Quasi-probability approach to Kochen-Specker theorem

Paweł Kurzyński (joint work with K. Onggadinata and D. Kaszlikowski)

QCQMB, 17 December 2022

Negative (quasi)probabilities

- violation of Bell/KS inequality ⇒ nonlocality/contextuality **or** lack or realism **or** lack of free will

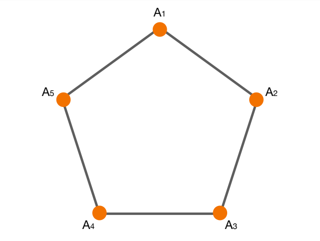
- if you stick to locality/non-contextuality, free will and Joint Probability Distribution (JPD) ⇒ Joint Quasiprobability Distribution (JQD) – negative probabilities

  S. Abramsky and A. Brandenburger, New J. Phys. 13, 113036 (2011)

  S. Abramsky and A. Brandenburger, in Horizons of the Mind (Springer, Cham, 2014)

- negativity ⇒ nonclassicality

Example – Wright/KCBS



$$\sum_{i=1}^{5} p(A_i) \leq 2$$

Joint event space

$$\{0,0,0,0,0\},\quad \{1,0,0,0,0\},\quad \{0,1,0,0,0\},$$
$$\{0,0,1,0,0\},\quad \{0,0,0,1,0\},\quad \{0,0,0,0,1\},$$
$$\{1,0,1,0,0\},\quad \{1,0,0,1,0\},\quad \{0,1,0,1,0\},$$
$$\{0,1,0,0,1\},\quad \{0,0,1,0,1\}$$

Example 1: $p(A_i) = 1/3$

$$\begin{aligned} p(1,0,1,0,0) &= 1/6, \\ p(1,0,0,1,0) &= 1/6, \\ p(0,1,0,1,0) &= 1/6, \\ p(0,1,0,0,1) &= 1/6, \\ p(0,0,1,0,1) &= 1/6, \\ p(0,0,0,0,0) &= 1/6. \end{aligned}$$

Example 2: $p(A_i) = 1/\sqrt{5}$

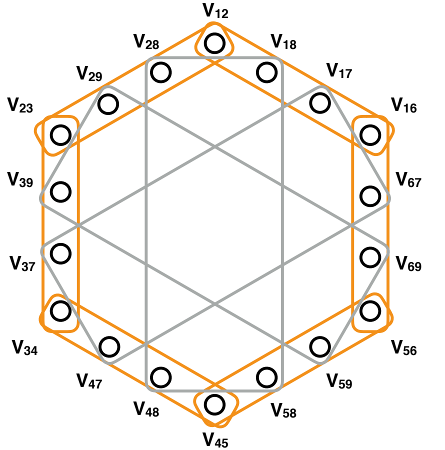
$$\begin{aligned} q(1,0,1,0,0) &= 1/2\sqrt{5}, \\ q(1,0,0,1,0) &= 1/2\sqrt{5}, \\ q(0,1,0,1,0) &= 1/2\sqrt{5}, \\ q(0,1,0,0,1) &= 1/2\sqrt{5}, \\ q(0,0,1,0,1) &= 1/2\sqrt{5}, \\ q(0,0,0,0,0) &= 1 - 5/2\sqrt{5} \approx -0.118 \end{aligned}$$

The Problem – KS sets and joint event space

1. *exclusivity* – for a jointly measurable subset of mutually exclusive events, corresponding to $\{A_1, A_2 \ldots, A_m\}$, at most one of them will occur at the same time, i.e., only the following outcome assignments $\{a_1, a_2, \ldots, a_m\}$ are allowed: $\{0, 0, \ldots, 0\}$, $\{1, 0, \ldots, 0\}$, $\{0, 1, \ldots, 0\}$, ..., $\{0, 0, \ldots, 1\}$.

2. *completeness* – for a complete jointly measurable subset of mutually exclusive events, corresponding to $\{A_1, A_2 \ldots, A_n\}$, exactly one of them will occur, i.e., only the following outcome assignments $\{a_1, a_2, \ldots, a_n\}$ are allowed: $\{1, 0, \ldots, 0\}$, $\{0, 1, \ldots, 0\}$, ..., $\{0, 0, \ldots, 1\}$.

The Problem – KS sets and joint event space



$V_{12} = (1,0,0,0)$
$V_{28} = (0,0,0,1)$
$V_{29} = (0,1,1,0)$
$V_{23} = (0,1,-1,0)$
$V_{39} = (1,0,0,1)$
$V_{37} = (1,1,1,-1)$
$V_{34} = (-1,1,1,1)$
$V_{47} = (1,1,-1,1)$
$V_{48} = (1,0,1,0)$
$V_{45} = (0,1,0,-1)$
$V_{58} = (1,0,-1,0)$
$V_{59} = (1,-1,1,-1)$
$V_{56} = (1,1,1,1)$
$V_{69} = (1,1,-1,-1)$
$V_{67} = (1,-1,0,0)$
$V_{16} = (0,0,1,-1)$
$V_{17} = (0,0,1,1)$
$V_{18} = (0,1,0,0)$

A. Cabello, J. M. Estebaranz, and G. Garcia Alcaine, Phys. Lett. A 212, 183 (1996)

Emergence of exclusivity and completeness from JQD

Example 3:

- consider two binary 0/1 random variables A and B
- set of possible measurement events: $\{00, 01, 10, 11\}$
- JPD: $\mathbf{p} = \{p_{00}, p_{01}, p_{10}, p_{11}\}$
- no exclusivity or completeness, i.e., $p_{11} \neq 0$ and $p_{00} \neq 0$

Emergence of exclusivity and completeness from JQD

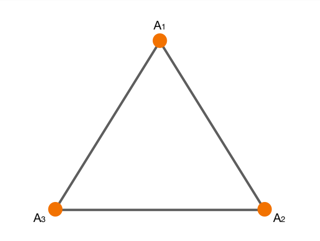
Example 4:

- consider three binary 0/1 random variables A, B and C
- set of possible measurement events: $\{000, 001, \ldots, 111\}$
- JQD: $\mathbf{q} = \{q_{000}, q_{001}, \ldots, q_{111}\}$

$$
\begin{aligned}
p_{00} &= q_{000} + q_{001} \geq 0, \\
p_{01} &= q_{010} + q_{011} \geq 0, \\
p_{10} &= q_{100} + q_{101} \geq 0, \\
p_{11} &= q_{110} + q_{111} \geq 0.
\end{aligned}
$$

- Exclusivity of A and B: $q_{111} = -q_{110}$
- Completeness of A and B: $q_{000} = -q_{001}$

Example – Specker's triangle



$$\sum_{i=1}^{3} p(A_i) \leq 1$$

Example 5: $p(A_i) = 1/2$

$$\begin{aligned} p_{00} &= q_{000} + q_{001} = 0, \\ p_{01} &= q_{010} + q_{011} = 1/2, \\ p_{10} &= q_{100} + q_{101} = 1/2, \\ p_{11} &= q_{110} + q_{111} = 0. \end{aligned}$$

$$\begin{aligned} q^{(ST)}_{010} &= q^{(ST)}_{100} = q^{(ST)}_{110} = q^{(ST)}_{001} = q^{(ST)}_{011} = q^{(ST)}_{101} = \frac{1}{4}, \\ q^{(ST)}_{000} &= q^{(ST)}_{111} = -\frac{1}{4}. \end{aligned}$$

JQD for KS sets (arXiv:2210.06822)

- consider a KS set of N events $\{A_1, A_2, \ldots, A_N\}$
- KS theorem $\Rightarrow$ $\{a_1, a_2, \ldots, a_N\}$ ($a_i \in \{0, 1\}$) not possible under exclusivity and completeness
- drop completeness and define outcome assignments

$$\omega_i \equiv \{\underbrace{0, \ldots, 0}_{i-1} 1 \underbrace{0, \ldots, 0}_{N-i}\}$$

$$\omega_0 \equiv \{\underbrace{0, \ldots, 0}_{N}\}$$

JQD for KS sets (arXiv:2210.06822)

- consider a scenario based on the KS set: $p_i \equiv p(A_i = 1) \geq 0$.
- important – for every complete measurement subset $\mathcal{C}$ we observe $\sum_{i \in \mathcal{C}} p_i = 1$
- e.g., in quantum theory A_i is a projector and $p_i \equiv \mathrm{T}r\{\rho A_i\}$
- assign probabilities

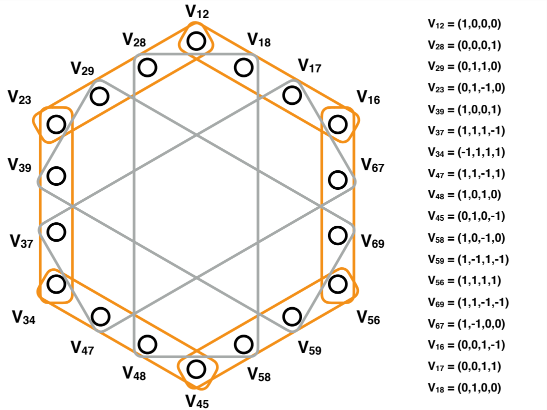
$$p(\omega_i) \equiv p_i, \qquad i \neq 0$$

- assign negative probability

$$p(\omega_0) \equiv p_0 \equiv 1 - \sum_{i=1}^{N} p_i < 0$$

JQD for KS sets – Example

Example 6: $\rho = \frac{I}{4}$



$$p_i = 1/4 \ (i \neq 0), \qquad p_0 = -14/4$$

Conclusions

- JQD construction applies to any KS set
- in fact it applies to any measurement set
- it also applies to a continuous set of measurements (Gleason)
- unified nonclassicality measure for both, S-D and S-I contextuality scenarios

Research supported by the Polish National Science Centre (NCN) under the Maestro Grant no.

DEC-2019/34/A/ST2/00081.