Simplicial quantum contextuality and applications

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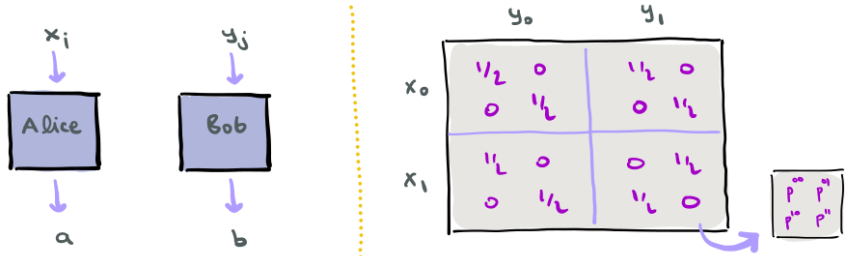
Quantum Contextuality in Quantum Mechanics and Beyond 2022

Outline

- Introduction to simplicial distributions
- Mermin polytopes
- Fine's theorem
- Λ-polytopes
- Further directions and connections

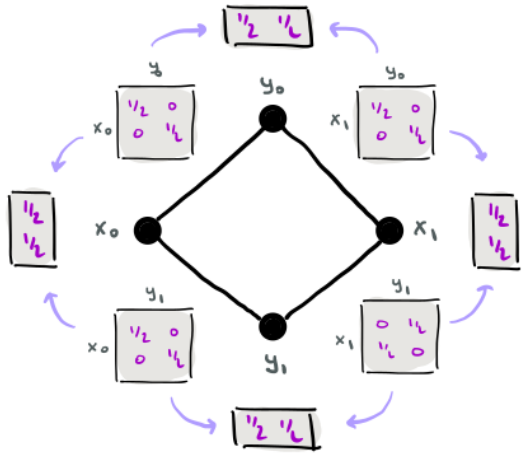
Simplicial distributions generalize nonsignaling distributions.



Nonsignaling conditions:

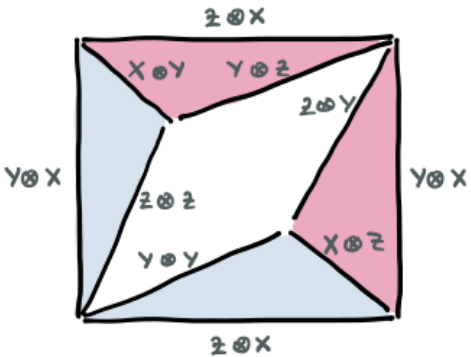
$$p^a_{x_i} = \sum_b p^{ab}_{x_i y_0} = \sum_b p^{ab}_{x_i y_1} \qquad p^b_{y_j} = \sum_a p^{ab}_{x_0 y_j} = \sum_a p^{ab}_{x_1 y_j}.$$

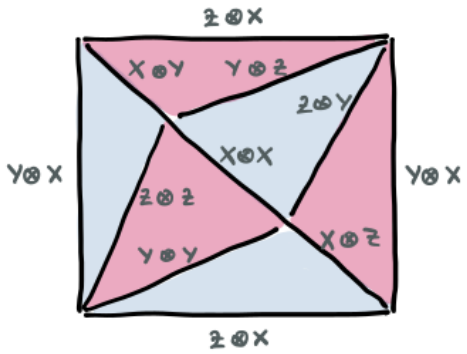
Nonsignaling conditions can be formulated using sheaves[1].



[1] S. Abramsky, A. Brandenburger, New Journal of Physics 13, no. 11 (2011)

Topological approach[2] organizes contexts into triangles and encodes the products $ABC = \pm\mathbb{1}$ as a cohomology class $[\beta]$.





[2] O., S. Roberts, S. D. Bartlett, R. Raussendorf, Quantum Information and Computation 17, 1135-1166 (2017)

Quick comparison of the two approaches

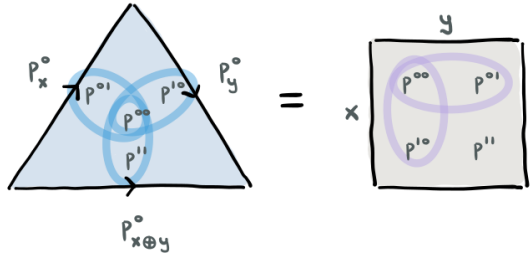
Sheaves: Measurements are represented as 0-dimensional simplices (points).

General enough to describe contextuality on the level of probabilities.

Topology: Measurements are represented as 1-dimensional simplices (edges).

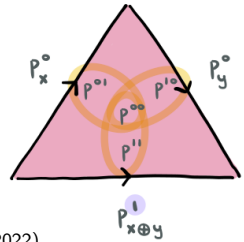
Can describe stronger versions of contextuality by encoding algebraic relations.

Simplicial distributions[3]



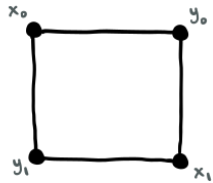
A simplicial distribution can be defined for an arbitrary **space** of measurements and outcomes.

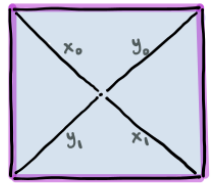
We can also allow for $\beta = 1$

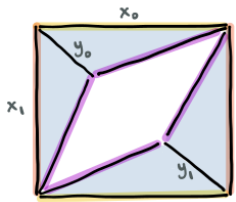


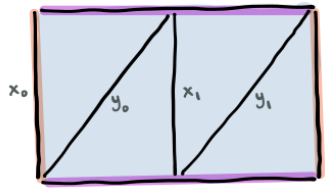
[3] O., A. Kharoof, S. Ipek, arXiv:2204.06648 (2022)

CHSH scenario

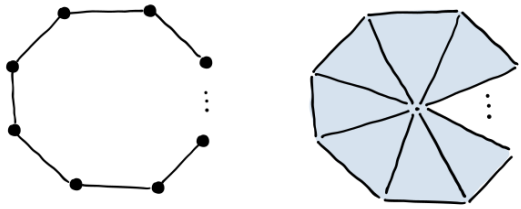




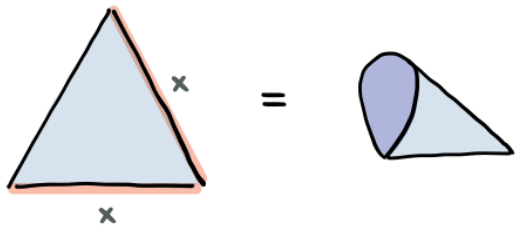




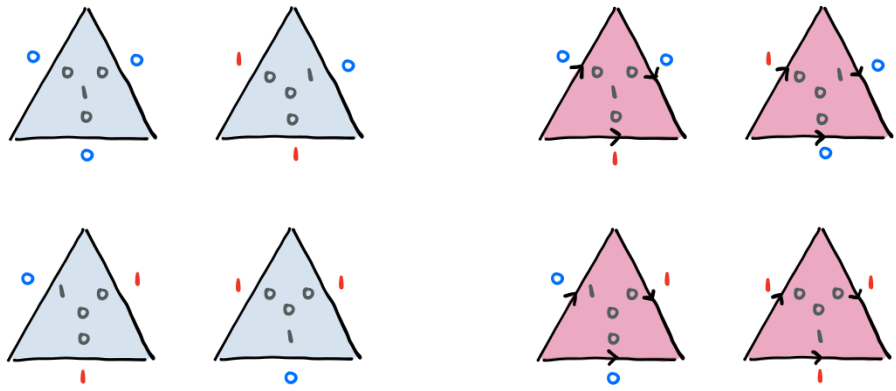
n-cycle scenario



We can also take $n = 1$

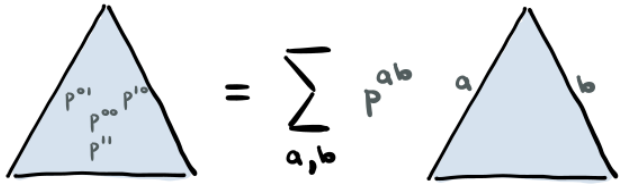


Deterministic distributions

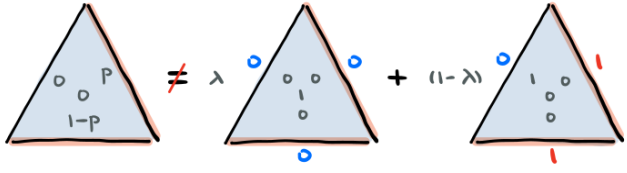


A simplicial distribution is called **noncontextual** if it can be expressed as a probabilistic mixture of deterministic distributions; otherwise it is called **contextual**.

Every n-simplex is noncontextual

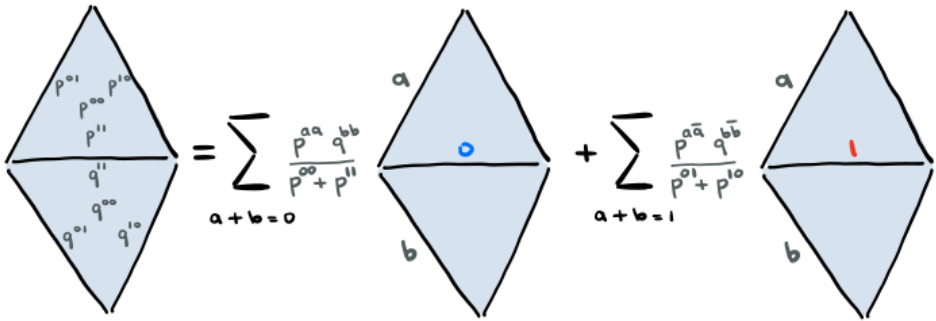


1-cycle scenario is contextual



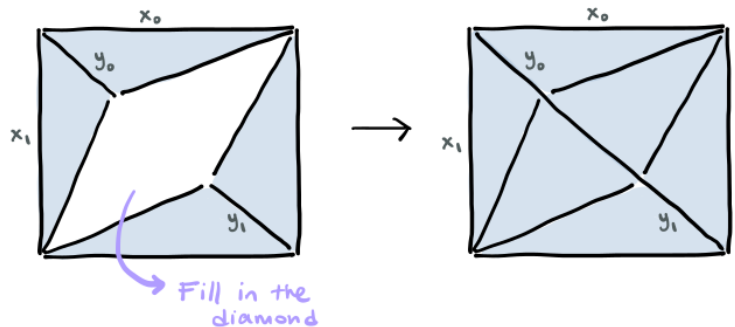
Gluing distributions

Let the measurement space X be obtained by gluing A and B along an n-simplex. A distribution p on X is noncontextual if and only if $p|_A$ and $p|_B$ are both noncontextual.

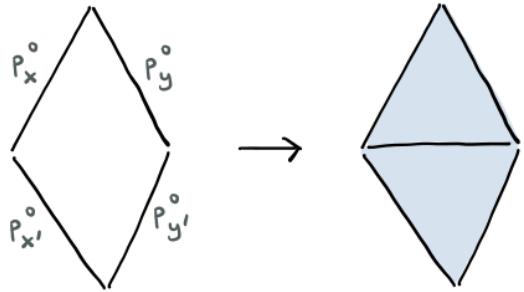


Extending distributions

Let $A \subset X$ be a subspace such that the deterministic distributions on A and X coincide. Assume that every distribution on X is noncontextual. Then a distribution p on A is noncontextual if and only if it extends to a distribution $\tilde{p}$ on X.



Extending to diamond



A distribution on the boundary of the diamond extends to the diamond if and only if it satisfies the CHSH inequalities

$$0 \leq p^0_x + p^0_y + p^0_{x'} - p^0_{y'} \leq 2$$

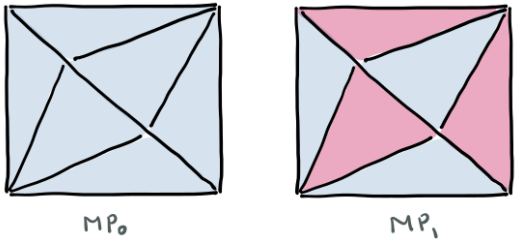
$$0 \leq p^0_x + p^0_y - p^0_{x'} + p^0_{y'} \leq 2$$

$$0 \leq p^0_x - p^0_y + p^0_{x'} + p^0_{y'} \leq 2$$

$$0 \leq -p^0_x + p^0_y + p^0_{x'} + p^0_{y'} \leq 2$$

Mermin polytope[4]

Let MP_β denote the polytope of simplicial distributions on the torus whose support respect β.



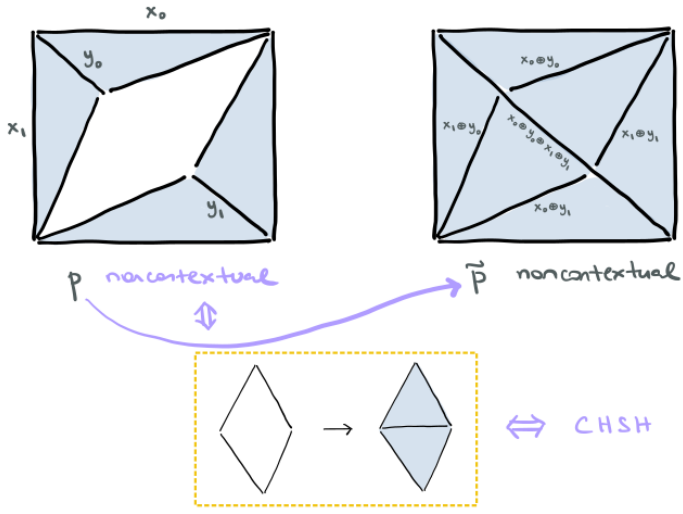
- Vertices of MP_0 are given by the deterministic distributions.
- Vertices of MP_1 are given by the maximal closed noncontextual (cnc) subsets[5].

[4] O., H. Chung, S. Ipek, arXiv:2210.10186 (2022)

[5] R. Raussendorf, J. Bermejo-Vega, E. Tyhurst, O., M. Zurel, Phys. Rev. A 101 (2020)

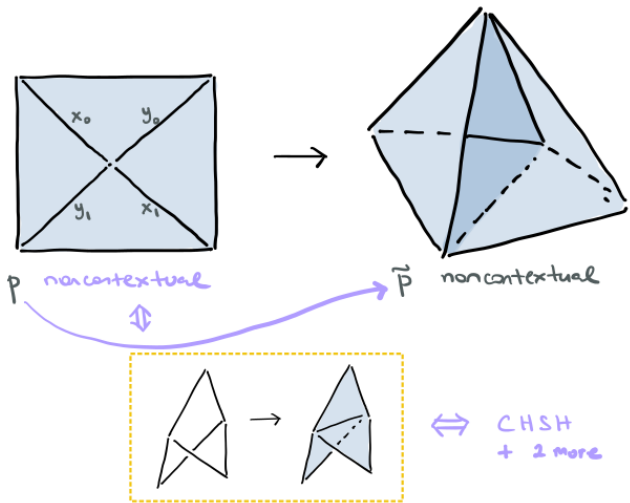
New proof of Fine's theorem[6]

A distribution on the CHSH scenario is noncontextual if and onlv if it satisfies the CHSH inequalities.

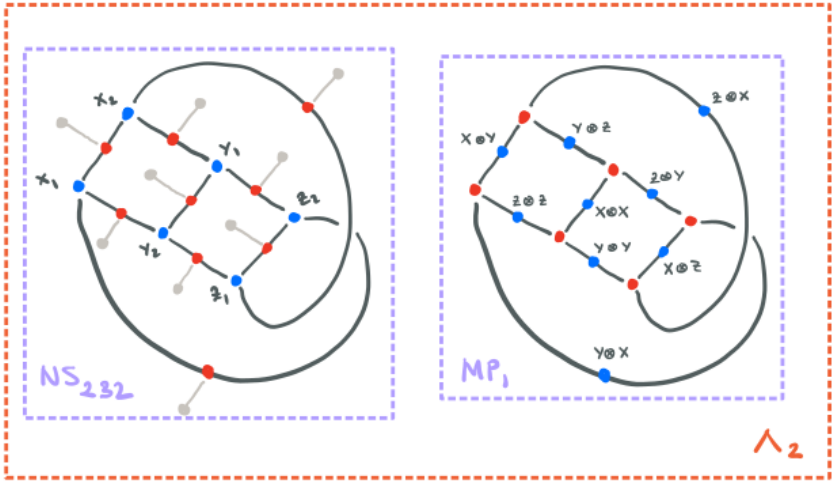


[6] A. Fine, Journal of Mathematical Physics, vol. 23, no. 7, pp. 1306-1310, (1982)

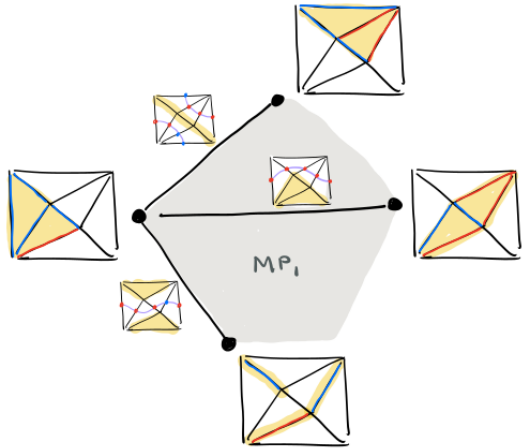
Fine's original argument is closer to



Decomposing Λ polytopes[7]



[7] M. Zurel, O., R. Raussendorf, Phys. Rev. Lett. 125 (2020)


MP1

Further directions and applications

- Monoid structure and simplicial homotopy[8]
- Bundle view and defining morphisms of simplicial distributions (joint with Rui Barbosa)
- Extension of Vorobev's theorem
- Extending cohomological framework for contextual MBQC[9]
- Causality and contextuality: causal sheaves[10]

[8] A. Kharoof, O., arXiv:2211.00571 (2022)
[9] R. Raussendorf, Quantum Information and Computation 19, 1141-1170 (2019)
[10] S. Gogioso, N. Pinzani, arXiv:2206.08911 (2022)

Thank you for your attention!