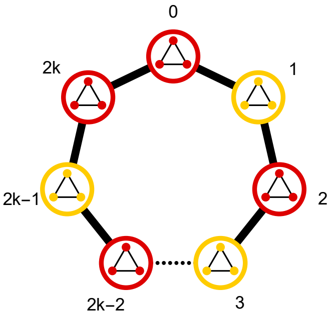
State-independent contextuality with nonunit rank

Matthias Kleinmann
University of Siegen

with **Pascal Höhn, Zheng-Peng Xu, Xiao-Dong Yu**



arXiv:2207.11183

NJP **23** (2021)

Motivation

☞ Why state-independent contextuality (SIC)?

- Characteristic part of quantum **theory**.
- Property of **projective** measurements.

☞ Why nonunit rank?

- Theory of **degenerate** measurements is poorly developed.
- The **smallest** SIC scenario might require nonunit rank.
- Useful for device-independent **certification** of nonunit-rank measurements?

Motivation

☞ Why state-independent contextuality (SIC)?

- Characteristic part of quantum **theory**.
- Property of **projective** measurements.

☞ Why nonunit rank?

- Theory of **degenerate** measurements is poorly developed.
- The **smallest** SIC scenario might require nonunit rank.
- Useful for device-independent **certification** of nonunit-rank measurements?

SIC inequalities

- The **exclusivity graph** G of $(\Pi_k)_k$ satisfies

$$(k,l) \text{ is an edge } \rightarrow \Pi_k \Pi_\ell = 0.$$

- The **stable set** $\mathrm{STAB}(G)$ is the convex hull of all $\{0,1\}$-vectors $\boldsymbol{x}$ with

$$(k,l) \text{ is an edge } \rightarrow x_k x_\ell = 0.$$

- **SIC** requires

$$\{ (\mathrm{tr}(\rho\Pi_k))_k \mid \rho \text{ is a state} \} \quad \text{is disjoint from} \quad \mathrm{STAB}(G).$$

$\hookrightarrow$ **Hyperplane separation** by a vector w,

$$\sum_k w_k \,\mathrm{tr}(\rho\Pi_k) > \sum_k w_k x_k \text{ for all states } \rho \text{ and } \boldsymbol{x} \in \mathrm{STAB}(G).$$

SIC inequalities

- The **exclusivity graph** G of $(\Pi_k)_k$ satisfies

$$(k,l) \text{ is an edge } \rightarrow \Pi_k \Pi_\ell = 0.$$

- The **stable set** $\mathrm{STAB}(G)$ is the convex hull of all $\{0,1\}$-vectors $\boldsymbol{x}$ with

$$(k,l) \text{ is an edge } \rightarrow x_k x_\ell = 0.$$

- **SIC** requires

$$\{ (\mathrm{tr}(\rho \Pi_k))_k \mid \rho \text{ is a state} \} \quad \text{is disjoint from} \quad \mathrm{STAB}(G).$$

$\hookrightarrow$ **Hyperplane separation** by a vector w,

$$\sum_k w_k \, \mathrm{tr}(\rho \Pi_k) > \sum_k w_k x_k \text{ for all states } \rho \text{ and } \boldsymbol{x} \in \mathrm{STAB}(G).$$

SIC inequalities

- The **exclusivity graph** G of $(\Pi_k)_k$ satisfies

$$(k,l) \text{ is an edge } \to \Pi_k \Pi_\ell = 0.$$

- The **stable set** $\mathrm{STAB}(G)$ is the convex hull of all $\{0,1\}$-vectors $\boldsymbol{x}$ with

$$(k,l) \text{ is an edge } \to x_k x_\ell = 0.$$

- **SIC** requires

$$\{ (\mathrm{tr}(\rho \Pi_k))_k \mid \rho \text{ is a state} \} \quad \text{is disjoint from} \quad \mathrm{STAB}(G).$$

$\hookrightarrow$ **Hyperplane separation** by a vector $\boldsymbol{w}$,

$$\sum_k w_k \,\mathrm{tr}(\rho \Pi_k) > \sum_k w_k x_k \text{ for all states } \rho \text{ and } \boldsymbol{x} \in \mathrm{STAB}(G).$$

SIC ratio

$$\sum_k w_k \operatorname{tr}(\rho \Pi_k) > \sum_k w_k x_k$$

SIC ratio

$$\eta = \max_{\boldsymbol{w}} \frac{\min \{ \sum_k w_k \operatorname{tr}(\rho \Pi_k) \mid \rho \text{ is a state} \}}{\max \{ \sum_k w_k x_k \mid \boldsymbol{x} \in \mathrm{STAB}(G) \}}$$

↪ $(\Pi_k)_k$ features SIC if and only if $\eta > 1$.

- w^* yields the **optimal** SIC inequality.

☞ η can be computed by a semi-definite program.

see also: [MK, Budroni, Larsson, Gühne & Cabello, PRL 2012]

SIC ratio

$$\sum_k w_k \operatorname{tr}(\rho\Pi_k) > \sum_k w_k x_k$$

SIC ratio

$$\eta = \max_{\boldsymbol{w}} \frac{\min\{\,\sum_k w_k \operatorname{tr}(\rho\Pi_k) \mid \rho \text{ is a state}\,\}}{\max\{\,\sum_k w_k x_k \mid \boldsymbol{x} \in \mathrm{STAB}(G)\,\}}$$

$\hookrightarrow$ $(\Pi_k)_k$ features SIC if and only if $\eta > 1$.

- $\boldsymbol{w}^*$ yields the **optimal** SIC inequality.

☞ η can be computed by a semi-definite program.

see also: [MK, Budroni, Larsson, Gühne & Cabello, PRL 2012]

SIC ratio

$$\sum_k w_k \operatorname{tr}(\rho\Pi_k) > \sum_k w_k x_k$$

SIC ratio

$$\eta = \max_{\boldsymbol{w}} \frac{\min\{\sum_k w_k \operatorname{tr}(\rho\Pi_k) \mid \rho \text{ is a state}\}}{\max\{\sum_k w_k x_k \mid \boldsymbol{x} \in \mathrm{STAB}(G)\}}$$

$\hookrightarrow$ $(\Pi_k)_k$ features SIC if and only if $\eta > 1$.

- $\boldsymbol{w}^*$ yields the **optimal** SIC inequality.

☞ η can be computed by a semi-definite program.

see also: [MK, Budroni, Larsson, Gühne & Cabello, PRL 2012]

Conditions for SIC

For G and $\operatorname{rank}\Pi_k = r_k$ define

- $d_\pi(G, \boldsymbol{r})$ as the minimal dimension
- $\eta(G, \boldsymbol{r})$ as the maximal SIC ratio

over all $(\Pi_k)_k$ with corresponding exclusivity graph G.

Theorem

$$\chi_{\mathrm{f}}(G, \boldsymbol{r}) \geq \eta(G, \boldsymbol{r}) d_\pi(G, \boldsymbol{r})$$

Necessary condition for SIC

$$\chi_{\mathrm{f}}(G, \mathbf{r}) > d_\pi(G, \mathbf{r}).$$

see also: [Ramanathan & Horodecki, PRL (2014)]

Conditions for SIC

For G and $\operatorname{rank} \Pi_k = r_k$ define

- $d_\pi(G, \boldsymbol{r})$ as the minimal dimension
- $\eta(G, \boldsymbol{r})$ as the maximal SIC ratio

over all $(\Pi_k)_k$ with corresponding exclusivity graph G.

Theorem

$$\chi_{\mathrm{f}}(G, \boldsymbol{r}) \geq \eta(G, \boldsymbol{r}) d_\pi(G, \boldsymbol{r})$$

Necessary condition for SIC

$$\chi_{\mathrm{f}}(G, \mathbf{r}) > d_\pi(G, \mathbf{r}).$$

see also: [Ramanathan & Horodecki, PRL (2014)]

Conditions for SIC

For G and $\operatorname{rank}\Pi_k = r_k$ define

- $d_\pi(G, \boldsymbol{r})$ as the minimal dimension
- $\eta(G, \boldsymbol{r})$ as the maximal SIC ratio

over all $(\Pi_k)_k$ with corresponding exclusivity graph G.

Theorem

$$\chi_{\mathrm{f}}(G, \boldsymbol{r}) \geq \eta(G, \boldsymbol{r}) d_\pi(G, \boldsymbol{r})$$

Necessary condition for SIC

$$\chi_{\mathrm{f}}(G, \mathbf{r}) > d_\pi(G, \mathbf{r}).$$

see also: [Ramanathan & Horodecki, PRL (2014)]

Computation of the SIC dimension

☞ How to compute $d_\pi(G, r)$?

- $d_\pi(G, \boldsymbol{r}) \geq \vartheta(\bar{G}, \boldsymbol{r})$
- $d_\pi(G, \boldsymbol{r}) = d_\pi(G^{\boldsymbol{r}}, 1)$
  where in $G^{\boldsymbol{r}}$, every vertex k is replaced by a clique of size r_k.

See-saw optimization

$d_\pi(G, 1) \leq d$ if and only if there exists a matrix M with

- $M_{k,l} = 0$ if (k, l) is an edge.
- $M \geq 0$ and $\operatorname{rank} M = d$

Alternating optimization:

1. Set $M_{k,l} = 0$ according to G.
2. Set all eigenvalues to 0 except at most d positive eigenvalues.

✔ Fast and reliable.

Computation of the SIC dimension

☞ How to compute $d_\pi(G, r)$?

- $d_\pi(G, \boldsymbol{r}) \geq \vartheta(\bar{G}, \boldsymbol{r})$
- $d_\pi(G, \boldsymbol{r}) = d_\pi(G^{\boldsymbol{r}}, 1)$
  where in $G^{\boldsymbol{r}}$, every vertex k is replaced by a clique of size r_k.

See-saw optimization

$d_\pi(G, 1) \leq d$ if and only if there exists a matrix M with

- $M_{k,l} = 0$ if (k, l) is an edge.
- $M \geq 0$ and $\mathrm{rank}\, M = d$

Alternating optimization:

1. Set $M_{k,l} = 0$ according to G.
2. Set all eigenvalues to 0 except at most d positive eigenvalues.

✔ Fast and reliable.

Computation of the SIC dimension

☞ How to compute $d_\pi(G, r)$?

- $d_\pi(G, \boldsymbol{r}) \geq \vartheta(\bar{G}, \boldsymbol{r})$
- $d_\pi(G, \boldsymbol{r}) = d_\pi(G^{\boldsymbol{r}}, 1)$
  where in $G^{\boldsymbol{r}}$, every vertex k is replaced by a clique of size r_k.

See-saw optimization

$d_\pi(G, 1) \leq d$ if and only if there exists a matrix M with

- $M_{k,l} = 0$ if (k, l) is an edge.
- $M \geq 0$ and $\operatorname{rank} M = d$

Alternating optimization:

1. Set $M_{k,l} = 0$ according to G.
2. Set all eigenvalues to 0 except at most d positive eigenvalues.

✔ Fast and reliable.

Computation of the SIC dimension

☞ How to compute $d_\pi(G, r)$?

- $d_\pi(G, \boldsymbol{r}) \geq \vartheta(\bar{G}, \boldsymbol{r})$
- $d_\pi(G, \boldsymbol{r}) = d_\pi(G^{\boldsymbol{r}}, 1)$
  where in $G^{\boldsymbol{r}}$, every vertex k is replaced by a clique of size r_k.

See-saw optimization

$d_\pi(G, 1) \leq d$ if and only if there exists a matrix M with

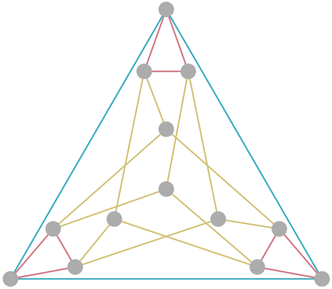
- $M_{k,l} = 0$ if (k, l) is an edge.
- $M \geq 0$ and $\operatorname{rank} M = d$

Alternating optimization:

1. Set $M_{k,l} = 0$ according to G.
2. Set all eigenvalues to 0 except at most d positive eigenvalues.

✔ Fast and reliable.

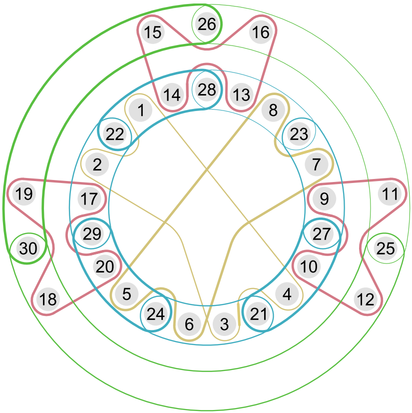
Search for the smallest SIC scenario



[Yu & Oh, PRL (2012)]

- There are **13** rank-1 projectors featuring SIC.
- For homogeneous rank 1,2,3, **no smaller set** exists.
- **At least 9** projectors are necessary.

Can higher rank help, at all?



[Toh, Chin. PL (2013)]

☞ Projective representation featuring SIC needs **rank 2**.

✘ Proof is complicated and particular.

OR-product

☞ Idea: Use the OR-product of graphs, $G \vee F$.

Intuitively:

- G is exclusivity graph of $(\Pi_k)_k$
- F is exclusivity graph of $(\Gamma_\ell)_\ell$

$\hookrightarrow$ $G \vee F$ is exclusivity graph of $(\Pi_k \otimes \Gamma_\ell)_{k,\ell}$

Definition

$G \vee F$ is the graph with

- vertices $V(G) \times V(G)$
- edges $[(u, v), (u', v')]$ where
  (u, u') is an edge of G **or** (v, v') is an edge of F.

OR-product

☞ Idea: Use the OR-product of graphs, $G \vee F$.

Intuitively:

- G is exclusivity graph of $(\Pi_k)_k$
- F is exclusivity graph of $(\Gamma_\ell)_\ell$

$\hookrightarrow$ $G \vee F$ is exclusivity graph of $(\Pi_k \otimes \Gamma_\ell)_{k,\ell}$

Definition

$G \vee F$ is the graph with

- vertices $V(G) \times V(G)$
- edges $[(u,v),(u',v')]$ where
  (u,u') is an edge of G **or** (v,v') is an edge of F.

General construction

Theorem

- ***Assume*** $\omega(G) < \chi_{\mathrm{f}}(G) < \omega(G) + \frac{1}{2}$ *and* $\eta(G, 1) > 1$.
- ***For any*** $r < \frac{1}{\kappa}$, *where* $\kappa = 2(\chi_{\mathrm{f}}(G) - \omega(G))$,
- ***choose*** k *such that* $k(1 - r\kappa) \geq r\chi_{\mathrm{f}}(F)$.

$\hookrightarrow$ $F = G \vee C_{2k+1}$ *enjoys*

$$\eta(F, r) \leq 1 \quad \textit{while} \quad \eta(F, k) > 1.$$

Examples

Graph	$\lvert V(G)\rvert$	$\eta(G,1)$	k_1	k_2	k_3	k_4	k_5
G_{BBC}	21	$1+\frac{1}{9}$	10	–	–	–	–
G_{YO}	13	$1+\frac{2}{33}$	5	24	–	–	–
G_{H}	18	$1+\frac{1}{75}$	6	17	50	–	–
G_{X}	18	$1+\frac{1}{42}$	6	20	87	–	–
G_{BBCr}	17	$1+\frac{1}{78}$	4	10	21	46	170

- G_{YO} Yu & Oh, PRL (2012)
- G_{BBC} Bengtsson, Blanchfield & Cabello, PRA (2012)
- G_H, G_X new SIC graphs
- G_{BBCr} is G_{BBC} with 4 vertices removed

Numerical examples

Graph	r	$\lvert V(\mathcal{G})\rvert$	$\chi_{\mathrm{f}}(\mathcal{G})$	$\eta(\mathcal{G}, r)$	d
$G_{\mathrm{YO}} \vee \mathrm{AR}(2)$	2	65	$7 + \frac{21}{22}$	$1 + \frac{2}{33}$	15
$\mathcal{R}(G_{\mathrm{YO}} \vee AR(2))$	2	39	$7 + \frac{71}{88}$	$1 + \frac{2}{65}$	15
$G_{\mathrm{CEG}} \vee \mathrm{AR}(2)$	2	90	$11 + \frac{1}{4}$	$1 + \frac{1}{8}$	20
$\mathcal{R}(G_{\mathrm{CEG}} \vee AR(2))$	2	54	$10 + \frac{3}{4}$	$1 + \frac{1}{17}$	20
$G_{\mathrm{YO}} \vee \mathrm{AR}(3)$	3	104	$8 + \frac{16}{33}$	$1 + \frac{2}{33}$	24

- G_{YO} Yu & Oh, PRL (2012)
- G_{CEG} Cabello, Estebaranz & Garcìa-Alcaine, PRA (1996)
- $AR(2) = C(5, (2, 3) = C_5$
- $AR(3) = C(8, [3, 4, 5])$

Summary

- **SIC ratio** η decides SIC for any set of projectors (SDP).
- $d_\pi(G)$ can be computed via a **see-saw** algorithm.
- **Smallest** SIC set: Still Yu&Oh.
- **OR-product** generates graphs requireing nonunit rank for SIC.



arXiv:2207.11183

NJP **23** (2021)