

# Impossibility theorem for extending contextuality to disturbing systems

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# Plan

Structure of the work

Basic framework

KS-compatibility

Monotonicity

    Nestedness

    Coarsening

    Relabeling

Post-processing

    Joining

Independence

Determinism

    Determinism

Main results

Discussion

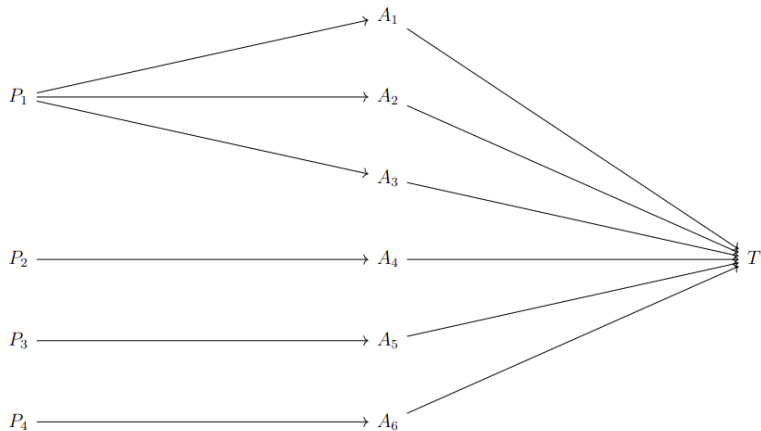
Appendix

# Structure

Principles

Axioms

Theorem



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# Basic framework

## Measurement scenario

# Basic framework

## Measurement scenario

measurements

$Q$

$q$

# Basic framework

## Measurement scenario

measurements

$\mathcal{Q}$

$q$

Outcomes

$O_q$

$a$

# Basic framework

## Measurement scenario

measurements

 $Q$  $q$ 

Outcomes

 $O_q$  $a$ 

Contexts

 $\mathcal{C}$  $c$



# Basic framework

## Measurement scenario

measurements

 $\mathcal{Q}$  $q$ 

Outcomes

 $O_q$  $a$ 

Contexts

 $\mathcal{C}$  $c$  $q \prec c$

# Basic framework

## Measurement scenario

measurements

 $Q$  $q$ 

Outcomes

 $O_q$  $a$ 

Contexts

 $\mathcal{C}$  $c$  $q \prec c$  $O^c \doteq \prod_{q \prec c} O_q.$

# Basic framework

## Measurement scenario

measurements

 $\mathcal{Q}$  $q$ 

Outcomes

 $O_q$  $a$ 

Contexts

 $\mathcal{C}$  $c$  $q \prec c$  $O^c \doteq \prod_{q \prec c} O_q.$ 

Measurement scenario

 $\mathcal{S} \equiv (\mathcal{Q}, \mathcal{C}, \prec, \mathcal{O})$

# Basic framework

## Behavior

# Basic framework

## Behavior

Context

*c*

# Basic framework

## Behavior

Context

$c$

Distribution

$$P(\cdot | c) : \mathcal{O}^c \rightarrow [0, 1]$$

# Basic framework

## Behavior

Context

$c$

Distribution

$$P(\cdot | c) : \mathcal{O}^c \rightarrow [0, 1]$$

Behavior

$$P \equiv (P(\cdot | c))_{c \in \mathcal{C}}$$

# Basic framework

## Marginalization



# Basic framework

## Marginalization

### Composite measurement

$$\mathbf{q} \equiv \{q_1, \dots, q_m\}$$
$$\mathbf{q} \prec c \Leftrightarrow \forall q \in \mathbf{q} : q \prec c$$

# Basic framework

## Marginalization

### Composite measurement

$$\mathbf{q} \equiv \{q_1, \dots, q_m\}$$
$$\mathbf{q} \prec c \Leftrightarrow \forall q \in \mathbf{q} : q \prec c$$

### Marginal distribution

$$P(\cdot | \mathbf{q}, c) : O_{\mathbf{q}} \rightarrow [0, 1]$$

# Basic framework

## Marginalization

Composite measurement

$$\mathbf{q} \equiv \{q_1, \dots, q_m\}$$
$$\mathbf{q} \prec c \Leftrightarrow \forall q \in \mathbf{q} : q \prec c$$

Marginal distribution

$$P(\cdot | \mathbf{q}, c) : O_{\mathbf{q}} \rightarrow [0, 1]$$

Single measurement

$$q \prec c$$

# Basic framework

## Marginalization

Composite measurement

$$\mathbf{q} \equiv \{q_1, \dots, q_m\}$$
$$\mathbf{q} \prec c \Leftrightarrow \forall q \in \mathbf{q} : q \prec c$$

Single measurement

$$q \prec c$$

Marginal distribution

$$P(\cdot | \mathbf{q}, c) : O_{\mathbf{q}} \rightarrow [0, 1]$$

Marginal distribution

$$P(\cdot | q, c) : O_q \rightarrow [0, 1]$$

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## KS-compatibility

Any extension of contextuality agrees with the standard definition whenever the latter applies

# KS-Compatibility

## KS-Compatibility

### KS-compatibility

A nondisturbing behavior is noncontextual if and only if it is KS-noncontextual

# Plan

Structure of the work

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## Monotonicity

Ignoring information in a noncontextual behavior cannot make it contextual

## Nestedness

Marginalizations of noncontextual behaviors are noncontextual

# Monotonicity

## Nestedness

# Monotonicity

## Nestedness

Behavior

$P$

# Monotonicity

## Nestedness

Behavior

$P$

---

- ◉ Remove some measurements

$$\mathcal{Q}' \subset \mathcal{Q}$$

# Monotonicity

## Nestedness

Behavior

$P$

- ◉ Remove some measurements
- ◉ Remove some contexts

$$\mathcal{Q}' \subset \mathcal{Q}$$

$$\mathcal{C}' \subset \mathcal{C}$$

# Monotonicity

## Nestedness

### Behavior

$P$

- Remove some measurements
- Remove some contexts
- Update  $\prec$

$$\mathcal{Q}' \subset \mathcal{Q}$$

$$\mathcal{C}' \subset \mathcal{C}$$

$$q \prec' c \text{ iff } q \prec c$$

# Monotonicity

## Nestedness

### Behavior

$P$

- Remove some measurements
- Remove some contexts
- Update  $\prec$
- Marginalize  $P$

$$\mathcal{Q}' \subset \mathcal{Q}$$

$$\mathcal{C}' \subset \mathcal{C}$$

$$q \prec' c \text{ iff } q \prec c$$



# Monotonicity

## Nestedness

Behavior

$P$

---

- Remove some measurements
- Remove some contexts
- Update  $\prec$
- Marginalize  $P$

$$\mathcal{Q}' \subset \mathcal{Q}$$

$$\mathcal{C}' \subset \mathcal{C}$$

$$q \prec' c \text{ iff } q \prec c$$

$P'$

---

# Monotonicity

## Nestedness

Behavior

$P$

Status

$\perp$

- Remove some measurements
- Remove some contexts
- Update  $\prec$
- Marginalize  $P$

$$\mathcal{Q}' \subset \mathcal{Q}$$

$$\mathcal{C}' \subset \mathcal{C}$$

$$q \prec' c \text{ iff } q \prec c$$

$P'$

# Monotonicity

## Nestedness

Behavior

$P$

Status

$\perp$

- Remove some measurements
- Remove some contexts
- Update  $\prec$
- Marginalize  $P$

$\mathcal{Q}' \subset \mathcal{Q}$

$\mathcal{C}' \subset \mathcal{C}$

$q \prec' c$  iff  $q \prec c$

$P'$

$\perp$

# Monotonicity

## Existing extensions

Axiom	
Extension	Nestedness
CbD 1.0	-
CbD 2.0	+
B-CbD	+
CB-CbD	+
$D \Rightarrow C$	+
$D \Rightarrow \neg C$	-
$D + CC \Rightarrow C$	-

## Coarsening

Any coarsening of a noncontextual behavior is also noncontextual

# Monotonicity

## Coarse-graining

# Monotonicity

## Coarse-graining

Measurement

$q$

# Monotonicity

## Coarse-graining

Measurement

$q$

Function

$$g : O_q \rightarrow O_{q'}$$



# Monotonicity

## Coarse-graining

Measurement

$q$

Function

$g : O_q \rightarrow O_{q'}$

Coarse-graining

$g(q) \equiv (q, g)$

# Monotonicity

## Coarsening

# Monotonicity

## Coarsening

Behavior

$P$

# Monotonicity

## Coarsening

Behavior

$P$

- Replace  $q$  with  $g(q)$

$$Q' \doteq Q \setminus \{q\} \cup \{g(q)\}$$

# Monotonicity

## Coarsening

### Behavior

$P$

- ◉ Replace  $q$  with  $g(q)$
- ◉ Update  $\prec$

$$\begin{aligned} Q' &\doteq Q \setminus \{q\} \cup \{g(q)\} \\ g(q) &\prec' c \text{ iff } q \prec c \end{aligned}$$

# Monotonicity

## Coarsening

### Behavior

$P$

⊙ Replace  $q$  with  $g(q)$

⊙ Update  $\prec$

⊙ Update  $P$

$$Q' \doteq Q \setminus \{q\} \cup \{g(q)\}$$

$$g(q) \prec' c \text{ iff } q \prec c$$

$$\forall c \succ q : P'(\cdot | g(q), c) \doteq P(g^{-1}(\cdot) | q, c)$$

# Monotonicity

## Coarsening

### Behavior

$P$

---

- ⊙ Replace  $q$  with  $g(q)$
- ⊙ Update  $\prec$
- ⊙ Update  $P$

$$\mathcal{Q}' \doteq \mathcal{Q} \setminus \{q\} \cup \{g(q)\}$$

$$g(q) \prec' c \text{ iff } q \prec c$$

$$\forall_{c \succ q} : P'(\cdot | g(q), c) \doteq P(g^{-1}(\cdot) | q, c)$$

---

$P'$

# Monotonicity

## Coarsening

Behavior

$P$

Status

$\perp$

- ◉ Replace  $q$  with  $g(q)$
- ◉ Update  $\prec$
- ◉ Update  $P$

$$Q' \doteq Q \setminus \{q\} \cup \{g(q)\}$$

$$g(q) \prec' c \text{ iff } q \prec c$$

$$\forall_{c \succ q} : P'(\cdot | g(q), c) \doteq P(g^{-1}(\cdot) | q, c)$$

$P'$



# Monotonicity

## Coarsening

Behavior

$P$

Status

$\perp$

⊙ Replace  $q$  with  $g(q)$

⊙ Update  $\prec$

⊙ Update  $P$

$Q' \doteq Q \setminus \{q\} \cup \{g(q)\}$

$g(q) \prec' c$  iff  $q \prec c$

$\forall_{c \succ q} : P'(\cdot | g(q), c) \doteq P(g^{-1}(\cdot) | q, c)$

$P'$

$\perp$

# Monotonicity

## Existing extensions

Extension	Axiom	
	N	Coarsening
CbD 1.0	-	-
CbD 2.0	+	-
B-CbD	+	+
CB-CbD	+	+
$D \Rightarrow C$	+	+
$D \Rightarrow \neg C$	-	-
$D + CC \Rightarrow C$	-	-

## Relabeling

Labeling a measurement differently in different contexts does not create contextuality

# Monotonicity

## Relabeling

# Monotonicity

## Relabeling

Behavior

$P$

# Monotonicity

## Relabeling

Behavior

$P$

---

- ⊙ Take  $q \prec c_1, c_2, c_3, c_4$

# Monotonicity

## Relabeling

Behavior

$P$

- ⊙ Take  $q \prec c_1, c_2, c_3, c_4$
- ⊙ Replace  $q$  with  $q_{1,2}$  and  $q_{3,4}$

$$\mathcal{Q}' \doteq \mathcal{Q} \setminus \{q\} \cup \{q_{1,2}, q_{3,4}\}$$

# Monotonicity

## Relabeling

### Behavior

$P$

- Take  $q \prec c_1, c_2, c_3, c_4$
- Replace  $q$  with  $q_{1,2}$  and  $q_{3,4}$
- Update  $\prec$

$$Q' \doteq Q \setminus \{q\} \cup \{q_{1,2}, q_{3,4}\}$$

$$q_{i,j} \prec c_i, c_j,$$



# Monotonicity

## Relabeling

### Behavior

$P$

---

- Take  $q \prec c_1, c_2, c_3, c_4$
  - Replace  $q$  with  $q_{1,2}$  and  $q_{3,4}$        $Q' \doteq Q \setminus \{q\} \cup \{q_{1,2}, q_{3,4}\}$
  - Update  $\prec$        $q_{i,j} \prec c_i, c_j,$
  - Relabel  $P$        $\forall_i \forall_{c \in \mathbf{c}_i} : P'(\cdot | q_{i,j}, c_{i,j}) \doteq P(\cdot | q, c)$
-

# Monotonicity

## Relabeling

### Behavior

$P$

---

- Take  $q \prec c_1, c_2, c_3, c_4$
  - Replace  $q$  with  $q_{1,2}$  and  $q_{3,4}$        $\mathcal{Q}' \doteq \mathcal{Q} \setminus \{q\} \cup \{q_{1,2}, q_{3,4}\}$
  - Update  $\prec$        $q_{i,j} \prec c_i, c_j,$
  - Relabel  $P$        $\forall_i \forall_{c \in \mathbf{c}_i} : P'(\cdot | q_{i,j}, c_{i,j}) \doteq P(\cdot | q, c)$
- 

$P'$

# Monotonicity

## Relabeling

Behavior

$P$

Status

$\perp$

- Take  $q \prec c_1, c_2, c_3, c_4$
- Replace  $q$  with  $q_{1,2}$  and  $q_{3,4}$        $\mathcal{Q}' \doteq \mathcal{Q} \setminus \{q\} \cup \{q_{1,2}, q_{3,4}\}$
- Update  $\prec$        $q_{i,j} \prec c_i, c_j,$
- Relabel  $P$        $\forall_i \forall_{c \in \mathbf{c}_i} : P'(\cdot | q_{i,j}, c_{i,j}) \doteq P(\cdot | q, c)$

$P'$

# Monotonicity

## Relabeling

Behavior

$P$

Status

$\perp$

⊙ Take  $q \prec c_1, c_2, c_3, c_4$

⊙ Replace  $q$  with  $q_{1,2}$  and  $q_{3,4}$

⊙ Update  $\prec$

⊙ Relabel  $P$

$\mathcal{Q}' \doteq \mathcal{Q} \setminus \{q\} \cup \{q_{1,2}, q_{3,4}\}$

$q_{i,j} \prec c_i, c_j,$

$\forall_i \forall_{c \in \mathbf{c}_i} : P'(\cdot | q_{i,j}, c_{i,j}) \doteq P(\cdot | q, c)$

$P'$

$\perp$

# Monotonicity

## Existing extensions

Extension	Axiom		
	N	C	Relabeling
CbD 1.0	-	-	-
CbD 2.0	+	-	+
B-CbD	+	+	+
CB-CbD	+	+	+
$D \Rightarrow C$	+	+	+
$D \Rightarrow \neg C$	-	-	-
$D + CC \Rightarrow C$	-	-	-

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## Post-processing

Classical post-processing of the output of a noncontextual behavior cannot result in a contextual behavior

## Joining

Taking composite measurements into account does not turn a noncontextual behavior into a contextual one



# Monotonicity

## Joining

# Monotonicity

## Joining

Behavior

$P$

# Monotonicity

## Joining

Behavior

$P$

---

- ⊙ Take  $\mathbf{q} \equiv \{q_1, q_2\}$

# Monotonicity

## Joining

### Behavior

$P$

---

- ⊙ Take  $\mathbf{q} \equiv \{q_1, q_2\}$
- ⊙ Define  $q' \doteq (q_1, q_2)$

$$O_{q'} \doteq O_{q_1} \times O_{q_2}$$

# Monotonicity

## Joining

### Behavior

$P$

---

- ⊙ Take  $\mathbf{q} \equiv \{q_1, q_2\}$
- ⊙ Define  $q' \doteq (q_1, q_2)$
- ⊙ Include  $(q_1, q_2)$

$$O_{q'} \doteq O_{q_1} \times O_{q_2}$$
$$\mathcal{Q}' \doteq \mathcal{Q} \cup \{(q_1, q_2)\}$$

# Monotonicity

## Joining

### Behavior

$P$

- Take  $\mathbf{q} \equiv \{q_1, q_2\}$
- Define  $q' \doteq (q_1, q_2)$
- Include  $(q_1, q_2)$
- Update  $\prec$

$$O_{q'} \doteq O_{q_1} \times O_{q_2}$$

$$\mathcal{Q}' \doteq \mathcal{Q} \cup \{(q_1, q_2)\}$$

$$(q_1, q_2) \prec' c \Leftrightarrow \{q_1, q_2\} \subset c$$

# Monotonicity

## Joining

### Behavior

$P$

- Take  $\mathbf{q} \equiv \{q_1, q_2\}$
- Define  $q' \doteq (q_1, q_2)$   $O_{q'} \doteq O_{q_1} \times O_{q_2}$
- Include  $(q_1, q_2)$   $\mathcal{Q}' \doteq \mathcal{Q} \cup \{(q_1, q_2)\}$
- Update  $\prec$   $(q_1, q_2) \prec' c \Leftrightarrow \{q_1, q_2\} \subset c$
- Update  $P$   $\forall c \succ_{q_1, q_2} : "P'((q_1, q_2) = \{q_1, q_2\}, c) = 1."$

# Monotonicity

## Joining

### Behavior

$P$

- Take  $\mathbf{q} \equiv \{q_1, q_2\}$
  - Define  $q' \doteq (q_1, q_2)$
  - Include  $(q_1, q_2)$
  - Update  $\prec$
  - Update  $P$
- $$O_{q'} \doteq O_{q_1} \times O_{q_2}$$
- $$\mathcal{Q}' \doteq \mathcal{Q} \cup \{(q_1, q_2)\}$$
- $$(q_1, q_2) \prec' c \Leftrightarrow \{q_1, q_2\} \subset c$$
- $$\forall c \succ_{q_1, q_2} : "P'((q_1, q_2) = \{q_1, q_2\}, c) = 1."$$

$P'$



# Monotonicity

## Joining

Behavior

Status

$P$

$\perp$

⊙ Take  $\mathbf{q} \equiv \{q_1, q_2\}$

⊙ Define  $q' \doteq (q_1, q_2)$

⊙ Include  $(q_1, q_2)$

⊙ Update  $\prec$

⊙ Update  $P$

$O_{q'} \doteq O_{q_1} \times O_{q_2}$

$\mathcal{Q}' \doteq \mathcal{Q} \cup \{(q_1, q_2)\}$

$(q_1, q_2) \prec' c \Leftrightarrow \{q_1, q_2\} \subset c$

$\forall c \succ_{q_1, q_2} : "P'((q_1, q_2) = \{q_1, q_2\}, c) = 1."$

$P'$

# Monotonicity

## Joining

Behavior

Status

$P$

$\perp$

⊙ Take  $\mathbf{q} \equiv \{q_1, q_2\}$

⊙ Define  $q' \doteq (q_1, q_2)$

⊙ Include  $(q_1, q_2)$

⊙ Update  $\prec$

⊙ Update  $P$

$O_{q'} \doteq O_{q_1} \times O_{q_2}$

$\mathcal{Q}' \doteq \mathcal{Q} \cup \{(q_1, q_2)\}$

$(q_1, q_2) \prec' c \Leftrightarrow \{q_1, q_2\} \subset c$

$\forall c \succ_{q_1, q_2} : "P'((q_1, q_2) = \{q_1, q_2\}, c) = 1."$

$P'$

$\perp$

# Post-processing

## Existing extensions

Extension	Axiom			
	N	C	R	Joining
CbD 1.0	-	-	-	-
CbD 2.0	+	-	+	-
B-CbD	+	+	+	$\emptyset$
CB-CbD	+	+	+	$\emptyset$
$D \Rightarrow C$	+	+	+	+
$D \Rightarrow \neg C$	-	-	-	+
$D + CC \Rightarrow C$	-	-	-	+

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## Independence

The joint realization of two statistically independent noncontextual systems is noncontextual

## Independence

The joint realization of two statistically independent noncontextual systems is noncontextual

# Independence

## Product scenario

# Independence

## Product scenario

Scenario 1

$$\mathcal{S}_1 \equiv (\mathcal{Q}_1, \mathcal{C}_1, \prec_1, \mathcal{O}_1)$$

Scenario 2

$$\mathcal{S}_2 \equiv (\mathcal{Q}_2, \mathcal{C}_2, \prec_2, \mathcal{O}_2)$$



# Independence

## Product scenario

Scenario 1

$$\mathcal{S}_1 \equiv (\mathcal{Q}_1, \mathcal{C}_1, \prec_1, \mathcal{O}_1)$$

Scenario 2

$$\mathcal{S}_2 \equiv (\mathcal{Q}_2, \mathcal{C}_2, \prec_2, \mathcal{O}_2)$$

Product scenario

# Independence

## Product scenario

Scenario 1

$$\mathcal{S}_1 \equiv (\mathcal{Q}_1, \mathcal{C}_1, \prec_1, \mathcal{O}_1)$$

Scenario 2

$$\mathcal{S}_2 \equiv (\mathcal{Q}_2, \mathcal{C}_2, \prec_2, \mathcal{O}_2)$$

Product scenario

Measurements

$$\mathcal{Q}_1 \cup \mathcal{Q}_2$$

# Independence

## Product scenario

Scenario 1

$$\mathcal{S}_1 \equiv (\mathcal{Q}_1, \mathcal{C}_1, \prec_1, \mathcal{O}_1)$$

Scenario 2

$$\mathcal{S}_2 \equiv (\mathcal{Q}_2, \mathcal{C}_2, \prec_2, \mathcal{O}_2)$$

Product scenario

Measurements

$$\mathcal{Q}_1 \cup \mathcal{Q}_2$$

Contexts

$$\mathcal{C}_1 \times \mathcal{C}_2$$

# Independence

## Product scenario

Scenario 1

$$\mathcal{S}_1 \equiv (\mathcal{Q}_1, \mathcal{C}_1, \prec_1, \mathcal{O}_1)$$

Scenario 2

$$\mathcal{S}_2 \equiv (\mathcal{Q}_2, \mathcal{C}_2, \prec_2, \mathcal{O}_2)$$

Product scenario

Measurements

$$\mathcal{Q}_1 \cup \mathcal{Q}_2$$

Contexts

$$\mathcal{C}_1 \times \mathcal{C}_2$$

Relation

$$q \prec (c_1, c_2) \Leftrightarrow \exists_k (q \prec_k c_k)$$

# Independence

## Product behavior

# Independence

## Product behavior

Behavior 1

$P_1$

Behavior 2

$P_2$

# Independence

## Product behavior

Behavior 1

 $P_1$ 

Behavior 2

 $P_2$ 

Product behavior

$$(P_1 \otimes P_2)((a_1, a_2)|(c_1, c_2)) = P_1(a_1|c_1)P_2(a_2|c_2)$$

# Independence

## Product behavior

Behavior 1

$$P_1 \\ \perp$$

Behavior 2

$$P_2 \\ \perp$$

Product behavior

$$(P_1 \otimes P_2)((a_1, a_2)|(c_1, c_2)) = P_1(a_1|c_1)P_2(a_2|c_2)$$



# Independence

## Product behavior

Behavior 1

 $P_1$  $\perp$ 

Behavior 2

 $P_2$  $\perp$ 

Product behavior

$$(P_1 \otimes P_2)((a_1, a_2)|(c_1, c_2)) = P_1(a_1|c_1)P_2(a_2|c_2)$$

 $\perp$

# Independence

## Existing extensions

Extension	Axiom				
	N	C	R	J	Independence
CbD 1.0	-	-	-	-	+
CbD 2.0	+	-	+	-	+
B-CbD	+	+	+	$\emptyset$	+
CB-CbD	+	+	+	$\emptyset$	-
$D \Rightarrow C$	+	+	+	+	+
$D \Rightarrow \neg C$	-	-	-	+	+
$D + CC \Rightarrow C$	-	-	-	+	+

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## Determinism

Deterministic behaviors are noncontextual

## Determinism

Deterministic behaviors are noncontextual

# Determinism

## Deterministic measurement

# Determinism

## Deterministic measurement

Measurement

$q$

# Determinism

## Deterministic measurement

Measurement

$q$

Behavior

$P$



# Determinism

## Deterministic measurement

Measurement

$q$

Behavior

$P$

Determinism

$$\forall c \exists u_c P(u_c | q, c) = 1$$

# Determinism

## Deterministic measurement

Measurement

$q$

Behavior

$P$

Determinism

$$\forall_c \exists_{u_c} P(u_c | q, c) = 1$$

Deterministic behavior

$$\forall_q \forall_c \exists_{u_c} P(u_c | q, c) = 1$$

# Determinism

## Determinism

### Status

Any deterministic behavior is noncontextual

# Determinism

## Existing extensions

Extension	Axiom					Determinism
	N	C	R	J	I	
CbD 1.0	-	-	-	-	+	+
CbD 2.0	+	-	+	-	+	+
B-CbD	+	+	+	$\emptyset$	+	+
CB-CbD	+	+	+	$\emptyset$	-	+
$D \Rightarrow C$	+	+	+	+	+	-
$D \Rightarrow \neg C$	-	-	-	+	+	+
$D + CC \Rightarrow C$	-	-	-	+	+	+

# Plan

Structure of the work

Basic framework

KS-compatibility

Monotonicity

    Nestedness

    Coarsening

    Relabeling

Post-processing

    Joining

Independence

Determinism

    Determinism

Main results

Discussion

Appendix

# Main results

## KS-contextuality

### Proposition 1

KS-contextuality satisfies all these axioms

# Main results

## Impossibility theorems

### Theorem 1

No extension of contextuality can simultaneously satisfy:

# Main results

## Impossibility theorems

### Theorem 1

No extension of contextuality can simultaneously satisfy:

- Nestedness
- Coarsening
- Relabeling



# Main results

## Impossibility theorems

### Theorem 1

No extension of contextuality can simultaneously satisfy:

- Nestedness
- Coarsening
- Relabeling
- Joining

# Main results

## Impossibility theorems

### Theorem 1

No extension of contextuality can simultaneously satisfy:

- Nestedness
- Coarsening
- Relabeling
- Joining
- Determinism

# Main results

## Impossibility theorems

### Theorem 1

No extension of contextuality can simultaneously satisfy:

- ◉ Nestedness
- ◉ Coarsening
- ◉ Relabeling
- ◉ Joining
- ◉ Determinism
- ◉ Independence

# Main results

## Proof

# Main results

## Proof

◉  $P_1$  (one measurement  $q_1$ )

KS-compatibility

# Main results

## Proof

- $P_1$  (one measurement  $q_1$ )
- $P_2$  (one measurement  $q_2$ )

KS-compatibility

Determinism

# Main results

## Proof

- $P_1$  (one measurement  $q_1$ )
- $P_2$  (one measurement  $q_2$ )
- $P_1 \otimes P_2$

KS-compatibility

Determinism

Independence

# Main results

## Proof

- $P_1$  (one measurement  $q_1$ )
- $P_2$  (one measurement  $q_2$ )
- $P_1 \otimes P_2$
- Take  $\{q_1, q_2\}$  into account

KS-compatibility

Determinism

Independence

Joining



# Main results

## Proof

- $P_1$  (one measurement  $q_1$ )
  - $P_2$  (one measurement  $q_2$ )
  - $P_1 \otimes P_2$
  - Take  $\{q_1, q_2\}$  into account
  - Replace  $(q_1, q_2)$  with  $q_3 \equiv g(q_1, q_2)$
- KS-compatibility  
Determinism  
Independence  
Joining  
Coarsening

# Main results

## Proof

- $P_1$  (one measurement  $q_1$ )
  - $P_2$  (one measurement  $q_2$ )
  - $P_1 \otimes P_2$
  - Take  $\{q_1, q_2\}$  into account
  - Replace  $(q_1, q_2)$  with  $q_3 \equiv g(q_1, q_2)$
  - Drop  $q_2$
- KS-compatibility  
Determinism  
Independence  
Joining  
Coarsening  
Nestedness

# Main results

## Proof

- $P_1$  (one measurement  $q_1$ )
  - $P_2$  (one measurement  $q_2$ )
  - $P_1 \otimes P_2$
  - Take  $\{q_1, q_2\}$  into account
  - Replace  $(q_1, q_2)$  with  $q_3 \equiv g(q_1, q_2)$
  - Drop  $q_2$
  - Relabel  $q_1$  and  $q_3$
- KS-compatibility  
Determinism  
Independence  
Joining  
Coarsening  
Nestedness  
Relabeling

# Main results

## Proof

- $P_1$  (one measurement  $q_1$ )
  - $P_2$  (one measurement  $q_2$ )
  - $P_1 \otimes P_2$
  - Take  $\{q_1, q_2\}$  into account
  - Replace  $(q_1, q_2)$  with  $q_3 \equiv g(q_1, q_2)$
  - Drop  $q_2$
  - Relabel  $q_1$  and  $q_3$
  - PR-box
- KS-compatibility  
Determinism  
Independence  
Joining  
Coarsening  
Nestedness  
Relabeling  
Contradiction

# Main results

## Impossibility theorems

### Theorem 2

No extension of contextuality can simultaneously satisfy:

# Main results

## Impossibility theorems

### Theorem 2

No extension of contextuality can simultaneously satisfy:

- Nestedness
- Coarsening
- Relabeling

# Main results

## Impossibility theorems

### Theorem 2

No extension of contextuality can simultaneously satisfy:

- Nestedness
- Coarsening
- Relabeling
- Joining

# Main results

## Impossibility theorems

### Theorem 2

No extension of contextuality can simultaneously satisfy:

- Nestedness
- Coarsening
- Relabeling
- Joining
- Deterministic redundancy



# Plan

Structure of the work

Basic framework

KS-compatibility

Monotonicity

    Nestedness

    Coarsening

    Relabeling

Post-processing

    Joining

Independence

Determinism

    Determinism

Main results

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# Discussion

Nullifying the impossibility theorem

# Discussion

## Interpreting contextuality

Extended contextuality is said in many ways

# Discussion

## Interpreting contextuality

Extended contextuality is said in many ways

- ⊙ “Contextuality is about identity of random variable” (E.N. Dzhafarov, J.V.Kujala, 2015)

# Discussion

## Interpreting contextuality

Extended contextuality is said in many ways

- ⊙ “Contextuality is about identity of random variable” (E.N. Dzhafarov, J.V.Kujala, 2015)
- ⊙ “Experimentally friendly framework for [Kochen-Specker] contextuality” (B. Amaral, C. Duarte, 2019)

# Discussion

## Interpreting contextuality

Extended contextuality is said in many ways

- ⊙ “Contextuality is about identity of random variable” (E.N. Dzhafarov, J.V.Kujala, 2015)
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- ⊙ “The necessity of stronger direct influences to model a full system than when considered individually.” (M. Jones, 2019)

# Discussion

## Interpreting contextuality

Extended contextuality is said in many ways

- ⊙ “Contextuality is about identity of random variable” (E.N. Dzhafarov, J.V.Kujala, 2015)
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- ⊙ “The necessity of stronger direct influences to model a full system than when considered individually.” (M. Jones, 2019)
- ⊙ “One may reject a statistical hypothesis that a studied population is described by a joint probability distribution of all these variables.” (M. Kupczynski, 2021)

# Discussion

## Rejecting post-processing

Rejecting post-processing



# Discussion

## Rejecting post-processing

### Contextuality and Informational Redundancy

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# Discussion

## Rejecting post-processing

“Experimental friendly framework to contextuality”

$(q_1, q_2)$

# Discussion

## Rejecting determinism

Rejecting determinism

# Discussion

## Rejecting determinism

### Causal interpretation

“In deterministic systems, all causal influences are fully observable.”

# Discussion

## Rejecting Coarsening

Rejecting coarsening

# Discussion

## Rejecting Coarsening

Kochen and Specker

Functional relations  $\Rightarrow$  Contextual structure

# Discussion

## Rejecting Coarsening

- Predetermined contextual structure (Measurement scenario)
- $g(q) \prec c \Leftrightarrow q \prec c$

# Discussion

## Rejecting Coarsening

Dichotomizations



# Discussion

## Rejecting isomorphism

Rejecting isomorphism

# Discussion

## Rejecting isomorphism

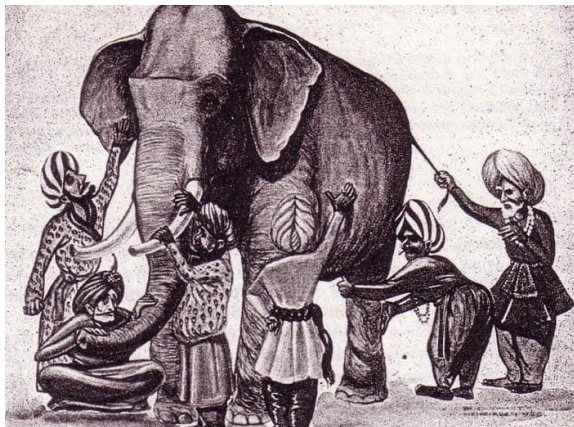
$$(\mathcal{S}, P) \mapsto \{\perp, \top\}$$

# Discussion

What else?

# Discussion

## Contextuality as a cluster concept



# Plan

Structure of the work

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# Appendix

## Proof of theorem 1

$$P_1(0|q_1) = P_1(1|q_1) = \frac{1}{2}$$
$$P_2(1|q_2, c_1) = P_2(1|q_2, c_2) = P_2(1|q_2, c_3) = P_2(0|q_2, c_4) = 1$$

# Appendix

## Proof of theorem 1

Define  $P_3 \doteq P_1 \otimes P_2$

$P_3(\cdot c_1)$	$q_2 = 0$	$q_2 = 1$
$q_1 = 0$	0	$\frac{1}{2}$
$q_1 = 1$	0	$\frac{1}{2}$

$P_3(\cdot c_2)$	$q_2 = 0$	$q_2 = 1$
$q_1 = 0$	0	$\frac{1}{2}$
$q_1 = 1$	0	$\frac{1}{2}$

$P_3(\cdot c_3)$	$q_2 = 0$	$q_2 = 1$
$q_1 = 0$	0	$\frac{1}{2}$
$q_1 = 1$	0	$\frac{1}{2}$

$P_3(\cdot c_4)$	$q_2 = 0$	$q_2 = 1$
$q_1 = 0$	$\frac{1}{2}$	0
$q_1 = 1$	$\frac{1}{2}$	0

# Appendix

## Proof of theorem 1

$q_3 = 0$  if  $q_1 \neq q_2$  and  $q_3 = 1$  if  $q_1 = q_2$

$P_4(\cdot   \mathbf{c}_1)$	$q_3 = 0$	$q_3 = 1$	$P_4(\cdot   \mathbf{c}_2)$	$q_3 = 0$	$q_3 = 1$
$q_1 = 0$	$\frac{1}{2}$	0	$q_1 = 0$	$\frac{1}{2}$	0
$q_1 = 1$	0	$\frac{1}{2}$	$q_1 = 1$	0	$\frac{1}{2}$

$P_4(\cdot   \mathbf{c}_3)$	$q_3 = 0$	$q_3 = 1$	$P_4(\cdot   \mathbf{c}_4)$	$q_3 = 0$	$q_3 = 1$
$q_1 = 0$	$\frac{1}{2}$	0	$q_1 = 0$	0	$\frac{1}{2}$
$q_1 = 1$	0	$\frac{1}{2}$	$q_1 = 1$	$\frac{1}{2}$	0



# Appendix

## proof of theorem 1

Relabel  $q_1$  as  $q_4$  in  $c_3$  and  $c_4$ , and relabel  $q_3$  as  $q_5$  in  $c_2$  and  $c_4$

$\mathbf{P}_5(\cdot \mathbf{c}_1)$	$q_3 = 0$	$q_3 = 1$	$\mathbf{P}_5(\cdot \mathbf{c}_2)$	$q_5 = 0$	$q_5 = 1$
$q_1 = 0$	$\frac{1}{2}$	0	$q_1 = 0$	$\frac{1}{2}$	0
$q_1 = 1$	0	$\frac{1}{2}$	$q_1 = 1$	0	$\frac{1}{2}$
$\mathbf{P}_5(\cdot \mathbf{c}_3)$	$q_3 = 0$	$q_3 = 1$	$\mathbf{P}_5(\cdot \mathbf{c}_4)$	$q_5 = 0$	$q_5 = 1$
$q_4 = 0$	$\frac{1}{2}$	0	$q_4 = 0$	0	$\frac{1}{2}$
$q_4 = 1$	0	$\frac{1}{2}$	$q_4 = 1$	$\frac{1}{2}$	0