Free choice, causality, contextuality, and signed measures

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System of random variables

$$\mathcal{R} = \left\{ R_q^c : c \in C, q \in Q, q \prec c \right\}$$

R_1^1	R_2^1		R_4^1		$c = 1$
	R_2^2	R_3^2		R_5^2	$c = 2$
R_1^3		R_3^3	R_4^3	R_5^3	$c = 3$
R_1^4	R_2^4	R_3^4	R_4^4	R_5^4	$c = 4$
$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q = 5$	

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R_1^1	R_2^1	$R_3^1 \equiv r_3^1$	R_4^1	$R_5^1 \equiv r_5^1$	$c = 1$
$R_1^2 \equiv r_1^2$	R_2^2	R_3^2	$R_4^2 \equiv r_4^2$	R_5^2	$c = 2$
R_1^3	$R_2^3 \equiv r_2^3$	R_3^3	R_4^3	R_5^3	$c = 3$
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System of random variables

$$R^c = \left\{R^c_q : q \in Q\right\}$$

	R^1_1	R^1_2	$R^1_3 \equiv r^1_3$	R^1_4	$R^1_5 \equiv r^1_5$	$c = 1$
	$R^2_1 \equiv r^2_1$	R^2_2	R^2_3	$R^2_4 \equiv r^2_4$	R^2_5	$c = 2$
R^3 :	R^3_1	$R^3_2 \equiv r^3_2$	R^3_3	R^3_4	R^3_5	$c = 3$
	R^4_1	R^4_2	R^4_3	R^4_4	R^4_5	$c = 4$
	$q = 1$	$q = 2$	$q = 3$	$q = 4$	$q = 5$	

Hidden Variable Model (HVM)

$$R^c \stackrel{d}{=} \{\text{function}\,(q, c, \text{hidden random variables}) : q \in Q\}$$

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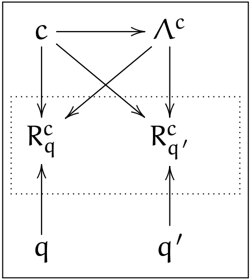
because we are free to choose the joint distribution of R^c and the hidden random variables.

General (universally applicable) HVM, HVM-Gen

$$R^c \stackrel{d}{=} \{\alpha\,(q, c, \Lambda^c) : q \in Q\}$$



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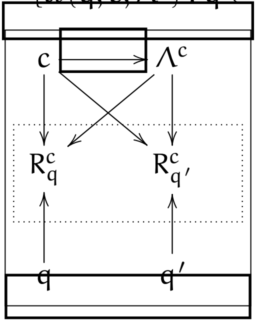

HVM-Gen generally violates both the assumption of context-independent (CI) mapping and the assumption of free choice (FC)

HVM with context-independent mapping (or local causality), HVM-CI

$$R^{c} \stackrel{d}{=} \{\beta\left(q, \Lambda^{c}\right) : q \in Q\}$$

$$\begin{tikzcd}
c \arrow[r] & \Lambda^{c} \arrow[dl] \arrow[d] \\
R^{c}_{q} & R^{c}_{q'} \\
q \arrow[u] & q' \arrow[u]
\end{tikzcd}$$

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Generally, HVM-CI allows for violations of the FC

HVM with free choice, HVM-FC

$$R^c \stackrel{d}{=} \{\gamma\,(q, c, \Lambda) : q \in Q\}$$

$$\begin{tikzcd}
c \arrow[d] \arrow[dr] & \Lambda \arrow[d] \arrow[dl] \\
R^c_q & R^c_{q'} \\
q \arrow[u] & q' \arrow[u]
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HVM with free choice, HVM-FC

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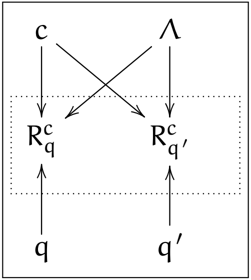
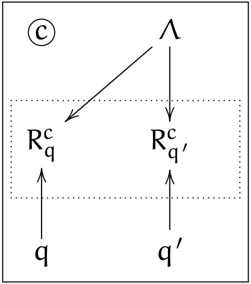


Generally, HVM-FC allows for violations of the CI mapping

HVM with context-independence and free choice, HV-Bell

$$R \stackrel{d}{=} \{\delta(q, \Lambda) : q \in Q\}$$



A concept needed: Coupling

A coupling of an indexed set of random variables $\{X_\phi\}_{\phi\in\Phi}$ is a set of jointly distributed random variables $\{Y_\phi\}_{\phi\in\Phi}$ such that

$$Y_\phi \stackrel{d}{=} X_\phi,$$

for all $\phi \in \Phi$.

An observation (trivial, but important):

Any set of jointly distributed random variables is a (single) random variable.

Theorem

HMV-CI

$$\boxed{R^c \overset{d}{=} \{\beta\,(q, \Lambda^c) : q \in Q\}}$$

$$\Updownarrow$$

$$\boxed{R^c \overset{d}{=} \{\gamma\,(q, c, \Lambda) : q \in Q\}}$$

HVM-FC

Theorem

$$
\begin{tikzcd}
 & \overset{\text{HMV-CI}}{\boxed{R^c \overset{d}{=} \{\beta\left(q, \Lambda^c\right) : q \in Q\}}} \arrow[dd, Leftrightarrow] \\
\overset{\text{HVM-Gen}}{\boxed{R^c \overset{d}{=} \{\alpha\left(q, c, \Lambda^c\right) : q \in Q\}}} \arrow[ur, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\
 & \underset{\text{HVM-FC}}{\boxed{R^c \overset{d}{=} \{\gamma\left(q, c, \Lambda\right) : q \in Q\}}}
\end{tikzcd}
$$

Main consequence of the theorem

HVM-Gen

$$\boxed{R^c \overset{d}{=} \{\alpha(q, c, \Lambda^c) : q \in Q\}}$$

HVM-Bell

$$\boxed{R \overset{d}{=} \{\delta(q, \Lambda) : q \in Q\}}$$

Any deviation of an HVM-Gen from HVM-Bell can be interchangeably interpreted/measured as restriction of FC or violation of CI.

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$$\boxed{R \overset{d}{=} \{\delta\left(q, \Lambda\right) : q \in Q\}}$$

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$$\boxed{R^c \overset{d}{=} \{\beta\left(q, \Lambda^c\right) : q \in Q\}}$$

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$$\boxed{R \overset{d}{=} \{\delta\left(q, \Lambda\right) : q \in Q\}}$$

$$\boxed{R^c \overset{d}{=} \{\gamma\left(q, c, \Lambda\right) : q \in Q\}}$$

HVM-FC

One and the same measure can be assigned to restrictions of FC and violations of CI.

Proof (1a)

$$\overset{\text{CI mapping (local)}}{\boxed{R^c \overset{d}{=} \{\beta\,(q, \Lambda^c) : q \in Q\}}} \Longrightarrow \overset{\text{Free choice}}{\boxed{R^c \overset{d}{=} \{\gamma\,(q, c, \Lambda) : q \in Q\}}}$$

Proof (1a)

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Form an arbitrary coupling Λ of the random variables $\{\Lambda^c : c \in C\}$. We have

$$\Lambda^c \overset{d}{=} \mathrm{Proj}_c(\Lambda) = \phi(c, \Lambda).$$

But then

$$\{\beta(q, \Lambda^c)\}_q \overset{d}{=} \{\beta(q, \phi(c, \Lambda))\}_q = \{\gamma(q, c, \Lambda)\}_q.$$

Proof (1b)

CI mapping (local)

Free choice

$$\boxed{R^c \stackrel{d}{=} \{\beta\left(q, \Lambda^c\right) : q \in Q\}} \Longleftarrow \boxed{R^c \stackrel{d}{=} \{\gamma\left(q, c, \Lambda\right) : q \in Q\}}$$

Proof (1b)

$$\underset{\text{CI mapping (local)}}{\boxed{R^c \overset{d}{=} \{\beta(q, \Lambda^c) : q \in Q\}}} \Longleftarrow \underset{\text{Free choice}}{\boxed{R^c \overset{d}{=} \{\gamma(q, c, \Lambda) : q \in Q\}}}$$

Define, for every c, the random variable

$$\Lambda^c := \{\gamma(q, c, \Lambda)\}_q$$

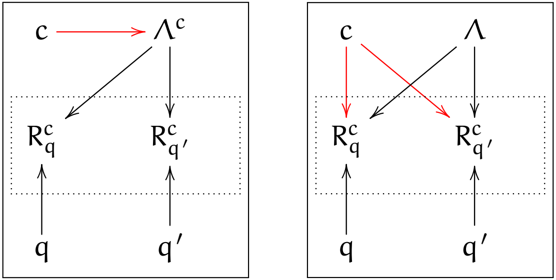
(whose components are jointly distributed because they are functions of one and the same Λ). The components $\gamma(q, c, \Lambda)$ in Λ^c are indexed by q, and

$$\gamma(q, c, \Lambda) = \mathrm{Proj}_q(\Lambda^c) = \beta(q, \Lambda^c).$$

But then

$$\{\gamma(q, c, \Lambda)\}_q \overset{d}{=} \{\beta(q, \Lambda^c)\}_q.$$

Proof (1a+1b)



Proof (2a)

$$\begin{array}{ccc} \text{General} & & \text{Free choice} \\ \boxed{R^c \overset{d}{=} \{\alpha\left(q, c, \Lambda^c\right) : q \in Q\}} & \Longrightarrow & \boxed{R^c \overset{d}{=} \{\gamma\left(q, c, \Lambda\right) : q \in Q\}} \end{array}$$

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Form an arbitrary coupling Λ of the random variables $\{\Lambda^c : c \in C\}$. We have

$$\Lambda^c \overset{d}{=} \mathrm{Proj}_c\left(\Lambda\right) = \phi\left(c, \Lambda\right).$$

But then

$$\{\alpha\left(q, c, \Lambda^c\right)\}_q \overset{d}{=} \{\alpha\left(q, c, \phi\left(c, \Lambda\right)\right)\}_q = \{\gamma\left(q, c, \Lambda\right)\}_q.$$

Proof (2b)

General		Free choice
$\boxed{R^c \overset{d}{=} \{\alpha\,(q, c, \Lambda^c) : q \in Q\}}$	$\Longleftarrow$	$\boxed{R^c \overset{d}{=} \{\gamma\,(q, c, \Lambda) : q \in Q\}}$

Proof (2b)

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Obvious.

Proof (2a+2b)

$$
\begin{tikzcd}
c \arrow[r] \arrow[d] \arrow[dr] & \Lambda^c \arrow[d] \arrow[dl] \\
R^c_q & R^c_{q'} \\
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\end{tikzcd}
\qquad
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QED

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\begin{tikzcd}
 & \begin{array}{c}\text{General}\\ \boxed{R^c \overset{d}{=} \{\alpha\,(q, c, \Lambda^c) : q \in Q\}}\end{array} \arrow[dl, Leftrightarrow] \arrow[dd, Leftrightarrow] \\
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 & \begin{array}{c}\boxed{R^c \overset{d}{=} \{\gamma\,(q, c, \Lambda) : q \in Q\}}\\ \text{Free choice}\end{array}
\end{tikzcd}
$$

Questions about hidden assumptions:

Couldn't the most general HVM formulated in terms of $\alpha\left(\mathsf{q}, \mathsf{c}, \Lambda^{\mathsf{c}}_{\mathsf{q}}\right)$ rather than $\alpha\left(\mathsf{q}, \mathsf{c}, \Lambda^{\mathsf{c}}\right)$?

Questions about hidden assumptions:

Couldn't the most general HVM formulated in terms of $\alpha\left(q, c, \Lambda_q^c\right)$ rather than $\alpha\left(q, c, \Lambda^c\right)$?

Yes, but they are trivially equivalent.

$$\begin{array}{c} \phi\left(q, c, \Lambda_q^c\right) \\ \Downarrow \\ \phi\left(q, c, \Lambda^c\right) \end{array} \qquad \Lambda^c = \left\{\Lambda_q^c : q \in Q\right\}$$

Questions about hidden assumptions:

Did we tacitly assume outcome determinism?

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No, it holds trivially. Using HVM-FC:

$$\begin{array}{c} \phi\left(q, c, \Lambda, \lambda_q^c\right) \\ \Downarrow \\ \phi\left(q, c, \Lambda^*\right) \end{array} \qquad \Lambda^* \stackrel{d}{=} \left\{\Lambda, \lambda_q^c : q \in Q, c \in C\right\}$$

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No, it holds automatically. Using HVM-FC:

$$\Pr\left[\left\{\gamma\left(\mathsf{q},\Lambda\right)=\gamma_{\mathsf{q}}:\mathsf{q}\in\mathsf{Q}\right\}\middle|\Lambda\right]=\prod_{\mathsf{q}\in\mathsf{Q}}\Pr\left[\gamma\left(\mathsf{q},\Lambda\right)=\gamma_{\mathsf{q}}\middle|\Lambda\right]$$

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Shouldn't the FC assumption be formulated in terms of the (non-)independence of the hidden variable Λ and context c treated as another random variable?

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Shouldn't the FC assumption be formulated in terms of the (non-)independence of the hidden variable Λ and context c treated as another random variable?

First, this makes no difference, and *second*, treating c as a random variable is conceptually dubious:
because clearly, variations of c in time and/or space can be made as *non-random* as one wishes.

Contextuality: Couplings for systems

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$$\Lambda = \left\{\Lambda_q^c : q \in Q\right\}$$

$$R^c \stackrel{d}{=} \{\mathsf{Proj}_{q,c}\left(\Lambda\right) : q \in Q\} \tag{coupling}$$

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$$R^c \overset{d}{=} \{\mathsf{Proj}_{q,c}\,(\Lambda) : q \in Q\} \qquad \text{(coupling)}$$

The system is noncontextual if Λ can be chosen so that, for all q and all pairs c, c',

$$\Pr\left[\Lambda_q^c = \Lambda_q^{c'} = 1\right] = \min\left(\Pr\left[\Lambda_q^c = 1\right], \Pr\left[\Lambda_q^{c'} = 1\right]\right).$$

(multimaximality property)

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(multimaximality property)

$$\Lambda = \mathsf{MMC}(\mathcal{R})$$

Contextuality: Quasi-couplings for systems

$$\Lambda : \mathfrak{S} \stackrel{\mathrm{id}}{\to} \mathfrak{S}, (\mathfrak{S}, \Sigma, \mu), \mathfrak{S} = \{-1, 1\}^{\|Q\| \times \|S\|}$$

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Redefine Λ by replacing measure μ with a signed measure $\mu^{\pm}$:

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$$R^{c} \overset{d}{=} \{\mathsf{Proj}_{q,c}(\Lambda) : q \in Q\} \qquad \text{(quasi-coupling)}$$

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$$\Lambda = \mathsf{MMQC}(\mathcal{R})$$

Contextuality: Degree of

Total variation:

$$\left|\mu^{\pm}\right| = \mu^{+} - \mu^{-} = \sup\left(\mu^{\pm}(A) : A \in \Sigma\right) - \sup\left(-\mu^{\pm}(A) : A \in \Sigma\right)$$

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$$\inf_{\mathsf{MMQC}(\mathcal{R})} \left|\mu^{\pm}\right| - 1 = \mathsf{CNT3}$$

Over all MMQC$(\mathcal{R})$, this infimum is a measure of contextuality in $\mathcal{R}$.