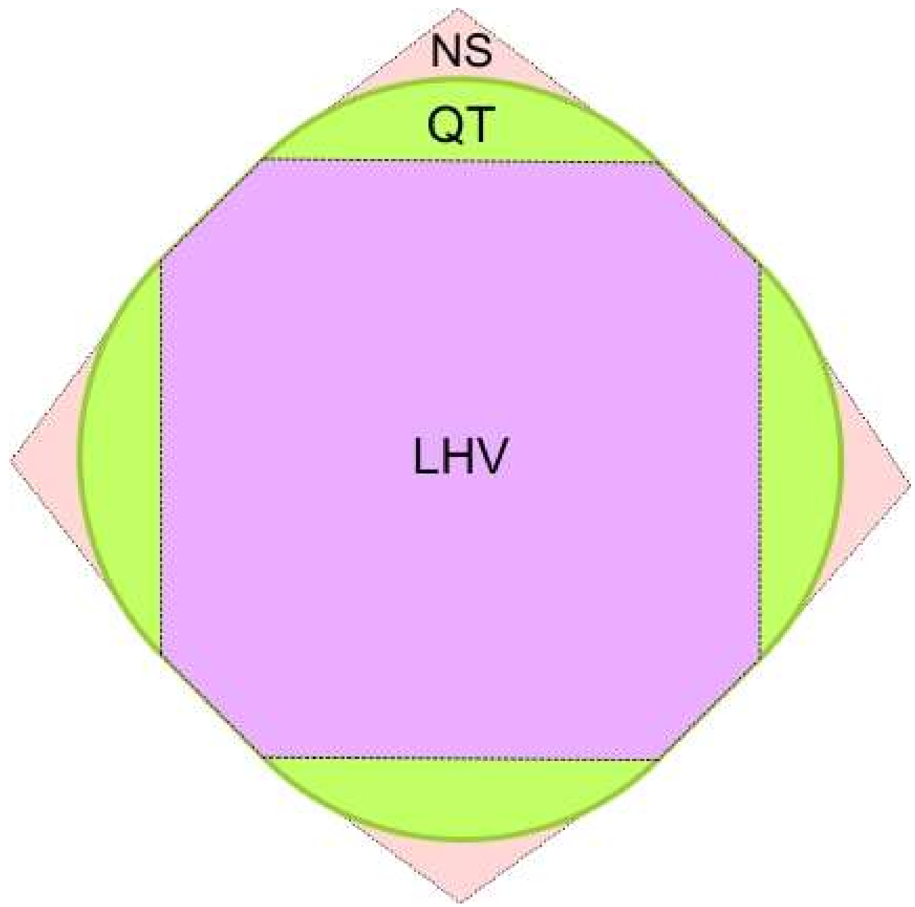
$$M_{a|x}M_{a'|x} = \delta_{aa'}M_{a|x}, \qquad \sum_a M_{a|x} = \mathbb{I}_A,$$

$$M_{b|y}M_{b'|y} = \delta_{bb'}M_{b|y}, \qquad \sum_b M_{b|y} = \mathbb{I}_B,$$

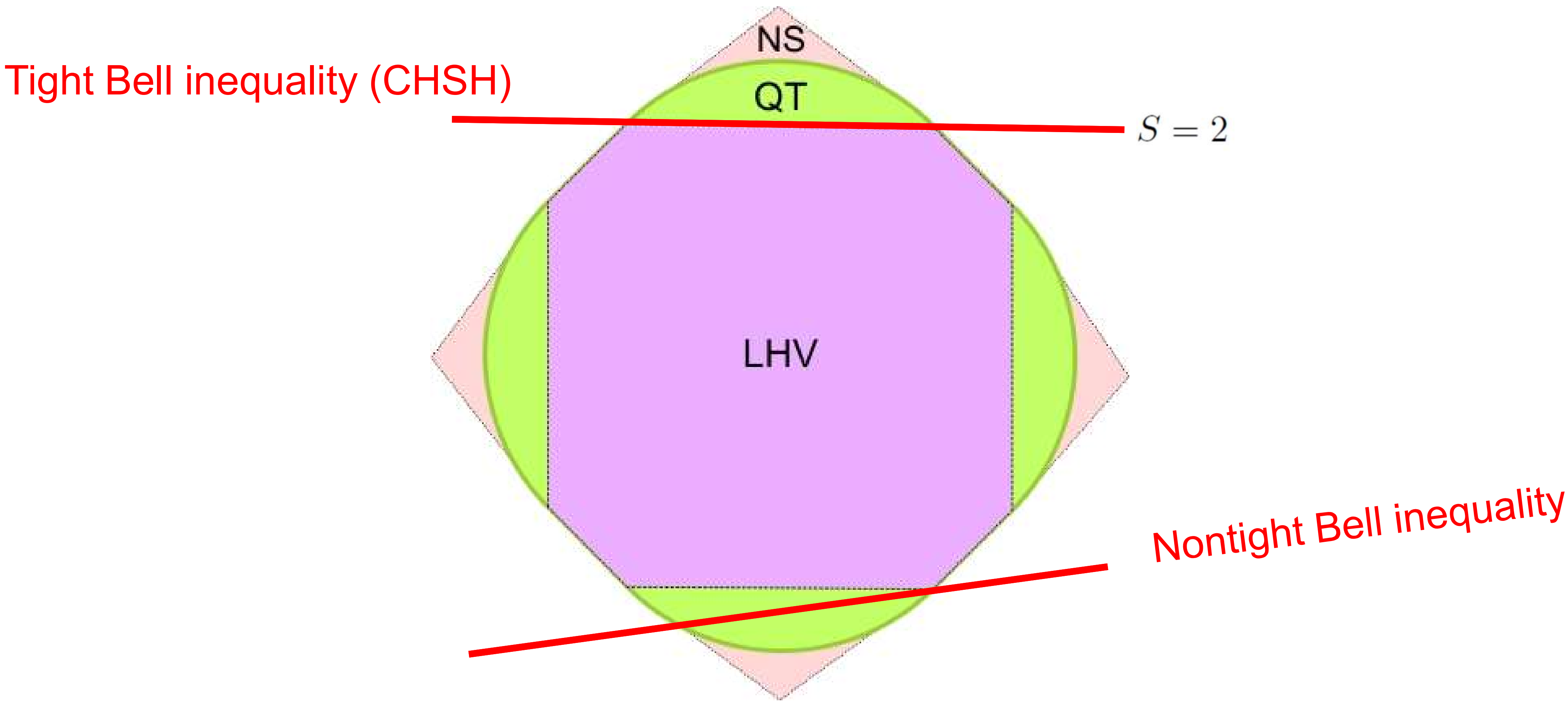
$$\left[M_{a|x}, M_{b|y}\right] = 0 \quad \forall a, b$$

$\mathbb{I}_A$ ($\mathbb{I}_B$ ) is the identity in the subspace $\mathcal{H}_A$ ($\mathcal{H}_B$) of $\mathcal{H}$.

# Geometry of the sets of correlations for (2,2,2)



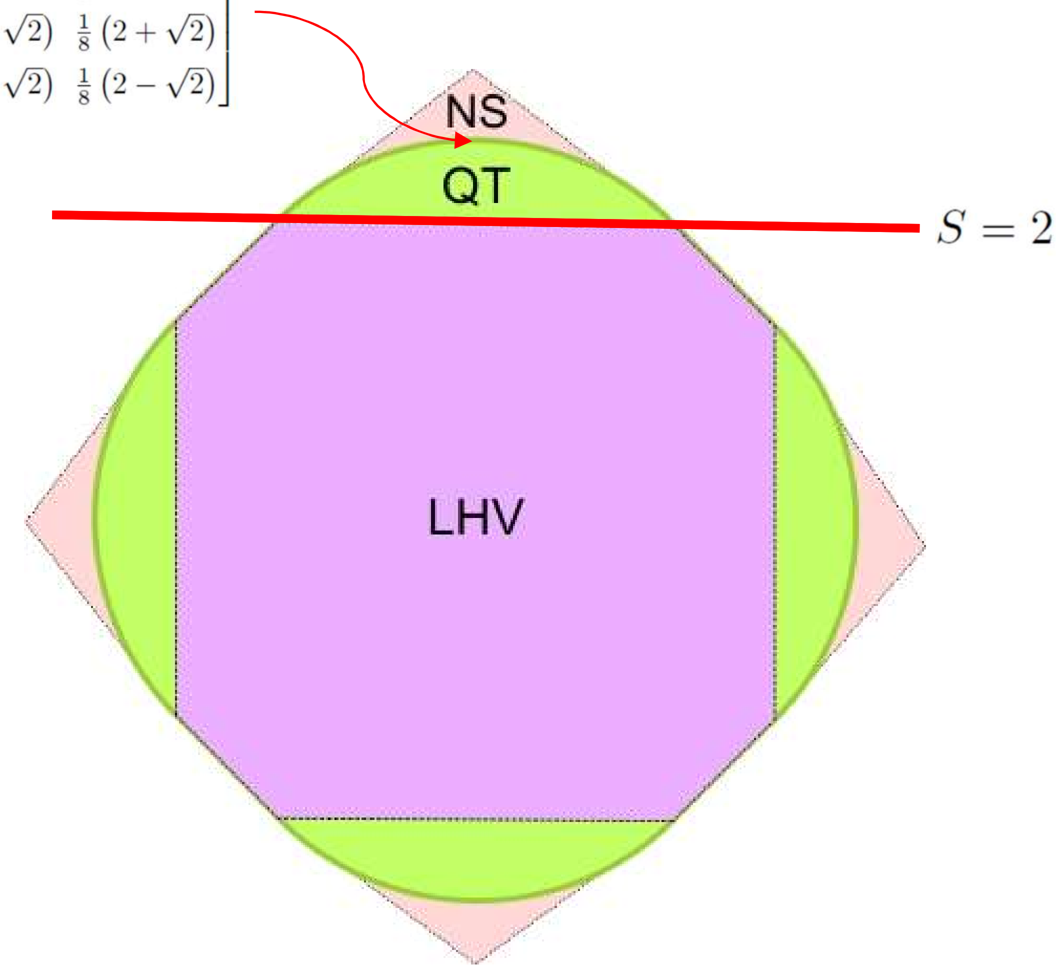
# Bell inequalities



# Maximal quantum violation of CHSH

$$\mathbf{p}_{\mathrm{T}} = \begin{bmatrix} \frac{1}{8}(2+\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2+\sqrt{2}) \\ \frac{1}{8}(2+\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2+\sqrt{2}) \\ \frac{1}{8}(2+\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2+\sqrt{2}) \\ \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2+\sqrt{2}) & \frac{1}{8}(2+\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) \end{bmatrix}$$

$$S(\mathbf{p}_{\mathrm{T}}) = 2\sqrt{2}$$



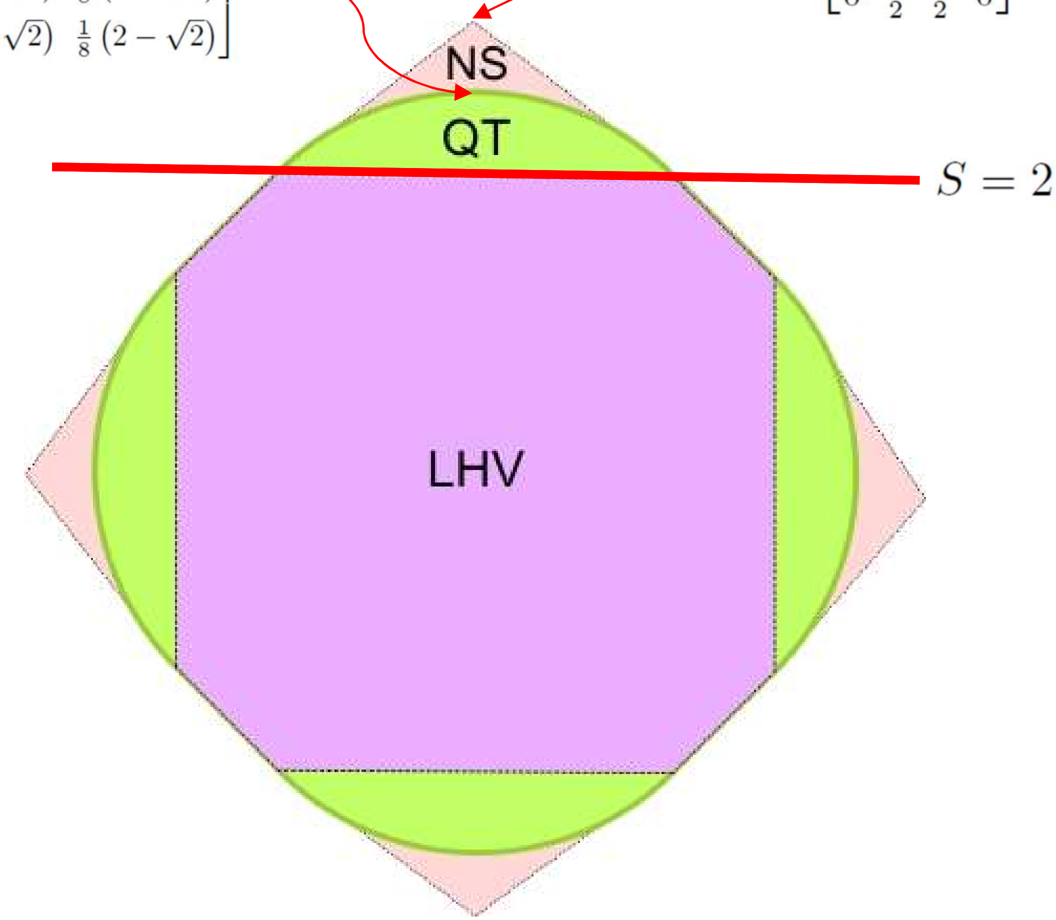
# Maximal violation allowed by no-signaling

$$\mathbf{p}_{\mathrm{T}} = \begin{bmatrix} \frac{1}{8}(2+\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2+\sqrt{2}) \\ \frac{1}{8}(2+\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2+\sqrt{2}) \\ \frac{1}{8}(2+\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2+\sqrt{2}) \\ \frac{1}{8}(2-\sqrt{2}) & \frac{1}{8}(2+\sqrt{2}) & \frac{1}{8}(2+\sqrt{2}) & \frac{1}{8}(2-\sqrt{2}) \end{bmatrix}$$

$$S(\mathbf{p}_{\mathrm{T}}) = 2\sqrt{2}$$

$$\mathbf{p}_{\mathrm{PR}} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$S(\mathbf{p}_{\mathrm{PR}}) = 4$$



# QM never reaches nonlocal vertices of the NS polytope

- No nonlocal vertex of the nonsignaling polytope of any Bell scenario can be achieved with a quantum behavior.

R. Ramanathan, J. Tuziemski, M. Horodecki, and P. Horodecki, Phys. Rev. Lett. **117**, 050401 (2016).

# Possible quantum behaviors

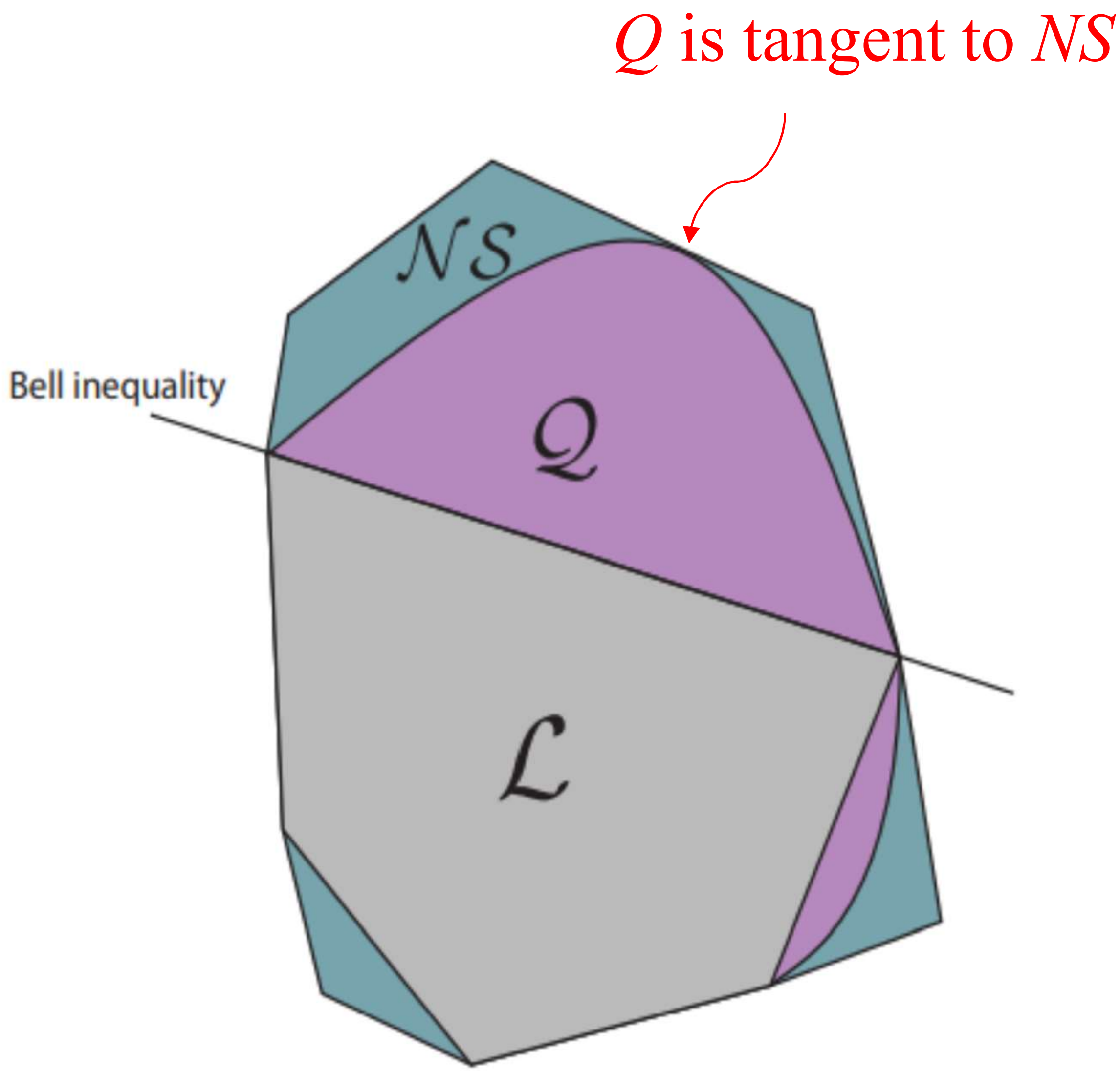
$$
\begin{array}{ll}
(1) & \beta_{\mathcal{L}} < \beta_{\mathcal{Q}} < \beta_{\mathcal{NS}} \,, \\
(2a) & \beta_{\mathcal{L}} = \beta_{\mathcal{Q}} < \beta_{\mathcal{NS}} \text{ and } \mathcal{F}_{\mathcal{L}} \subsetneq \mathcal{F}_{\mathcal{Q}} \,, \\
(2b) & \beta_{\mathcal{L}} = \beta_{\mathcal{Q}} < \beta_{\mathcal{NS}} \text{ and } \mathcal{F}_{\mathcal{L}} = \mathcal{F}_{\mathcal{Q}} \,, \\
(3a) & \beta_{\mathcal{L}} < \beta_{\mathcal{Q}} = \beta_{\mathcal{NS}} \text{ and } \mathcal{F}_{\mathcal{Q}} \subsetneq \mathcal{F}_{\mathcal{NS}} \,, \\
*(3b) & \beta_{\mathcal{L}} < \beta_{\mathcal{Q}} = \beta_{\mathcal{NS}} \text{ and } \mathcal{F}_{\mathcal{Q}} = \mathcal{F}_{\mathcal{NS}} \,, \\
(4a) & \beta_{\mathcal{L}} = \beta_{\mathcal{Q}} = \beta_{\mathcal{NS}} \text{ and } \mathcal{F}_{\mathcal{L}} \subsetneq \mathcal{F}_{\mathcal{Q}} \subsetneq \mathcal{F}_{\mathcal{NS}} \,, \\
(4b) & \beta_{\mathcal{L}} = \beta_{\mathcal{Q}} = \beta_{\mathcal{NS}} \text{ and } \mathcal{F}_{\mathcal{L}} = \mathcal{F}_{\mathcal{Q}} \subsetneq \mathcal{F}_{\mathcal{NS}} \,, \\
*(4c) & \beta_{\mathcal{L}} = \beta_{\mathcal{Q}} = \beta_{\mathcal{NS}} \text{ and } \mathcal{F}_{\mathcal{L}} \subsetneq \mathcal{F}_{\mathcal{Q}} = \mathcal{F}_{\mathcal{NS}} \,, \\
(4d) & \beta_{\mathcal{L}} = \beta_{\mathcal{Q}} = \beta_{\mathcal{NS}} \text{ and } \mathcal{F}_{\mathcal{L}} = \mathcal{F}_{\mathcal{Q}} = \mathcal{F}_{\mathcal{NS}} \,.
\end{array}
$$

The face $\mathcal{F}_{\mathcal{NS}}$ is the convex hull of some vertices of the no-signalling polytope. If all these vertices are local, we must have $\mathcal{F}_{\mathcal{L}} = \mathcal{F}_{\mathcal{NS}}$

* not realizable

K. T. Goh, J. m. k. Kaniewski, E. Wolfe, T. Vértesi, X. Wu, Y. Cai, Y.-C. Liang, and V. Scarani, Phys. Rev. A **97**, 022104 (2018).

# For some Bell scenarios (but no CHSH)



# Question

- What is the simplest bipartite Bell scenario for which $Q$ is nonlocal and tangent to $NS$?

- This is a fundamental question in physics.

# 1989: All versus nothing (GHZ)

- *x* in { $X$ , $Z$ }.
- *y* in { $X$ , $Z$ }.
- *z* in { $X$ , $Z$ }.
- *a*, *b*, *c* in {1,-1}.

- Quantum realization (GHZ state)

$$|\phi\rangle = -\frac{i}{\sqrt{2}}\left(|\oplus\oplus\oplus\rangle - |\ominus\ominus\ominus\rangle\right)$$

$$X = XII, \qquad X = IXI, \qquad X = IIX,$$
$$Z = ZII, \qquad Z = IZI, \qquad Z = IIZ.$$

$$|\oplus\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right)$$
$$|\ominus\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

D. M. Greenberger, M. A. Horne, and A. Zeilinger, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe* (Springer, 1989) pp. 69–72.
N. D. Mermin, Phys. Today **43**, 9 (1990).
N. D. Mermin, Am. J. Phys. **58**, 731 (1990).

# All versus nothing (GHZ)

- Alice's, Bob's, Charlie's outcomes satisfy

$$\begin{aligned} ZXX &= 1, \\ XZX &= 1, \\ XXZ &= 1, \\ ZZZ &= -1. \end{aligned}$$

- It is impossible to assign 1 or -1 satisfying all conditions.
- *Proof:* Multiply all the conditions, 1 = -1.

# All versus nothing (GHZ)

- If we multiply the three first quantum predictions

$$ZXX = 1,$$
$$XZX = 1,$$
$$XXZ = 1,$$

- we obtain

$$ZZZ = \quad 1.$$

- But the fourth quantum prediction is

$$ZZZ = -1.$$

N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).

# 1992: Local fraction. Full nonlocality

- Given a nonsignaling behavior, let us consider all possible decompositions

$$P(a,b|x,y) = q_L P_L(a,b|x,y) + (1-q_L) P_{NL}(a,b|x,y)$$

- in local behaviors $P_L(a,b|x,y)$ and nonlocal nonsignaling behaviors $P_{NL}(a,b|x,y)$, with $0 \leq q_L \leq 1$. The local fraction of $P(a,b|x,y)$ is

$$p_L \doteq \max_{\{P_L, P_{NL}\}} q_L.$$

- Fully nonlocal behaviors are those in which $p_L = 0$.

A. C. Elitzur, S. Popescu, and D. Rohrlich, Phys. Lett. A **162**, 25 (1992).

# Local fraction

$$p_L \leq \frac{\beta_{NS} - \beta_Q}{\beta_{NS} - \beta_L}$$

L. Aolita, R. Gallego, A. Acín, A. Chiuri, G. Vallone, P. Mataloni, and A. Cabello, Phys. Rev. A **85**, 032107 (2012).

# Bell inequality associated to GHZ

$$\langle ZXX \rangle + \langle XZX \rangle + \langle XXZ \rangle - \langle ZZZ \rangle \leq 2$$

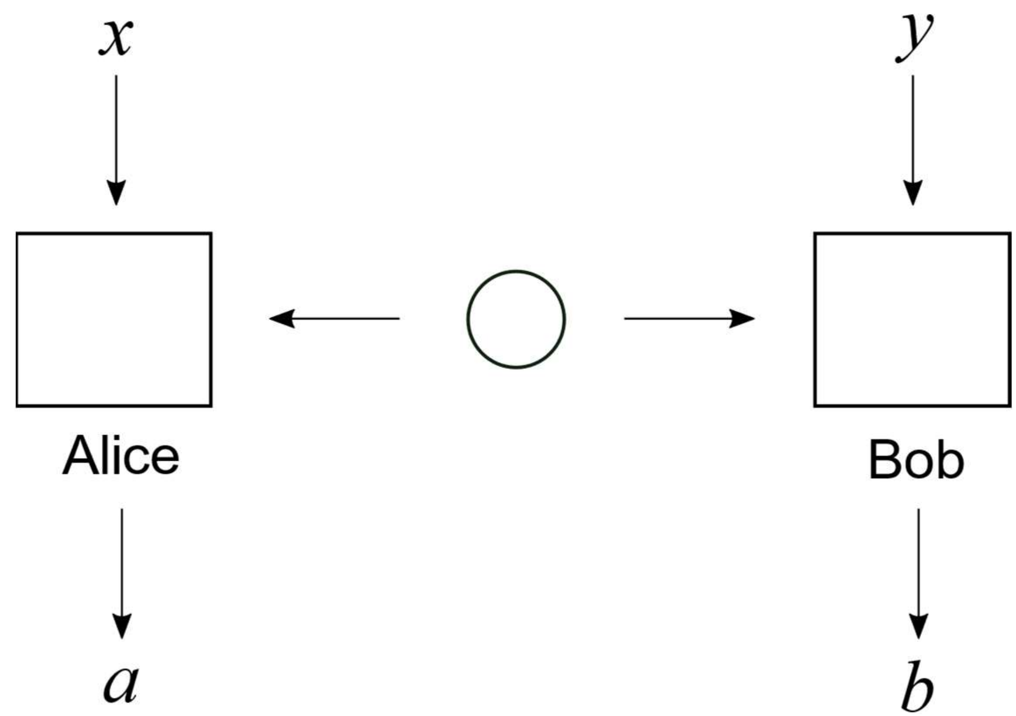
N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).

- GHZ is the simplest example of tripartite full nonlocality.

$$p_L \leq \frac{\beta_{NS} - \beta_Q}{\beta_{NS} - \beta_L}$$

# 2001: Bipartite AVN

- $x$ in $\{(A, a), (B, \beta), (c, \gamma)\}$
- $y$ in $\{(A, B), (a, c), (\beta, \gamma)\}$
- $a$, $b$ in {(1,1), (1,-1), (-1,1), (-1,-1)}.




A. Cabello, Phys. Rev. Lett. **86**, 1911 (2001).
A. Cabello, Phys. Rev. Lett. **87**, 010403 (2001).

# Bipartite AVN

- Quantum realization

$$|\psi\rangle = \frac{1}{4}\sum_{i=0}^{3}|ii\rangle$$

$$A = ZIII, \quad A = IIZI,$$
$$a = IXII, \quad a = IIIX,$$
$$B = IZII, \quad B = IIIZ,$$
$$\beta = XZII, \quad \beta = IIXZ,$$
$$c = XXII, \quad c = IIXX,$$
$$\gamma = YYII, \quad \gamma = IIYY,$$

$IIXZ = \mathbb{1} \otimes \mathbb{1} \otimes \sigma_x \otimes \sigma_z$

A. Cabello, Phys. Rev. Lett. **86**, 1911 (2001).

# Bipartite AVN

- Outcomes satisfy:

$$A = A,$$
$$a = a,$$
$$B = B,$$
$$\beta = \beta,$$
$$c = c,$$
$$\gamma = \gamma,$$
$$Aa = \beta\gamma,$$
$$B\beta = ac,$$
$$c\gamma = -AB.$$

- It is impossible to assign 1 or -1 satisfying all conditions.
- *Proof:* Multiply all the conditions, 1 = -1.

# Bell inequality associated to the bipartite AVN

$$\langle AA\rangle + \langle aa\rangle + \langle BB\rangle + \langle \beta\beta\rangle + \langle cc\rangle + \langle \gamma\gamma\rangle + \langle Aa\beta\gamma\rangle + \langle B\beta ac\rangle - \langle c\gamma AB\rangle \leq 7$$

- The bipartite AVN is the simplest example known of bipartite full nonlocality.

$$p_L \leq \frac{\beta_{NS} - \beta_Q}{\beta_{NS} - \beta_L}$$

A. Cabello, Phys. Rev. Lett. **86**, 1911 (2001).

# Bipartite AVN (and PT). Experiments

- C. Cinelli, M. Barbieri, R. Perris, P. Mataloni, and F. De Martini, Phys. Rev. Lett. **95**, 240405 (2005).
- T. Yang, Q. Zhang, J. Zhang, J. Yin, Z. Zhao, M. Żukowski, Z.-B. Chen, and J.-W. Pan, Phys. Rev. Lett. **95**, 240406 (2005).
- L. Aolita, R. Gallego, A. Acín, A. Chiuri, G. Vallone, P. Mataloni, and A. Cabello, Phys. Rev. A **85**, 032107 (2012).
- J.-M. Xu, Y.-Z. Zhen, Y.-X. Yang, Z.-M. Cheng, Z.-C. Ren, K. Chen, X.-L. Wang, and H.-T. Wang, Phys. Rev. Lett. **129**, 050402 (2022).

# 2003: Pseudo telepathy

**Definition 1.** *[3] A* bipartite game *$G = (I, O, W)$ is a set of inputs $I = X \times Y$, a set of outputs $O = A \times B$, and a relation $W \subseteq I \times O$ that inputs and outputs should satisfy to declare the game won.*

**Definition 2.** *[3] A* winning strategy *for a bipartite game $G = (I, O, W)$ is a strategy according to which, for every $x \in X$ and $y \in Y$, Alice and Bob output $a$ and $b$, respectively, such that $(x, y, a, b) \in W$.*

**Definition 3.** *[3] A bipartite game $G$ exhibits* pseudo-telepathy *(or* perfect pseudo-telepathy *[1]) if a behavior can yield a winning strategy, whereas no classical strategy that does not involve communication between the players is a winning strategy.*


[1] G. Brassard, A. Broadbent, and A. Tapp, in *Algorithms and Data Structures*, edited by F. Dehne, J.-R. Sack, and M. Smid (Springer Berlin Heidelberg, Berlin, Heidelberg, 2003) pp. 1–11.

[2] G. Brassard, A. Broadbent, and A. Tapp, Found. Phys. **35**, 1877 (2005).

[3] N. Gisin, A. A. Méthot, and V. Scarani, Int. J. Quant. Inf. **5**, 525 (2007).

# Pseudo telepathy magic square = Bipartite AVN

**Remark 1.** *As Brassard, Broadbent, and Tapp pointed out, the two-player PT game in Ref. [2] and the game used in the "all-vs-nothing" nonlocality proof in [8] "are totally equivalent!" [2].*

*[8]* A. Cabello, Phys. Rev. Lett. **86**, 1911 (2001).

[2] G. Brassard, A. Broadbent, and A. Tapp, Found. Phys. **35**, 1877 (2005).

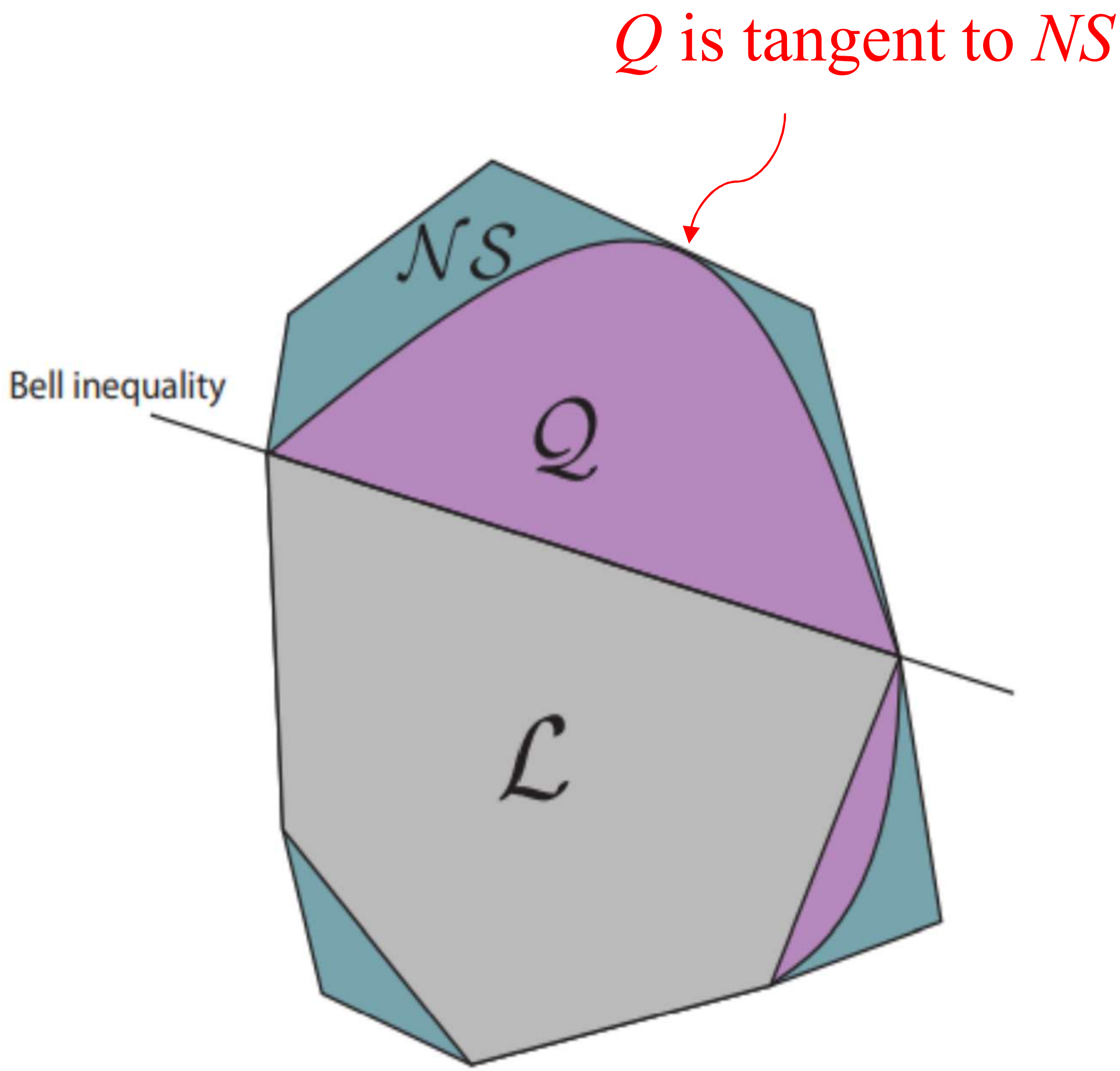
# 2011: Strong contextuality

- Strong contextuality is the impossibility of any deterministic assignment that is consistent with the support of a distribution. "Consistent" means that whenever an event is not in the support we assign the value 0 to it; when it is in the support, we may assign 0 or 1.

- A probabilistic model is strongly contextual if and only if it is maximally contextual (i.e., fully nonlocal in the case of Bell scenarios).

S. Abramsky and A. Brandenburger, New J. Phys. **13**, 113036 (2011).

# AVN, FN, PT, SC


$Q$ is tangent to $NS$
$\mathcal{NS}$
Bell inequality
$\mathcal{Q}$
$\mathcal{L}$

# Minimal requirements for bipartite PT…

- The minimal entanglement that is necessary and sufficient for bipartite pseudo-telepathy is qutrit-qutrit entanglement.

R. Renner and S. Wolf, in *International Symposium on Information Theory, 2004. ISIT 2004. Proceedings.*, IEEE (IEEE, 2004) p. 322,

- The minimal input cardinality that is necessary for bipartite pseudo-telepathy is $3 \times 3$.

- The minimal output cardinality that is necessary for bipartite pseudo-telepathy is $3 \times 2$.

R. Cleve, P. Høyer, B. Toner, and J. Watrous, in *Proceedings. 19th IEEE Annual Conference on Computational Complexity* (2004) pp. 236–249.

# Bipartite PT with minimal requirements?

**Question 4.** *An "open question of interest is whether we can find a bipartite pseudo-telepathy game which only uses the minimal requirements" (i.e.,* $3 \times 3$ *inputs and* $3 \times 2$ *outputs) [17].*

N. Gisin, A. A. Méthot, and V. Scarani, Int. J. Quant. Inf. **5**, 525 (2007).

# There is no bipartite PT with minimal requirements

**Theorem 4.** *Quantum mechanics does not allow for a bipartite PT game associated to a facet of the local polytope and using "the minimal requirements" (i.e., $3 \times 3$ inputs and $3 \times 2$ outputs).*

Emmanuel Zambrini Cruzeiro, Junior R. Gonzales-Ureta, AC

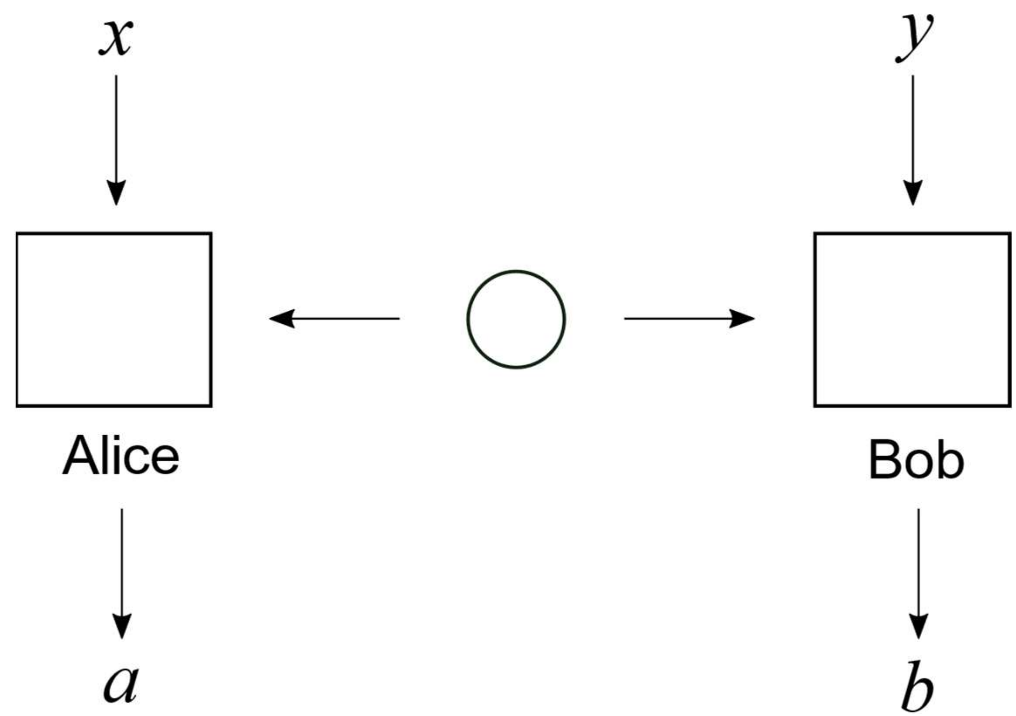
# Proof

*Proof.* The set of classical correlations for the $(3, 3, 3, 2)$ Bell scenario is fully described by a set of 25 (classes of) Bell inequalities, the facets of the corresponding local polytope. These 25 facets were first presented in Ref. [19], without solving the question of whether or not the list was complete. The proof that the list is actually complete was presented in Ref. [20]. Since we have the half-space representation of the local polytope (i.e., we have the local polytope defined as an intersection of a finite number of half-spaces), we can address Question 4 calculating, for every facet, the corresponding quantum bound (or an upper bound of it) using the NPA hierarchy [21]. We have found that for every facet, the quantum bound is strictly smaller than the nonsignaling one. As a consequence, the answer to Question 4 is negative. □

[19] T. Cope and R. Colbeck, Phys. Rev. A **100**, 022114 (2019).
[20] J. Jesus and E. Zambrini Cruzeiro, (2022), arXiv:arXiv:2212.03212 [quant-ph].
[21] M. Navascués, S. Pironio, and A. Acín, New J. Phys. **10**, 073013 (2008).

# Conjecture

- $x$ in $\{(A, a), (B, \beta), (c, \gamma)\}$
- $y$ in $\{(A, B), (a, c), (\beta, \gamma)\}$
- $a$, $b$ in {(1,1), (1,-1), (-1,1), (-1,-1)}.




- is the simplest bipartite AVN, FN, PT, and SC in nature.