Causality and Contextuality II

QCQMB Prague Workshop, December 2022

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December 17, 2022

1. Motivation
2. Recap
3. Mapping Measurement Scenarios
4. Transporting Results from Flat Scenarios to Other Scenarios

Motivation

 = Agent

 = Measurement

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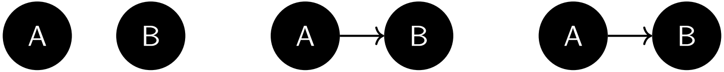
No-signalling | Signalling | Adaptive



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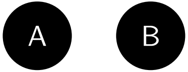

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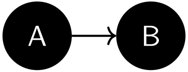

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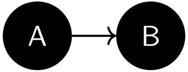
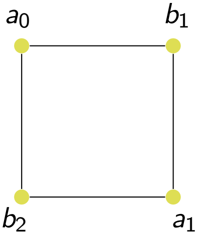
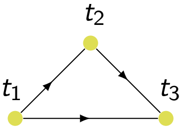






Motivation

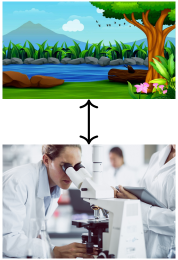
The Research Question

Can we introduce a framework general enough to capture dependency between measurements (via relaxations on no-signalling), adaptivity protocols *and* retain the generality that contextuality has over non-locality?

Contextuality as a two player game

E	N	E	N
$a_0?$	$(a_0, 0)$	$b_0?$	$(b_0, 0)$

Table: Contextuality as a two player game



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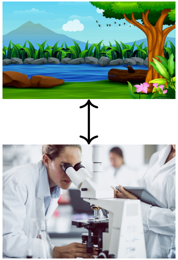

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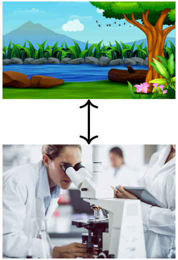

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There are also rules about how a non-contextual N can respond:

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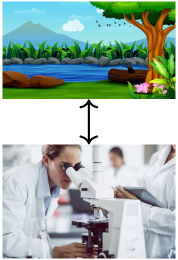

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Moral: There are noncontextual N- and E-strategies

Global Sections and Hidden Variables

Recall:

Global sections of the sheaf are in one-to-one correspondence with deterministic hidden variables for the system (Abramsky and Brandenburger, 2011)

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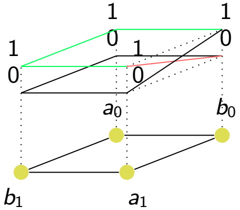

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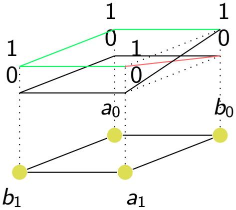

Contextuality is about abscence of *global* sections.

Histories for a Measurement Scenario

Histories are sets of measurement events in which

1. All measurements are consistent
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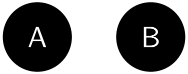

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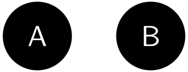

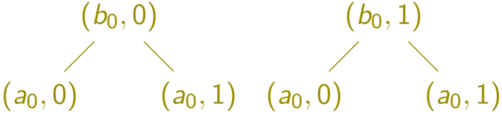

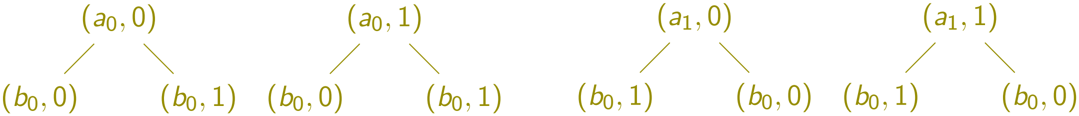

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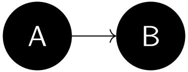

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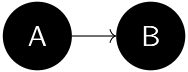

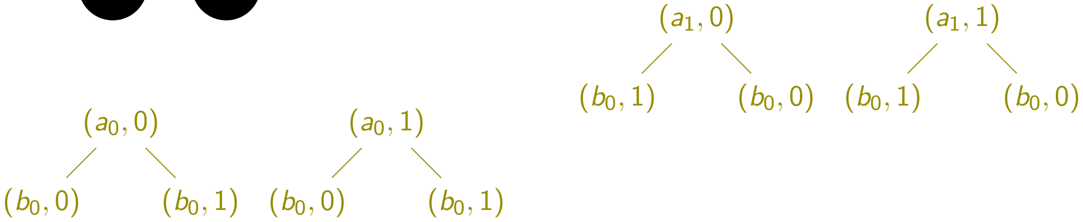

Nature strategies for flat scenarios

The measurement scenario is $(M, \mathcal{C})$ where $M = (X, O, \vdash)$ with

1. $X = \{a_1, a_2, b_1, b_2\}$
2. $\forall i \in X . O_i = \{0, 1\}$
3. $\forall i \in X . \emptyset \vdash i$
4. $\mathcal{C} = \{\emptyset, \{a_0, b_0\}, \{a_0, b_1\}, \{a_1, b_0\}, \{a_1, b_1\}\}$

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Strategy σ over $\{a_0, a_1, b_0\}$

$$(a_0, 0) \qquad (a_1, 0)$$

$$(b_0, 0)$$

$$\begin{tikzcd}[column sep=tiny] & (a_0,0) \arrow[dl, no head] \arrow[dr, no head] & \\ (b_0,0) & & (b_0,1) \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny] & (a_1,0) \arrow[dl, no head] \arrow[dr, no head] & \\ (b_0,0) & & (b_0,1) \end{tikzcd}$$

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Example 2: GP scenarios

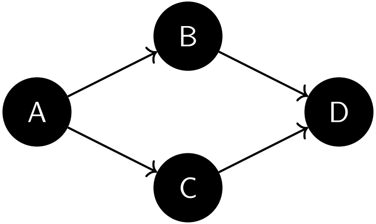

Figure: Example setup from Gogioso and Pinzani, 2021

A GP measurement scenario is a triple $\langle \Omega, \underline{I}, \underline{O} \rangle$ where

1. Ω is a set of agents
2. $\underline{I}$ consists of a set of inputs I_ω for each agent $\omega \in \Omega$
3. $\underline{O}$ consists of a set of outputs O_ω for each agent $\omega \in \Omega$

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Strategy σ over $\{a_0, a_1, b_0\}$

$(a_0, 0)$ $(a_1, 0)$

- $(a_0, 0)$
  - $(b_0, 0)$
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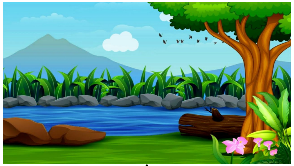
Experimenter Strategies



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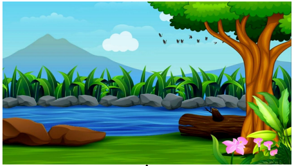

Strategy τ

$(a_0, 0)$ $(a_0, 1)$

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Playing off Strategies

- N- strategies branch at measurements
- E- strategies branch at outcomes

N-strategy σ over $\{a_0, b_0\}$:

$$(a_0, 0)$$
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N-strategy σ over $\{a_1, b_0\}$:

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E-strategy τ over $\{a_i, b_0\}$:

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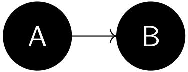
The result is a deterministic sequence $\langle \sigma || \tau \rangle = (a_0 = 0)(b_0 = 0)$ over $\{a_0, b_0\}$ and $\langle \sigma || \tau \rangle = (a_1 = 0)(b_0 = 1)$ over $\{a_1, b_0\}$.

Non-contextuality of signalling Alice Bob

Recall the setup:



Proposition

Any empirical model on this setup is noncontextual.

Non-contextuality of signalling Alice Bob

Recall the setup:

$$A \longrightarrow B$$

Proposition

Any empirical model on this setup is noncontextual.

How can we generalise this statement?

Mapping of Empirical Models

1. Given a GP scenario $\mathcal{M} = \langle X, \underline{I}, \underline{O} \rangle$ we can define a scenario $\mathcal{M}' = \langle (X', \vdash, O'), \mathcal{C} \rangle$.
2. Only a subset of empirical models on $\mathcal{M}'$ will arise from empirical models on $\mathcal{M}$.

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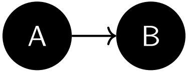
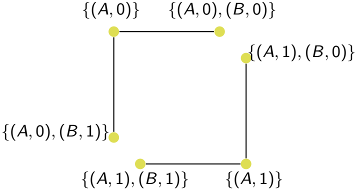


$$\xrightarrow{\text{preserves noncontextuality}}$$



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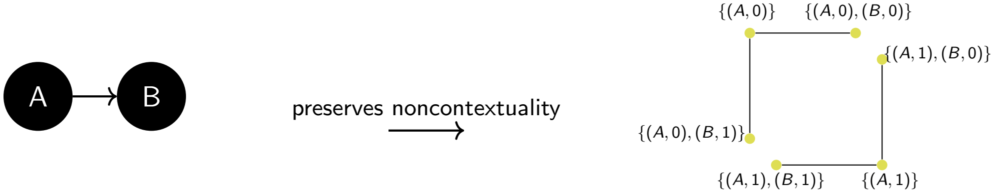

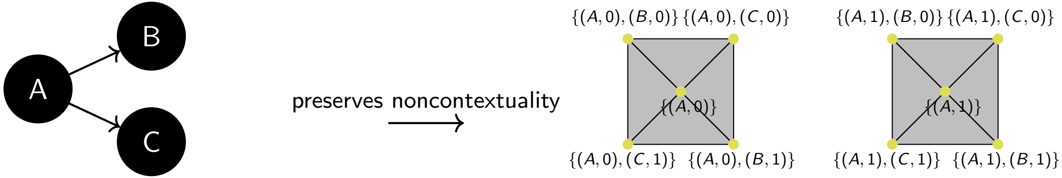

A small detour: Vorob'ev's Theorem

Recipe for attaching data to a space:

1. Define the values each vertex can take on
2. For each face of the simplicial complex, define a probability distribution on mappings on that face.
3. Ensure marginalisation of probability distributions agree on overlap

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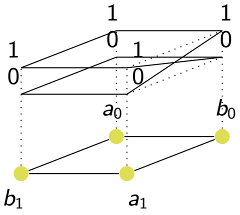

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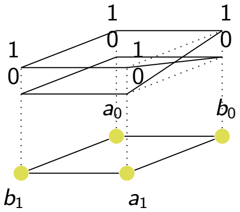

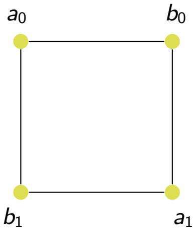

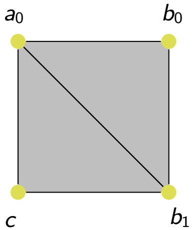

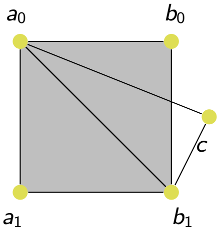

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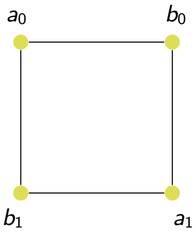

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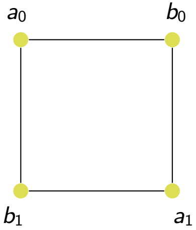
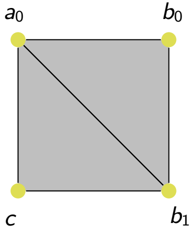


Acyclic



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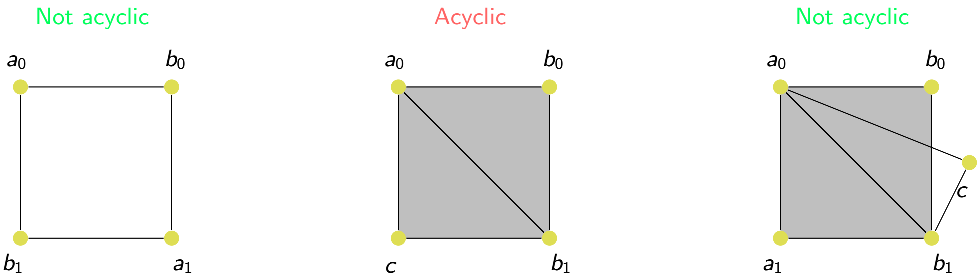

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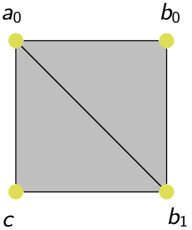

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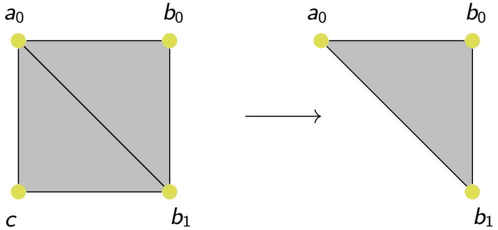

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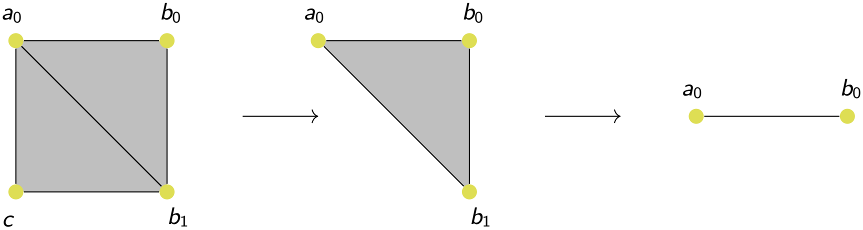
A simplicial complex is acyclic if it can be Graham reduced to the simplicial complex which is just a single vertex.



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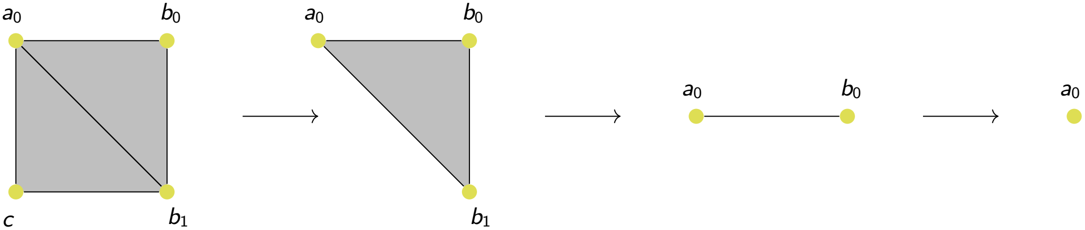
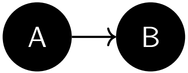
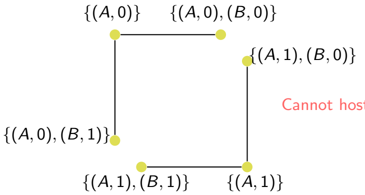


Applying Vorob'ev to GP Scenarios



preserves noncontextuality $\longrightarrow$



Cannot host contextuality

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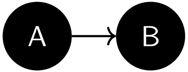

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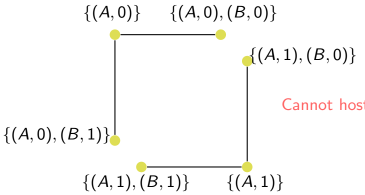

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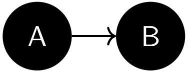

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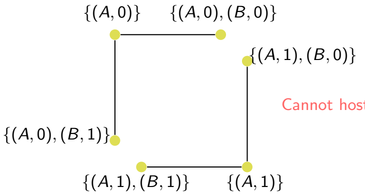

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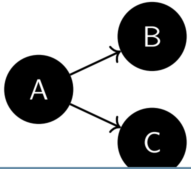

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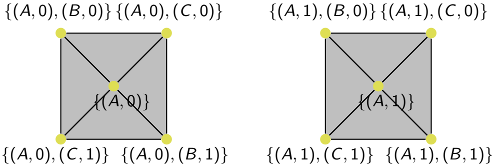
Can host contextuality



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2. Framing contextuality setups in this way makes conversions between different types of measurement scenarios possible.
3. Allowing for N-strategies which can 'see' all measurements which occurred previously (the entire measurement history) seems to make it easier to classically simulate empirical data.

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5. Memory costs (and possible relation to Sivert's work on shallow circuits)

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