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Empirical Recovery of Response Time Decomposition Rules II. Discriminability of Serial and Parallel Architectures

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Among the possible response time (RT) decomposition rules, three are of a traditional interest: addition (serial RT architecture), minimum (parallel-OR architecture), and maximum (parallel-AND architecture). Given RT samples, one can decide which of these three operation is the true decomposition rule by choosing the operation producing the smallest Smirnov distance between the RT samples combined in a certain way, as described by E. N. Dzhafarov and J. M. Cortese (1996, *Journal of Mathematical Psychology* **40**, 185–202). By means of Monte-Carlo simulations, we determine at what sample sizes this decision identifies the true decomposition rule reliably. The results indicate that for a broad class of RT distribution functions the sample sizes required are by an order of magnitude larger when the component times are stochastically independent than when they are perfectly positively stochastically interdependent. In both cases, however, the required sample sizes are realistically achievable in an experiment, provided the experimental factors selectively influencing component times are sufficiently effective. Addition and maximum are generally more difficult to discriminate than addition and minimum, which in turn are more difficult to discriminate than maximum and minimum.

PROBLEM

In this paper, we address the problem of how to identify the response time (RT) decomposition rule within a list of operations containing this decomposition rule, given that RTs are only known on a sample level. Although the general logic of this study applies to all RT architectures that are within the scope of Dzhafarov and Schweickert's (1995) theory of decomposition tests, we specifically focus on three architectures of traditional interest: serial architectures (where the decomposition rule is arithmetic addition), parallel-OR architectures (where the decomposition rule is minimum), and parallel-AND architectures (where the decomposition rule is maximum).

A rigorous statement of this problem follows Dzhafarov and Schweickert's theory of decomposition tests and their sample-level version presented in Dzhafarov and Cortese (1996). A familiarity with these papers is desirable, but we include just enough introductory information to make the present paper readable independently.

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Let $\mathbf{T}_{ij}$ be a RT whose distribution depends on two experimental factors, i and j being levels of the two factors in a crossed factorial design. We consider decompositions of $\mathbf{T}_{ij}$ by means of an operation $\blacklozenge$ into component times $\mathbf{A}_i$ and $\mathbf{B}_j$ that are selectively influenced by these factors and have a given form of stochastic relationship $-\overset{\mathrm{s}}{}-$ between them:

$$\mathbf{T}_{ij} \overset{\mathrm{d}}{=} \mathbf{A}_i \blacklozenge \mathbf{B}_j,\ \mathbf{A}_i -\overset{\mathrm{s}}{}- \mathbf{B}_j\,. \tag{1}$$

Here, $\overset{\mathrm{d}}{=}$ means "is distributed as," and $-\overset{\mathrm{s}}{}-$ denotes one of two simple forms of stochastic relationship investigated in Dzhafarov (1992), Dzhafarov and Rouder (1996), and Dzhafarov and Schweickert (1995): the "classical" stochastic independence between $\mathbf{A}_i$ and $\mathbf{B}_j$ (denoted $\mathbf{A}_i \perp \mathbf{B}_j$ and abbreviated s.-independence), and the perfect positive stochastic interdependence between $\mathbf{A}_i$ and $\mathbf{B}_j$ (denoted $\mathbf{A}_i \parallel \mathbf{B}_j$ and abbreviated p.p.s.-interdependence), when $\mathbf{A}_i$ and $\mathbf{B}_j$ are increasing functions of a single random variable (a common "internal source of variability").

Let the true decomposition rule $\blacklozenge$ be known (assumed) to belong to a list of k competing operations $\{\diamondsuit_1, ..., \diamondsuit_k\}$, and let the RTs $\mathbf{T}_{ij}$ be represented by random samples of size n, $\{\mathbf{T}_{ij}^1, ..., \mathbf{T}_{ij}^n\}$. The problem is to determine a procedure by which one can identify $\blacklozenge$ among $\{\diamondsuit_1, ..., \diamondsuit_k\}$. We specifically focus on the list containing three operations, $\{+, \min, \max\}$, or any two of them. The theoretical importance of these operations is due to the possibility of the following well-known interpretation: if $\mathbf{A}_i$ and $\mathbf{B}_j$ are the durations of two processes (selectively influenced by the index factors), then $\mathbf{A}_i + \mathbf{B}_j$ is the duration of their serial connection, $\max\{\mathbf{A}_i, \mathbf{B}_j\}$ is the duration of their parallel-AND connection, and $\min\{\mathbf{A}_i, \mathbf{B}_j\}$ is the duration of their parallel-OR connection (in the latter case the duration of the longer process should be viewed as virtual, or potential; alternatively, the longer process may be viewed as continuing after the response is given). The present work, therefore, relates to the vast literature on empirical discriminability of serial and parallel architectures of processing times (Nozawa, 1992;

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Schweickert, 1978; Townsend & Ashby, 1983; Townsend & Nozawa, 1988; for surveys, see Luce, 1986; Massaro & Cowan, 1993; Schweickert, 1992; Townsend, 1990).

However, Dzhafarov & Schweickert (1995) suggested that the component times $\mathbf{A}_i$ and $\mathbf{B}_j$ can be viewed as time-dimensioned aspects of a single process developing until it reaches a critical level (see Dzhafarov, 1993), rather than durations of separate processes. In such an interpretation the operations plus, minimum, and maximum have no privileged status compared to other operations (e.g., multiplication), except in the following purely mathematically sense. For any operation within the scope of Dzhafarov & Schweickert's theory (termed "simple operations"), if $c = a \diamond b$, then either $g(c) = g(a) + g(b)$, by a strictly monotonic transformation g, or $\diamond$ is minimum or maximum.

The problem of identifying the true decomposition rule $\blacklozenge$ among the operations $\{+, \min, \max\}$ is well-defined, because according to the uniqueness theorems of Dzhafarov and Schweickert's theory, under a mild constraint imposed on the RT distributions, if (1) holds for one of these operations then it cannot hold for another: thus (min)-decomposability excludes both $(+)$-decomposability and (max)-decomposability, etc. The following statement is referred to as the (population-level) decomposition test for $\diamond$, or $(\diamond)$-test. For any 2×2 (subset of a) crossed factorial design producing the RTs $\mathbf{T}_{ij}$ (i.e., putting $i = 1, 2$, $j = 1, 2$), provided that the RTs $\mathbf{T}_{ij}$ are decomposable at all, they are $(\diamond)$-decomposable (i.e., $\diamond$ is the true decomposition rule $\blacklozenge$) if and only if

$$d[U_\diamond(t), C_\diamond(t)] = 0, \tag{2}$$

where $C_\diamond(t)$ is the distribution function of the "cross" combination $\mathbf{T}_{12} \diamond \mathbf{T}_{21}$ ($\mathbf{T}_{12} -\!\!\overset{s}{-}\!\!- \mathbf{T}_{21}$), $U_\diamond(t)$ the distribution function of the "uncross" combination $\mathbf{T}_{11} \diamond \mathbf{T}_{22}$ ($\mathbf{T}_{11} -\!\!\overset{s}{-}\!\!- \mathbf{T}_{22}$), and $d[U_\diamond(t), C_\diamond(t)]$ is any distance function on the space of distribution functions (symmetrical, satisfying the triangle inequality, and vanishing if and only if the two distribution functions are identical). A decomposition test is only designed to identify the true operation $\blacklozenge$, not the component times themselves (that generally cannot be recovered uniquely), or the form of stochastic relationship $-\!\!\overset{s}{-}\!\!-$ (that should be viewed as part of the definition of the component times to be connected by the operation sought).

Based on formulation (2), Dzhafarov and Cortese (1996) proposed the following sample-level version of a $(\diamond)$-test. Given random samples $\{\mathbf{T}_{ij}^1, ..., \mathbf{T}_{ij}^n\}$ from $\mathbf{T}_{ij}$ ($i = 1, 2$, $j = 1, 2$), let $\{\mathbf{T}_{ij}^{(1)}, ..., \mathbf{T}_{ij}^{(n)}\}$ be the same samples arranged in an increasing order. Form the sequences

$$\{\mathbf{T}_{11}^1 \diamond \mathbf{T}_{22}^1, ..., \mathbf{T}_{11}^n \diamond \mathbf{T}_{22}^n\}$$

and

$$\{\mathbf{T}_{12}^1 \diamond \mathbf{T}_{21}^1, ..., \mathbf{T}_{12}^n \diamond \mathbf{T}_{21}^n\}$$

if the stochastic relationship is s.-independence ($\mathbf{A}_i \perp \mathbf{B}_j$), or form the sequences of paired empirical quantiles

$$\{\mathbf{T}_{11}^{(1)} \diamond \mathbf{T}_{22}^{(1)}, ..., \mathbf{T}_{11}^{(n)} \diamond \mathbf{T}_{22}^{(n)}\}$$

and

$$\{\mathbf{T}_{12}^{(1)} \diamond \mathbf{T}_{21}^{(1)}, ..., \mathbf{T}_{12}^{(n)} \diamond \mathbf{T}_{21}^{(n)}\}$$

if the stochastic relationship is p.p.s.-interdependence ($\mathbf{A}_i \parallel \mathbf{B}_j$). Let $\mathbf{U}_\diamond^n(t)_\perp$ and $\mathbf{C}_\diamond^n(t)_\perp$ be (random) empirical distribution functions corresponding to the former two sequences, respectively; analogously, $\mathbf{U}_\diamond^n(t)_\parallel$ and $\mathbf{C}_\diamond^n(t)_\parallel$ correspond to the latter two sequences. Compute the Smirnov distance

$$\mathbf{D}_\diamond = \sup_t |\mathbf{U}_\diamond^n(t)_\perp - \mathbf{C}_\diamond^n(t)_\perp|$$

or

$$\mathbf{D}_\diamond = \sup_t |\mathbf{U}_\diamond^n(t)_\parallel - \mathbf{C}_\diamond^n(t)_\parallel|,$$

depending on the form of stochastic relationship. Dzhafarov and Cortese (1996) show that these distances are strongly consistent estimators of the corresponding distances between the population distribution functions, which means that

$$\begin{aligned} \sup_t |\mathbf{U}_\diamond^n(t)_\perp - \mathbf{U}_\diamond^n(t)_\perp| &\xrightarrow{\text{a.s.}} \sup_t |U_\diamond(t)_\perp - C_\diamond(t)_\perp| \\ \sup_t |\mathbf{U}_\diamond^n(t)_\parallel - \mathbf{U}_\diamond^n(t)_\parallel| &\xrightarrow{\text{a.s.}} \sup_t |U_\diamond(t)_\parallel - C_\diamond(t)_\parallel| \end{aligned} \tag{3}$$

(a.s. stands for almost sure convergence). Also Dzhafarov and Cortese (1996) derive the asymptotic sampling distribution for $\mathbf{D}_\diamond$ under the hypothesis that $\diamond$ is the true decomposition rule $\blacklozenge$: for any observed value $d_\diamond$ of $\mathbf{D}_\diamond$, the asymptotic p-value

$$\lim_{n \to \infty} \text{Prob}\{\mathbf{D}_\blacklozenge > d_\diamond\}$$

can be evaluated (or enclosed between bounds), and it does not depend on $\blacklozenge$; whereas for any incorrect operation $*$ (i.e., any operation different from the true decomposition rule $\blacklozenge$),

$$\lim_{n \to \infty} \text{Prob}\{\mathbf{D}_* > d_\diamond\} > \lim_{n \to \infty} \text{Prob}\{\mathbf{D}_\blacklozenge > d_\diamond\}.$$

The hypothesis that $\diamond$ is the true decomposition rule $\blacklozenge$ (against the generic alternative that it is not) then can be

tested by comparing, in the conventional way, the observed p-value with a significance level α.

This decision rule, however, is not directly applicable to our present problem—to identify $\blacklozenge$ within a given list of operations that is assumed to contain $\blacklozenge$, say, $\{+, \min, \max\}$, $\{\min, \max\}$, $\{+, \min\}$, or $\{+, \max\}$. It is quite possible that a given value of α falls below the p-values for more than one operation from the list, or exceeds the p-values for all these operations. Most importantly, however, this decision rule, as well as the sampling distribution theory upon which it is based, does not take into account that the Smirnov distances for different operations, say, $\mathbf{D}_{\min}$, $\mathbf{D}_{+}$, and $\mathbf{D}_{\max}$, are all computed from one and the same quadruple of RT samples, $\{\mathbf{T}_{ij}^1, ..., \mathbf{T}_{ij}^n\}$, $i=1, 2$, $j=1, 2$. They are not, therefore, stochastically independent.

An analytic approach to this problem requires that one evaluate a joint sampling distribution of the vector $\{\mathbf{D}_{\blacklozenge}, \mathbf{D}_{*_1}, ..., \mathbf{D}_{*_{k-1}}\}$ computed from one and the same quadruple of samples selected from ($\blacklozenge$)-decomposable RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$. Such a theory is not available. A suitable decision rule, however, follows immediately from the strong consistency property (3): identify $\blacklozenge$ as the operation for which the observed value of the Smirnow distance is the smallest. Indeed, as the sample sizes for the RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ increase beyond bound,

$$\text{Prob}\{\min[\mathbf{D}_{\blacklozenge}, \mathbf{D}_{*_1}, ..., \mathbf{D}_{*_{k-1}}] = \mathbf{D}_{\blacklozenge}\} \to 1 \qquad (4)$$

for all operations $\{*_1, ..., *_{k-1}\}$ different from these true decomposition rule $\blacklozenge$. This follows from the fact that, due to (3), for any $\varepsilon > 0$, $\text{Prob}\{\mathbf{D}_{\blacklozenge} < \varepsilon\}$ converges to 1 (as sample size increases), whereas, for any incorrect operation $*$, one can chose a sufficiently small $\varepsilon > 0$ for which $\text{Prob}\{\mathbf{D}_{*} < \varepsilon\} \to 0$. It follows from (4) that to make $\text{Prob}\{\min[\mathbf{D}_{\blacklozenge}, \mathbf{D}_{*_1}, ..., \mathbf{D}_{*_{k-1}}] = \mathbf{D}_{\blacklozenge}\}$ arbitrarily close to 1 one "merely" has to chose sufficiently large samples. From an empirical point of view, of course, the main question is whether such sample sizes are realistically achievable in an experiment. This question is central for the present work. The results suggest a cautiously affirmative answer, at least if one confines one's attention to the operations $\{+, \min, \max\}$ and if the distributions used in our study are not all very dissimilar to empirically observed ones.

The minimum sample size (for convenience, let it be the same for all four RT samples) at which the probability in (4) reaches a given level can be considered a measure of the empirical identifiability of $\blacklozenge$ within a given list of operations: the larger this minimum sample size, the less identifiable is the operation empirically (as opposed to the theoretical, population-level identifiability established by the uniqueness theorems of Dzhafarov & Schweickert, 1995). If, as in the simulation study described below, all operations within a list, say, $\{+, \min, \max\}$, take turns in serving as the true decomposition rule $\blacklozenge$, then the minimum sample size at which all three probabilities,

$$\text{Prob}\{\min[\mathbf{D}_{+}, \mathbf{D}_{\max}, \mathbf{D}_{\min}] = \mathbf{D}_{+} \mid \blacklozenge \text{ is } +\},$$
$$\text{Prob}\{\min[\mathbf{D}_{+}, \mathbf{D}_{\max}, \mathbf{D}_{\min}] = \mathbf{D}_{\max} \mid \blacklozenge \text{ is max}\},$$
$$\text{Prob}\{\min[\mathbf{D}_{+}, \mathbf{D}_{\max}, \mathbf{D}_{\min}] = \mathbf{D}_{\min} \mid \blacklozenge \text{ is min}\}$$

(or analogous probabilities for any other list of operations) exceed a given level can be considered a measure of empirical discriminability of these operations: the larger this sample size, the less discriminable they are empirically. As the context precludes ambiguity, hereafter we omit the qualifier "empirical" when referring to the identifiability and discriminability.

It is important to keep in mind that the identifiability, or discriminability, within a given list of operations is not generally invariant with respect to possible distributions of the RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$. One aspect of this assertion relates to the effectiveness of the index factors. The first (or second) index factor $i=1, 2$ ($j=1, 2$) is ineffective if, respectively,

$$\begin{cases} F_{11}(t) \equiv F_{21}(t) \\ F_{22}(t) \equiv F_{12}(t) \end{cases} \quad \text{or} \quad \begin{cases} F_{11}(t) \equiv F_{12}(t) \\ F_{22}(t) \equiv F_{21}(t), \end{cases} \qquad (5)$$

where $F_{ij}(t)$ is the distribution function for $\mathbf{T}_{ij}$. In either of these cases

$$\sup_t |U_{\diamond}(t) - C_{\diamond}(t)| = 0$$

for any operation $\diamond$, and consequently, no two operations are discriminable at all (Dzhafarov & Schweickert, 1995). A "gradualized" version of this statement is that the less effective one of the index factors, the less identifiable is the true decomposition rule within a given list of operations. Unfortunately, there seems to be no universal measure of effectiveness for the index factors that could allow one to predict the value of $\text{Prob}\{\min[\mathbf{D}_{\blacklozenge}, \mathbf{D}_{*_1}, ..., \mathbf{D}_{*_{k-1}}] = \mathbf{D}_{\blacklozenge}\}$, or even to rank-order such probabilities for different lists of operations. In a sense, this probability itself can be viewed as a list-specific measure for the effectiveness of the index factors: the higher this probability, the more effective the factors are (with respect to a given list of operations).

This logical circularity is, obviously, unsatisfactory: to be of use, a measure of effectiveness should be computable from the RT distribution functions. One can only hope to find such a measure, however, by confining one's analysis to a particular list of operations, say, $\{+, \min, \max\}$, and to a particular class of RT distributions (such as the combinations of the Weibull-distributed component times used in our Monte-Carlo simulations). As shown below, for instance, the correct identification probabilities in our

simulations are in the ordinal correspondence with certain measures (different for the two forms of stochastic relationship) computed from the supremal distances

$$\sup_t |F_{11}(t) - F_{12}(t)|, \quad \sup_t |F_{22}(t) - F_{21}(t)|,$$

$$\sup_t |F_{11}(t) - F_{21}(t)|, \quad \sup_t |F_{22}(t) - F_{12}(t)|.$$

Another, perhaps less obvious, aspect of the assertion that the identifiability, or discriminability, within a given list of operations is not generally invariant with respect to possible RT distributions, relates to the comparative statements like "operation $\diamond_1$ is more discriminable from operation $\diamond_2$ than from operation $\diamond_3$." The truth value of such statements can also change from one quadruple of RTs to another. For example, in the Monte-Carlo simulation results presented below, addition and maximum are usually less discriminable than addition and minimum, but the reverse is true in at least one case. Assuming that the main determinant of the discriminability of the true decomposition rule $\blacklozenge$ from an incorrect operation $*$ is the magnitude of $\sup_t |U_*(t) - C_*(t)|$ (see Dzhafarov & Cortese, 1996), such cases can easily be constructed analytically. One can even construct a case when the operations maximum and minimum are more discriminable from addition than they are from each other. Consider RTs $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ with distribution functions $F_{11}(t), F_{12}(t), F_{21}(t), F_{22}(t)$ such that (5) does not hold, but

$$\begin{aligned} \max\{F_{12}(t), F_{21}(t)\} &\equiv \max\{F_{11}(t), F_{22}(t)\} \\ \min\{F_{12}(t), F_{21}(t)\} &\equiv \min\{F_{11}(t), F_{22}(t)\}. \end{aligned} \tag{6}$$

In the language of Dzhafarov and Schweickert's theory, this means that the two index factors are effective, but the pairs of the distribution functions $\{F_{11}(t), F_{22}(t)\}$ and $\{F_{12}(t), F_{21}(t)\}$ are cross-over rearrangements of each other. In such a situation, under s.-independence,

$$\sup_t |U_{\max}(t) - C_{\max}(t)| = 0, \quad \sup_t |U_{\min}(t) - C_{\min}(t)| = 0,$$

but generally

$$\sup_t |U_+(t) - C_+(t)| > 0.$$

Then, obviously, the difference between $\mathbf{D}_{\max}$ and $\mathbf{D}_{\min}$ will be stochastically smaller than that between either $\mathbf{D}_{\max}$ and $\mathbf{D}_+$ or $\mathbf{D}_{\min}$ and $\mathbf{D}_+$. Strictly speaking, this example is not legitimate as stated, because Dzhafarov and Schweickert's theory excludes the case of cross-over rearrangements from consideration. However, it is easy to realize that the distribution functions $\{F_{11}(t), F_{22}(t)\}$ can always be chosen arbitrarily close but not identical to cross-over rearrangements of $\{F_{12}(t), F_{21}(t)\}$, resulting in the same outcome: a better discriminability of maximum and minimum from addition than from each other.

MONTE-CARLO SIMULATION

The logic of this simulation study is similar to that in Dzhafarov and Cortese's (1996) analysis of the statistical power of the sample-level decomposition tests. Let $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$ be component times with known distributions, and let the form of stochastic relationship $-\overset{s}{-}-$ be given. We choose one of the three operations, minimum, maximum, or addition, to serve as the true decomposition rule $\blacklozenge$, with the remaining two serving as competing, "incorrect" alternatives $*_1$, $*_2$. Using the true decomposition rule, we form RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ from $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$ according to (1). Then we select a quadruple of samples of size n from these RTs, and we form their "uncross" and "cross" combinations by means of all three operations,

$$\{t_{11}^{(1)} + t_{22}^{(n)}, ..., t_{11}^{(n)} + t_{22}^{(n)}\}$$

$$\text{and } \{t_{12}^{(1)} + t_{21}^{(1)}, ..., t_{12}^{(n)} + t_{21}^{(n)}\}$$

$$\{\min[t_{11}^{(1)}, t_{22}^{(1)}], ..., \min[t_{11}^{(n)}, t_{22}^{(n)}]\}$$

$$\text{and } \{\min[t_{12}^{(1)}, t_{21}^{(1)}], ..., \min[t_{12}^{(n)}, t_{21}^{(n)}]\}$$

$$\{\max[t_{11}^{(1)}, t_{22}^{(1)}], ..., \max[t_{11}^{(n)}, t_{22}^{(n)}]\}$$

$$\text{and } \{\max[t_{12}^{(1)}, t_{21}^{(1)}], ..., \max[t_{12}^{(n)}, t_{21}^{(n)}]\}$$

in the case of p.p.s.-interdependence, and

$$\{t_{11}^{1} + t_{22}^{1}, ..., t_{11}^{n} + t_{22}^{n}\}$$

$$\text{and } \{t_{12}^{1} + t_{21}^{1}, ..., t_{12}^{n} + t_{21}^{n}\}$$

$$\{\min[t_{11}^{1}, t_{22}^{1}], ..., \min[t_{11}^{n}, t_{22}^{n}]\}$$

$$\text{and } \{\min[t_{12}^{1}, t_{21}^{1}], ..., \min[t_{12}^{n}, t_{21}^{n}]\}$$

$$\{\max[t_{11}^{1}, t_{22}^{1}], ..., \max[t_{11}^{n}, t_{22}^{n}]\}$$

$$\text{and } \{\max[t_{12}^{1}, t_{21}^{1}], ..., \max[t_{12}^{n}, t_{21}^{n}]\}$$

in the case of s.-independence. Then we compute the Smirnov distances $d_\blacklozenge$, d_{*_1}, and d_{*_2} and verify whether

$$d_\blacklozenge < d_{*_1} \quad \text{and/or} \quad d_\blacklozenge < d_{*_2},$$

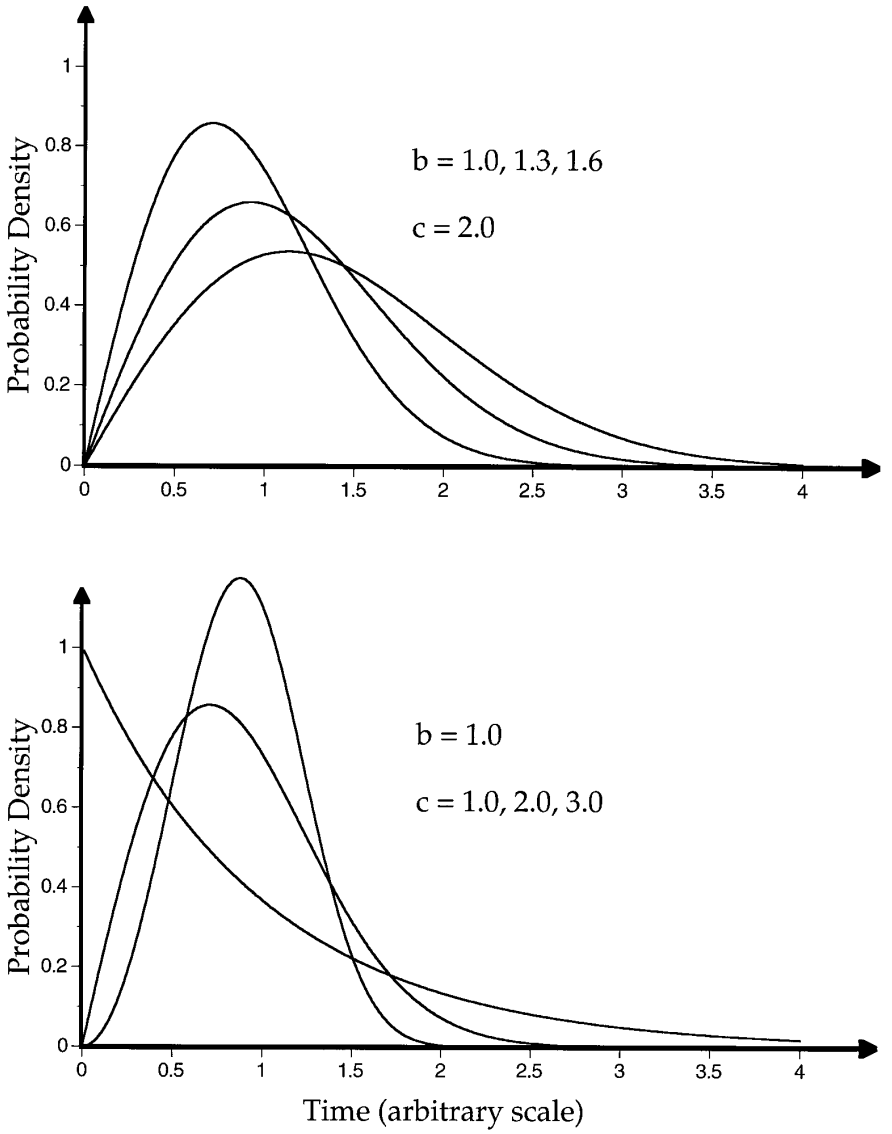

FIG. 1. Six different Weibull distributions used for the component times in the Monte-Carlo study.

(i.e., whether the true decomposition rule "wins" in a competition with an incorrect operation). This procedure is repeated 2500 times, allowing us to reliably estimate (asymptotically, ignoring the probability of ties)

$$\text{Prob}\{\min[\mathbf{D}_\blacklozenge, \mathbf{D}_{*1}, \mathbf{D}_{*2}] = \mathbf{D}_\blacklozenge\},$$

as well as the pairwise probabilities

$$\text{Prob}\{\min[\mathbf{D}_\blacklozenge, \mathbf{D}_{*1}] = \mathbf{D}_\blacklozenge\}$$

and

$$\text{Prob}\{\min[\mathbf{D}_\blacklozenge, \mathbf{D}_{*2}] = \mathbf{D}_\blacklozenge\}.$$

The estimates of these probabilities are obtained for all three operations as each serves in turn as the true decomposition rule, for both forms of stochastic relationship, for different sample sizes n, and for a variety of distributions for the component times $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$.

For reasons discussed below, we chose Weibull-distributed component times in our simulations:

$$\begin{cases} \mathbf{A}_1 \stackrel{\text{d}}{=} b_{\text{A1}}[-\log(1-\mathbf{P})]^{c_{\text{A1}}^{-1}} \\ \mathbf{A}_2 \stackrel{\text{d}}{=} b_{\text{A2}}[-\log(1-\mathbf{P})]^{c_{\text{A2}}^{-1}} \\ \mathbf{B}_1 \stackrel{\text{d}}{=} b_{\text{B1}}[-\log(1-\mathbf{P})]^{c_{\text{B1}}^{-1}} \\ \mathbf{B}_2 \stackrel{\text{d}}{=} b_{\text{B2}}[-\log(1-\mathbf{P})]^{c_{\text{B2}}^{-1}}, \end{cases}$$

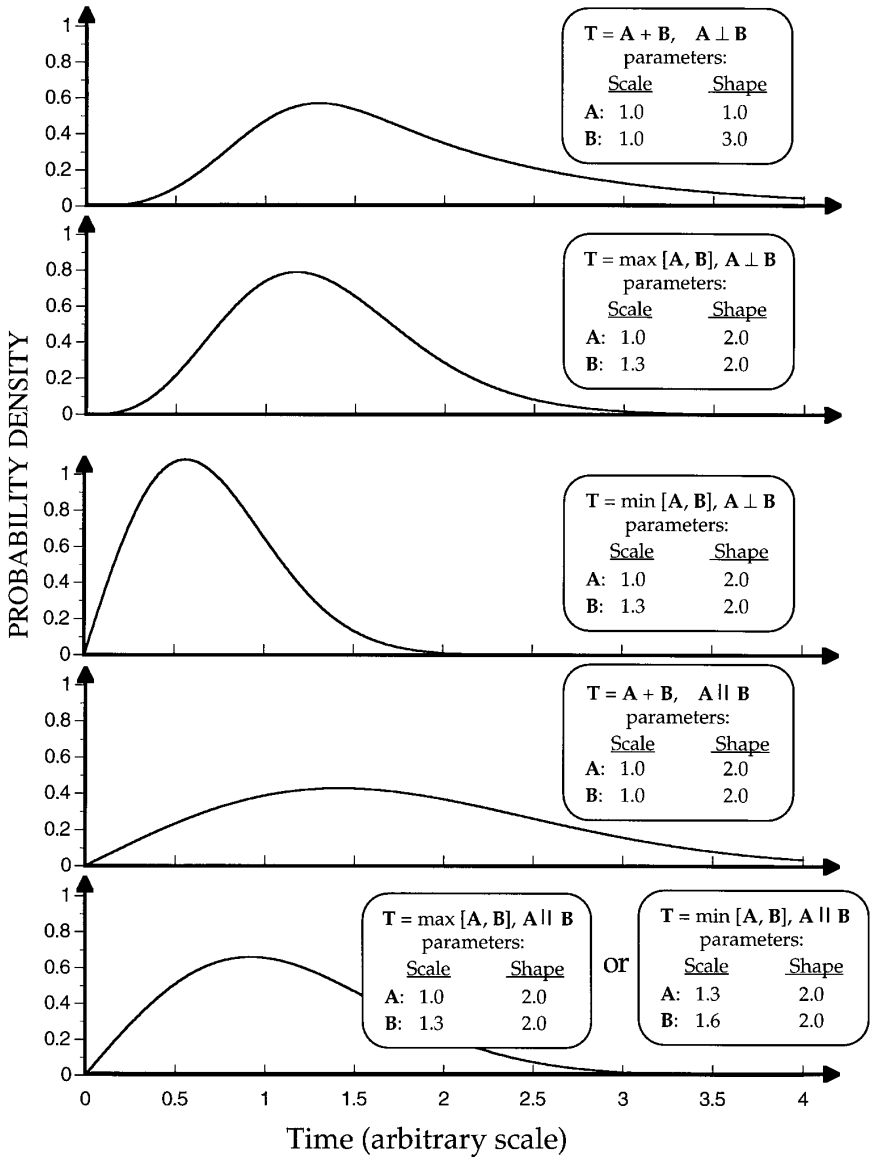

FIG 2. Examples of density functions for simulated RTs.

where $\mathbf{P}$ is unit-uniformly distributed, and b and c are referred to as the scale and shape parameters, respectively. The simulated RTs, therefore, are formed as

$$\mathbf{T}_{ij} \stackrel{\mathrm{d}}{=} b_{\mathrm{A}i}[-\log(1-\mathbf{P})]^{c_{\mathrm{A}i}^{-1}} \blacklozenge b_{\mathrm{B}j}[-\log(1-\mathbf{Q})]^{c_{\mathrm{B}j}^{-1}} \quad (i=1,2, j=1,2),$$

where $\mathbf{P}$ and $\mathbf{Q}$ are unit-uniformly distributed: under p.p.s.-interdependence, $\mathbf{P}=\mathbf{Q}$, whereas under s.-independence, $\mathbf{P} \perp \mathbf{Q}$.

The six Weibull distributions used in our simulations are shown in Fig. 1. To limit the number of combinations, in computing the RTs $\mathbf{T}_{11}, \mathbf{T}_{12}, \mathbf{T}_{21}, \mathbf{T}_{22}$ each of the two parameters was fixed in turn while the other varied on three levels, as shown by grouping of the distributions in the upper and lower panels of Fig. 1. There were 12 different quadruples of component time distributions, 6 with varying scale parameter and 6 with varying shape parameter, yielding the total of 72 quadruples of RT distributions: 12 times 2 forms of stochastic relationship times 3 operations taking turns in serving as the decomposition rule (a few out of these 72 cases, however, had to be excluded for reasons indicated in Appendices 1–4). Each quadruple of the RT distributions was examined at 3 to 5 sample sizes, based on preliminary estimates of their order of magnitude.

As follows from the discussion in the introductory section, two considerations should guide one's choice of the

distributions for the component times $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{A}_2$, $\mathbf{B}_2$. First, while the simulated RTs $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ computed from these component times should form a sufficiently wide variety of distribution shapes, some of them should be sufficiently similar to empirically observed RT distributions. The presence of such "realistic looking" distributions would ensure that the simulation results are transferable to experimental data; at the same time, the variety of different shapes among the simulated quadruples of RTs, provided that the simulation results are essentially the same for all of them, would ensure that the results are robust (so that whatever dissimilarity there is between the "realistic looking" simulated distributions and the empirically observed ones, it should not lead to dramatically different outcomes). The second consideration is that the effectiveness of the index factors in $\mathbf{T}_{11}$, $\mathbf{T}_{12}$, $\mathbf{T}_{21}$, $\mathbf{T}_{22}$ should be on the order of or smaller than that expected to be found in an experiment. This would ensure that the simulation-based identifiability and discriminability estimates are realistic or conservative.

Unfortunately, these requirements are too informal to allow one to verify one's compliance with them. There are many conflicting ways of assessing similarity between distributions (e.g., see Luce's 1986 discussion of the comparisons of distribution functions versus densities versus hazard functions), and we do not know all the critical features that make a random variable similar to an empirically observed RT. Nor is it possible to construct a general algorithm for assessing the effectiveness of index factors (see the introductory section), and even if some measure of effectiveness is adopted, there is no way of assessing its "typical" values in an experiment. As a result, the conformity of our simulations with the requirements above is more of an assumption than a fact.

We assume, for instance, that the simulated RT distributions shown in Fig. 2 are sufficiently similar to empirically observed RTs (see Luce, 1986, for a review). A Weibull distribution with the shape parameter $c = 1$ is exponential, and as c increases, the density function becomes progressively more normal-like, with relatively little skewness at c-values greater than about 3. Under s.-independence, therefore, the distributions of RTs generated from Weibull-distributed component times can provide fair imitations of distributions that have been previously used for modeling RTs, ranging from generalized two-stage gamma (McGill, 1963) to ex-Gaussian (Hohle, 1965; Ratcliff, 1978). Weibull distributions themselves, that may be obtained as RT distributions under p.p.s.-interdependence when the decomposition rule is maximum or minimum (Fig. 2, bottom panel), have also been proposed as approximations for RT distributions (Maloney & Wandel, 1984).

On the other hand, Weibull-distributed component times produce in some cases quadruples of RTs whose distributions are clearly different from those of empirically observed RTs. As an example, under p.p.s.-interdependence, the maximum or minimum of two Weibull-distributed component times with the same scale parameter but different shape parameters has a density function with a distinct discontinuity. Also, under s.-independence, when the operation is minimum and one of the components is exponential, the resulting density function is nonzero at the origin. The fact that the results reported below do not differ for "realistic" and "unrealistic" simulated RTs can be interpreted as an indication of the results' robustness.

We relegate to the subsequent discussion the issue of whether the effectiveness of the index factors in our simulated RT quadruples was reasonably low to ensure that our identifiability and discriminability estimates are conservative.

RESULTS AND DISCUSSION

For the triple comparisons, the estimates of

$$\text{Prob}\{\min[\mathbf{D}_{\blacklozenge}, \mathbf{D}_{*1}, \mathbf{D}_{*2}] = \mathbf{D}_{\blacklozenge}\}$$

plotted against sample size per treatment are shown in Figs. 3–6. All estimates, based on 2500 replications, are highly reliable: when computed from two non-overlapping blocks of 1000 replications, the estimates generally differ by less than two percentage points.

We found empirically that the probability of correct identification for a given sample size within each panel (but not across the panels) is monotonically related to the following two measures of the effectiveness of index factors,

$$\min\{\max\{d_{12}^{11}, d_{21}^{22}\}, \max\{d_{21}^{11}, d_{12}^{22}\}\} \tag{7}$$

in the case of p.p.s.-interdependence, and

$$(d_{12}^{11} + d_{21}^{22})(d_{21}^{11} + d_{12}^{22}) \tag{8}$$

in the case of s.-independence, where

$$d_{kl}^{ij} = \sup_t |F_{ij}(t) - F_{kl}(t)|$$

$$(i = 1, 2, j = 1, 2, k = 1, 2, l = 1, 2).$$

These measures are well-constructed because they only vanish when (5) is satisfied, that is, when at least one of the two index factors is ineffective. The two measures, however, have very different properties: the possible values of (7) range from 0 to 1, achieving the maximal value when both pairs $\{d_{12}^{11}, d_{21}^{22}\}$ and $\{d_{21}^{11}, d_{12}^{22}\}$ contain distances equal to 1; the possible values of (8) range from 0 to 4, achieving the maximal value when all four distances $\{d_{12}^{11}, d_{21}^{22}, d_{21}^{11}, d_{12}^{22}\}$ equal 1. The values of d_{kl}^{ij} and the overall effectiveness

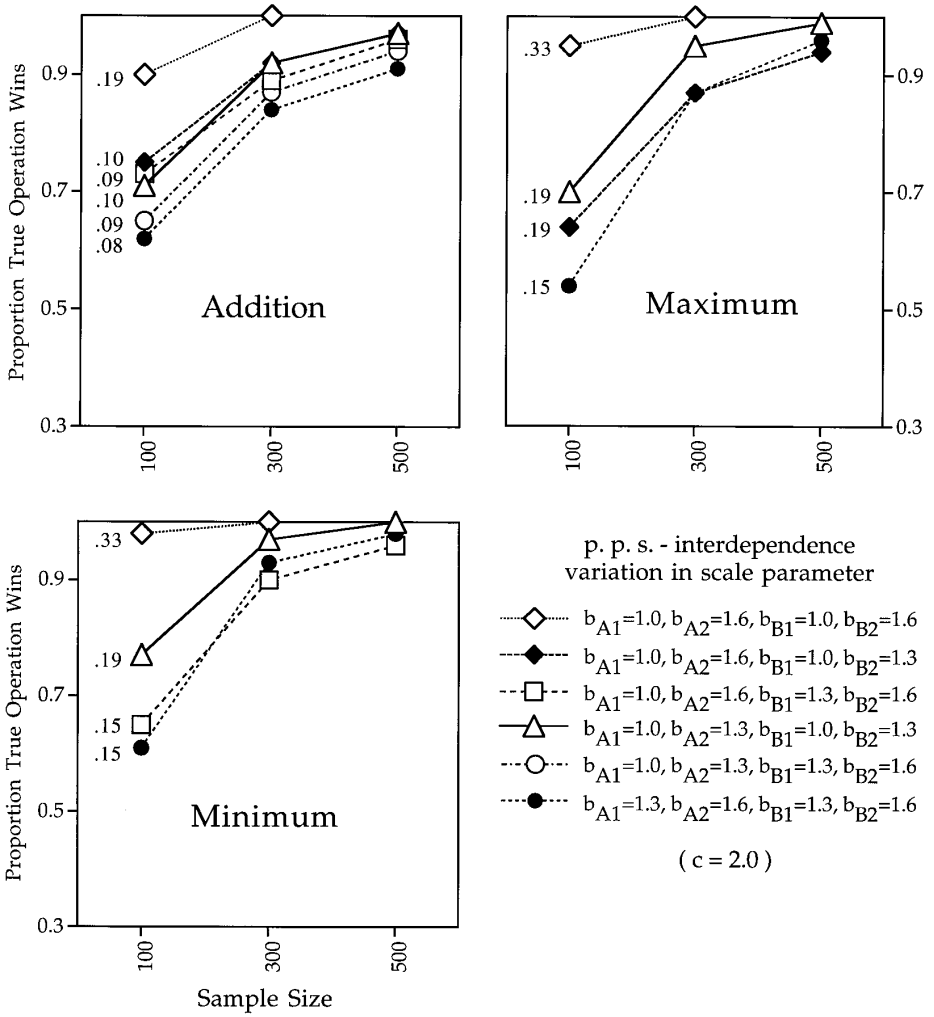

FIG 3. Estimates of the probability with which the true decompostion rule produces a smaller Smirnov distance than the two competing operations. The results are for p.p.s.-interdependent components, whose distributions have the same shape. The missing curves correspond to the cases excluded for the reasons indicated in the Appendices

measures for all simulated RT quadruples are shown in Appendices 1–4.

To ascertain that the effectiveness values in the simulated RT quadruples were not unrealistically high, we estimated the distances $\{d_{12}^{11}, d_{21}^{22}, d_{21}^{11}, d_{12}^{22}\}$ from the graphs of RT distributions reported in Roberts and Sternberg (1992)—the only study so far where a decomposition test (for addition, under the assumption of s.-independence) was applied to experimental data. Although their particular design involves a Cartesian product scheme (discussed below) and is further complicated by the involvement of a third experimental factor as well as averaging across subjects, which does not permit a direct comparison of their sample sizes with ours, the distances $\{d_{12}^{11}, d_{21}^{22}, d_{21}^{11}, d_{12}^{22}\}$ in their graphs can still serve as a reference point. Computed according to (8), as it should be under s.-independence, the effectiveness measure in their experiments 1, 2a, and 2b equals, respectively, 0.68, 0.20, and 1.77. Only two of the effectiveness values in Figs. 5 and 6 exceed the lowest of these numbers (Fig. 5, with addition and maximum as true decomposition rules), and these are the cases with very high identification probabilities (exceeding 0.9) achieved well below 1000 observations per treatment. If the effectiveness of the index factors in Roberts and Sternberg's graphs is computed according to (7), that is, under the assumption of p.p.s.-interdependence, then the estimated values are 0.29, 0.19, and 0.49 (for their experiments 1, 2a, and 2b, respectively). Again, the lowest of these estimates lies well within the upper quartile of the effectiveness values associated with the curves shown in Figs. 3 and 4. Finally, without resorting to any single measure of effectiveness, a simple comparison of the distances $\{d_{12}^{11}, d_{21}^{22}, d_{21}^{11}, d_{12}^{22}\}$ estimated from Roberts and Sternberg's graphs with those in Appendices 1–4 clearly show that overwhelmingly most of the distances in our

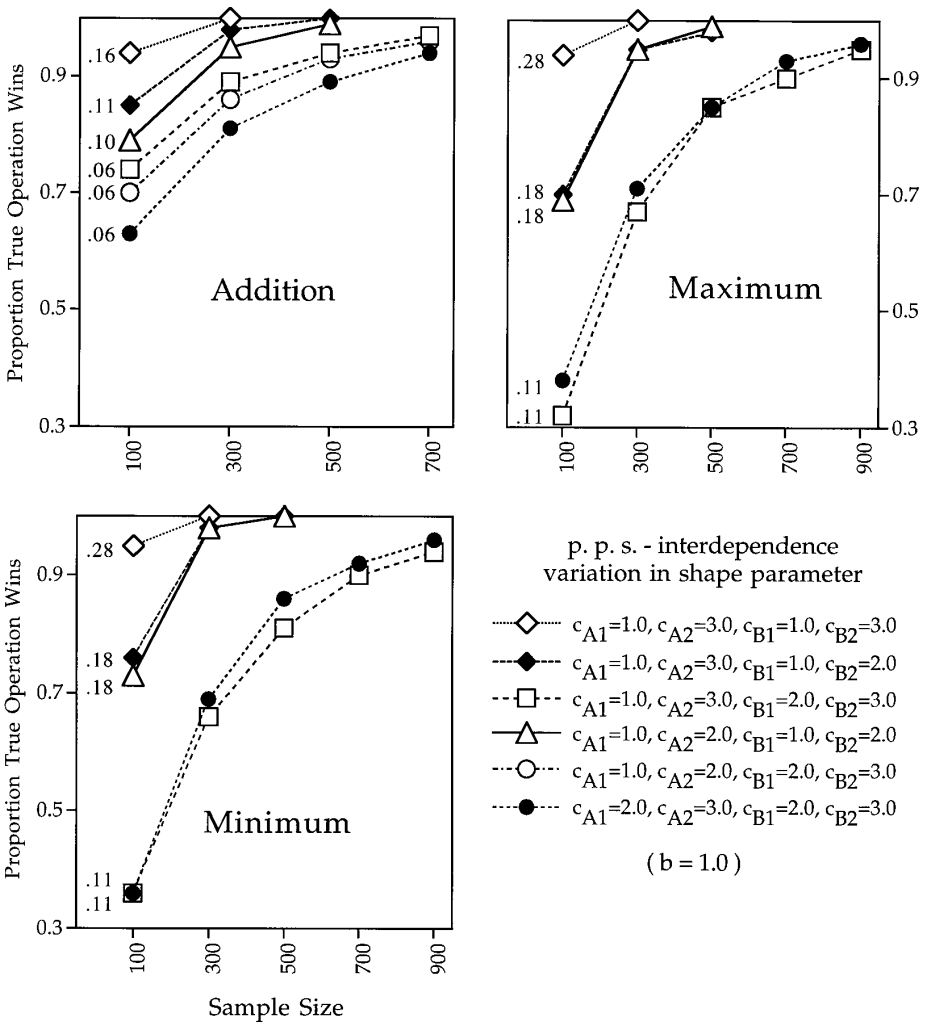

FIG. 4. Same as in Fig. 3, except that the component time distributions have different shapes.

simulations were smaller. We conclude that the simulation-based estimates of the sample sizes required to achieve high identification probabilities can be considered conservative.

The results presented in Figs. 3–6 indicate that sample sizes at which a true operation, plus, minimum, or maximum, reliably wins over its two competitors (say, with probability 0.75) are by an order of magnitude larger when the component times are s.-independent (about 5000 observations per treatment) than when they are p.p.s.-interdependent (about 500 observations per treatment). This result is in good agreement with the statistical power analysis in Dzhafarov and Cortese (1996). There it is shown that the superiority of the p.p.s.-interdependence is due to the fact that in this case the differences

$$|U_{*_1}(t) - C_{*_1}(t)| \quad \text{and} \quad |U_{*_2}(t) - C_{*_2}(t)|$$

are much larger than in the case of s.-independence. The suprema of these differences are typically on the order of 0.1 under p.p.s.-interdependence, but only on the order of 0.01 under s.-independence. Dzhafarov and Cortese point out that this fact should only be taken as an empirical finding pertaining to a limited, though perhaps very broad from an experimentalist's point of view, class of distribution functions. Thus one can see in Figs. 3–6 that the identification probabilities for the lowest values of the effectiveness of the index factors under p.p.s.-interdependence come very close to those for the highest effectiveness values under s.-independence.

Dzhafarov and Cortese (1996) also consider and dismiss as inappropriate the possibility that the statistical power (in our case, the correct identification probabilities) under s.-independence can be increased by computing the Smirnov distances between empirical distribution functions corresponding to two Cartesian product combinations

$$\{\mathbf{T}_{11}^{(k)} \diamond \mathbf{T}_{22}^{(l)}\} \quad \text{and} \quad \{\mathbf{T}_{12}^{(k)} \diamond \mathbf{T}_{21}^{(l)}\},$$

$$k = 1, ..., n; \quad l = 1, ..., n,$$

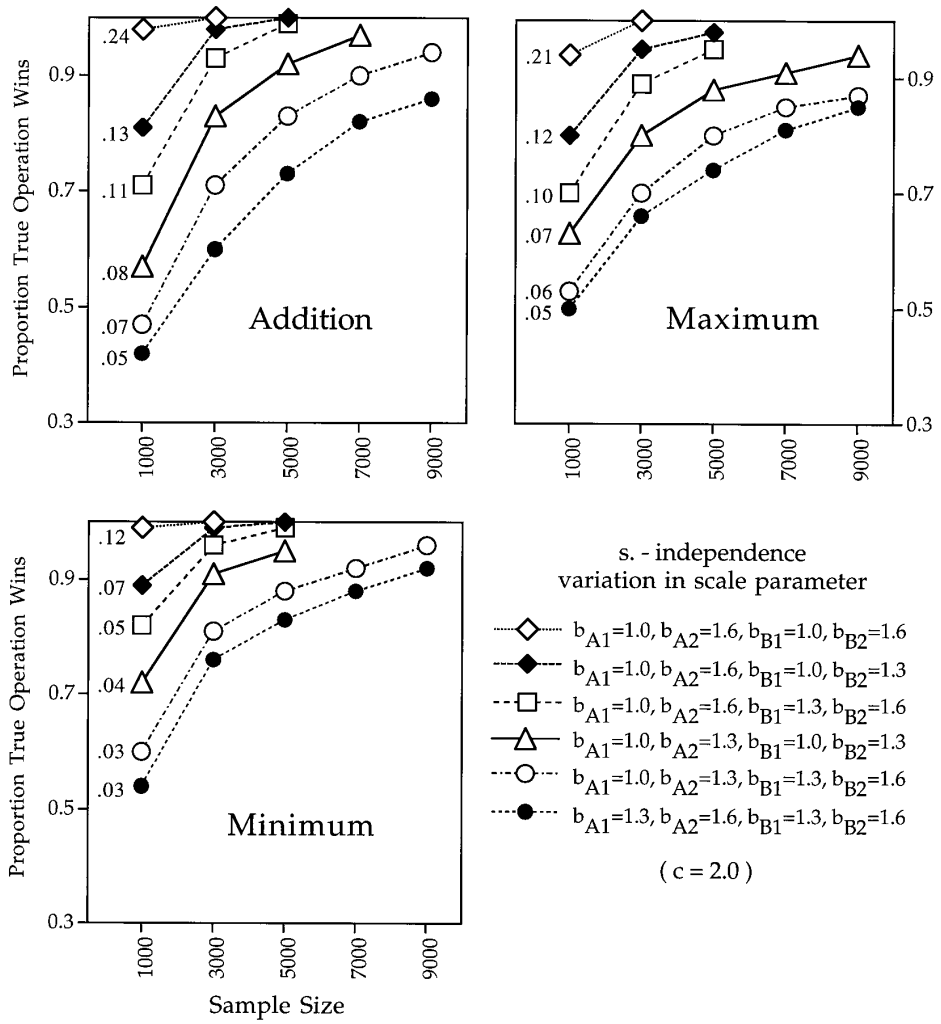

FIG. 5. Same as in Fig. 3, but for s.-independence.

instead of two "linear" pairings,

$$\{\mathbf{T}_{11}^k \diamond \mathbf{T}_{22}^k\} \quad \text{and} \quad \{\mathbf{T}_{12}^k \diamond \mathbf{T}_{21}^k\}, \quad k = 1, ..., n.$$

For smaller sample sizes (on the order of 500 observations per treatment) we directly verified that the Cartesian product scheme leads to the operation plus consistently winning over minimum and maximum,

$$\min[\mathbf{D}_+, \mathbf{D}_{\max}, \mathbf{D}_{\min}] = \mathbf{D}_+,$$

even when maximum or minimum is the true decomposition rule. It is easy to prove that this trend should disappear at much larger sample sizes, but then the Cartesian product scheme becomes computationally unmanageable; it also becomes unnecessary, because at large sample sizes a reliable discriminability is also achieved by the "linear" pairing scheme.

Another possible attempt to increase the identifiability of the true decomposition rule under s.-independence is related to the fact that for a given quadruple of random samples $\{\mathbf{T}_{ij}^1, ..., \mathbf{T}_{ij}^n\}$ $(i = 1, 2, j = 1, 2)$, one can form $(n!)^2$ different pairings

$$\{\mathbf{T}_{11}^k \diamond \mathbf{T}_{22}^{\pi_1(k)}\} \quad \text{and} \quad \{\mathbf{T}_{12}^k \diamond \mathbf{T}_{21}^{\pi_2(k)}\}, \quad k = 1, ..., n,$$

where π_1 and π_2 are two independent permutations of $\{1, ..., n\}$. The strategy used in our simulations was to base our decision on just one randomly chosen pair of such permutations. Consider, however, the following alternative: for each pair of permutations π_1 and π_2 one computes the triad

$$\{\mathbf{D}_+^{\pi_1, \pi_2}, \mathbf{D}_{\max}^{\pi_1, \pi_2}, \mathbf{D}_{\min}^{\pi_1, \pi_2}\}$$

and determines which operation corresponds to the smallest value; across all $(n!)^2$ pairs of permutations (or some large

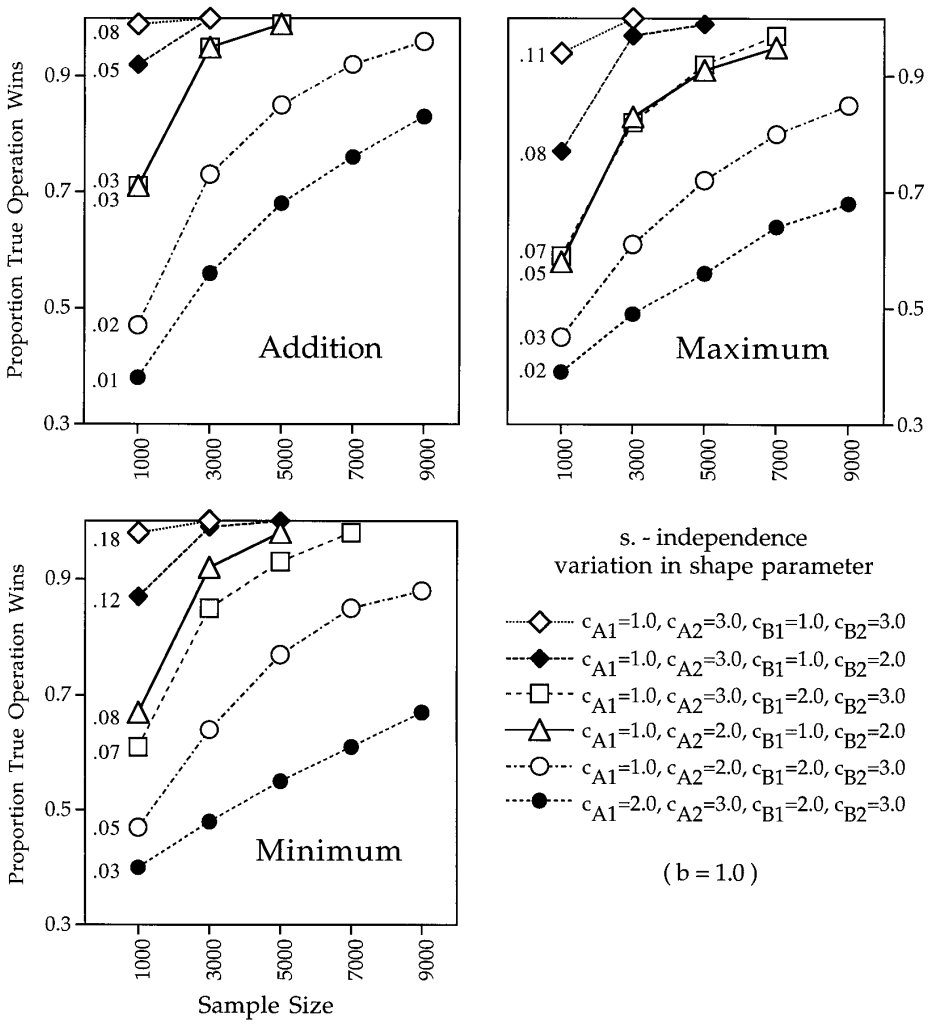

FIG. 6. Same as in Fig. 4, but for s.-independence.

random subset thereof) one computes the proportions $\mathbf{P}_+$, $\mathbf{P}_{\max}$, $\mathbf{P}_{\min}$ of times when this smallest value corresponds to operations plus, maximum, and minimum, respectively (these proportions are random variables when considered as functions of the initial random samples); the operation corresponding to the largest of these proportions is declared a winner; the whole procedure is repeated for different quadruples of samples $\{\mathbf{T}_{ij}^1, ..., \mathbf{T}_{ij}^n\}$ until one achieves a reliable estimate of

$$\mathrm{P}_{\mathrm{multiple}} = \mathrm{Prob}\{\max[\mathbf{P}_\blacklozenge, \mathbf{P}_{*1}, \mathbf{P}_{*2}] = \mathbf{P}_\blacklozenge\}.$$

We refer to this scheme as the multiple pairings one, as opposed to the single pairing scheme used in our simulations. Clearly, the random variables

$$\min\{\mathbf{D}_\blacklozenge^{\pi_1,\pi_2}, \mathbf{D}_{*1}^{\pi_1,\pi_2}, \mathbf{D}_{*2}^{\pi_1,\pi_2}\}$$

computed for different permutations but one and the same quadruple of samples

$$\{\mathbf{T}_{ij}^1, ..., \mathbf{T}_{ij}^n\}$$

are identically distributed but stochastically interdependent. Lacking a sampling distribution theory for this statistic, one cannot predict, therefore, the difference between $\mathrm{P}_{\mathrm{multiple}}$ and

$$\mathrm{P}_{\mathrm{single}} = \mathrm{Prob}\{\min[\mathbf{D}_\blacklozenge, \mathbf{P}_{*1}, \mathbf{D}_{*2}] = \mathbf{D}_\blacklozenge\}.$$

in the single pairing scheme, although it is clear that

$$\mathrm{P}_{\mathrm{multiple}} \geqslant \mathrm{P}_{\mathrm{single}}.$$

We estimated these probabilities in a Monte-Carlo simulation where the two pairing schemes were applied to

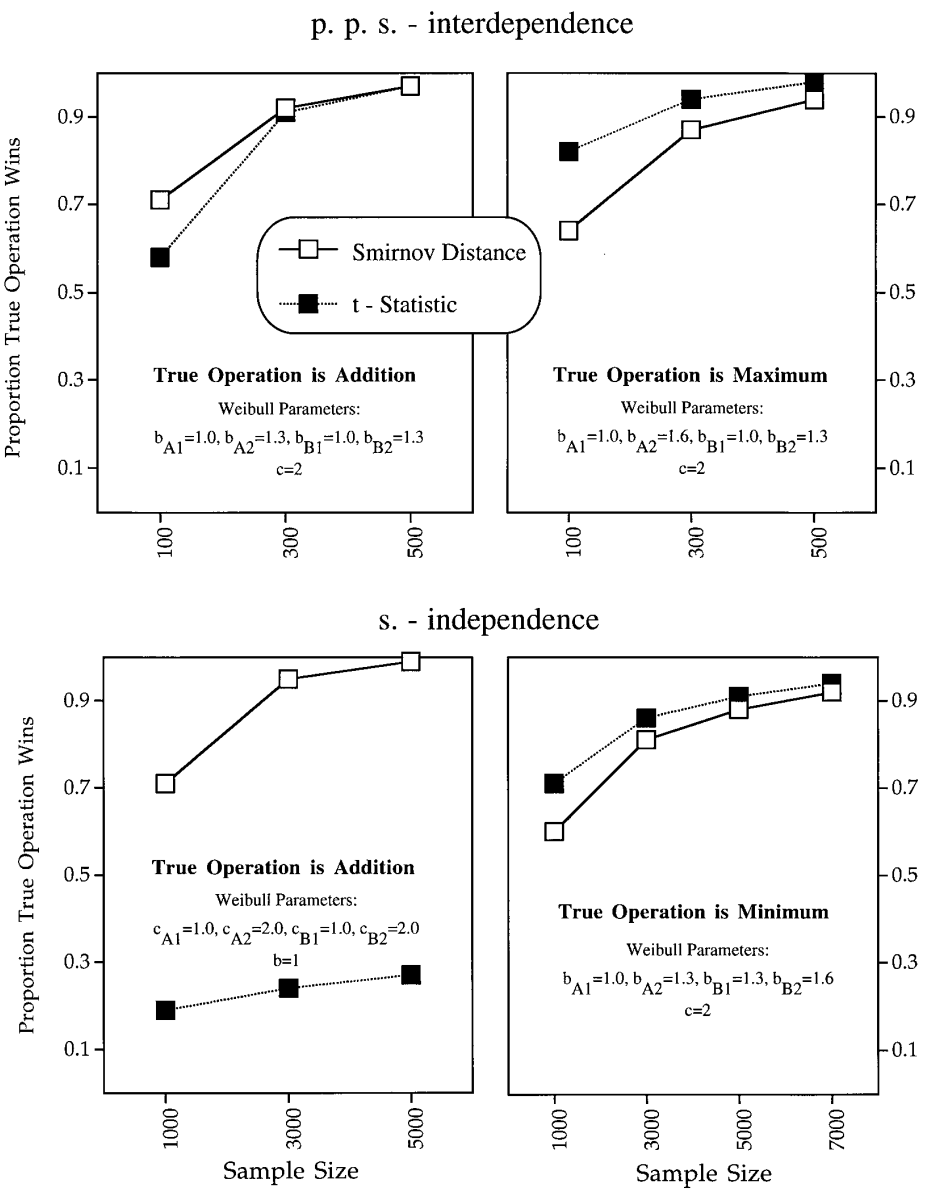

FIG. 7. Comparison of the identification probability curves obtained by means of the Smirnov distance with those obtained by means of the *t*-statistic.

RTs formed from two quadruples of component times $\mathbf{A}_1, \mathbf{B}_1, \mathbf{A}_2, \mathbf{B}_2$. The sample size in these simulations was 1000 observations/treatment. In the multiple pairings scheme, the winning operation (for a given quadruple of RT samples) was determined from 400 different pairings randomly selected from the astronomic number of $(1000!)^2$ possible pairings. The number of replications of the procedure, for both pairing schemes, was 2500, as in the main simulation study. For one of the quadruples (with Weibull parameters $b_{A1} = 1.0$, $b_{A2} = 1.3$, $b_{B1} = 1.3$, $b_{B2} = 1.6$; $c = 2.0$), the estimates were

$$P_{single} = 0.53, \quad P_{multiple} = 0.59,$$

for another (with Weibull parameters $c_{A1} = 1.0$, $c_{A2} = 3.0$, $c_{B1} = 2.0$, $c_{B2} = 3.0$; $b = 1.0$),

$$P_{single} = 0.71, \quad P_{multiple} = 0.82.$$

(The estimates for $P_{multiple}$ are reasonably reliable, in spite of being based on 400 pairings only: we found by looking at many blocks of 400 different pairings, all on the same quadruple of samples, that the proportion of the pairings within a block won by any one operation generally varies by less than 5 or 6%, and usually one operation wins an overwhelming proportion of the pairings within a block.)

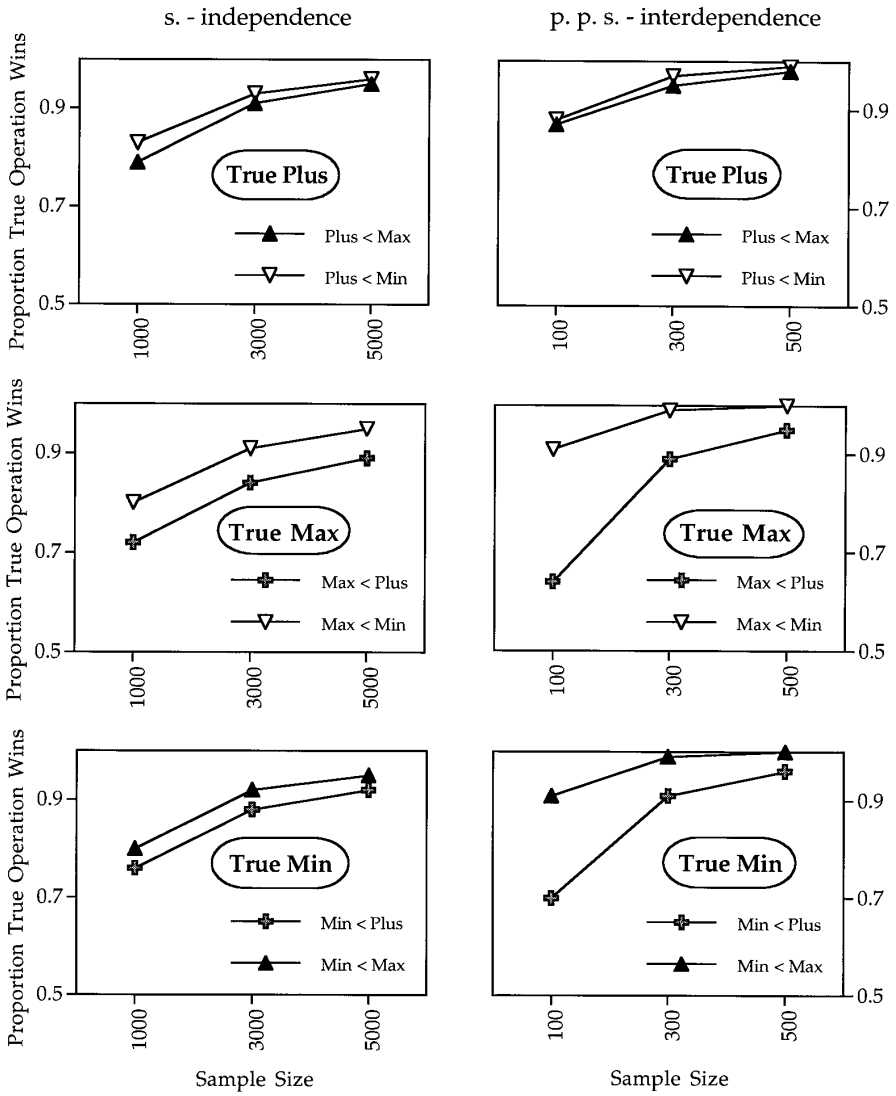

FIG. 8. Estimates of the probability with which the true decomposition rule produces a smaller Smirnov distance than each of the two competing operations in turn. The results are averaged across all quadruples of the simulated RT distributions.

These findings indicate that the gain in the identifiability provided by the multiple pairings scheme is too small to warrant the enormous increase in the computational time it entails. The use of the single pairing scheme in Monte-Carlo simulations is therefore justifiable. For an experimenter with a single data set, however, it may be relatively easy, in terms of computational time, to examine many different pairings. The estimates given in the present paper can then be viewed as lower bounds for achievable levels of discriminability.

The multiple pairings scheme is not, of course, the only way of increasing discriminability. Another possibility is to use dissimilarity measures between "cross" and "uncross" combinations of RT samples other than the Smirnov distance. These dissimilarity measures do not even have to be true distances. If, for example, one expects that the experimental factor levels strongly affect RT means, one might use the traditional t-statistic,

$$t = \frac{|\bar{t}_U - \bar{t}_C|}{\sqrt{s_U^2 + s_C^2}},$$

where the symbols represent, in the conventional way, the means and variances of the "cross" and "uncross" combinations of RT samples. (This measure is not a distance in the space of samples, as it does not satisfy the triangle inequality and it may vanish even when the samples are not identical.) To compare the performance of this measure with that of the Smirnov distance, we repeated our simulation study using the t-statistic instead. The examples chosen for Fig. 7

show that the *t*-statistic may perform very poorly, but in some cases it does increase the identification probabilities.

Returning to our main study, the results presented in Figs. 3–6 should be complemented by the estimates of

$$\text{Prob}\{\min[\mathbf{D}_{\blacklozenge}, \mathbf{D}_{*_1}] = \mathbf{D}_{\blacklozenge}\}$$

and

$$\text{Prob}\{\min[\mathbf{D}_{\blacklozenge}, \mathbf{D}_{*_2}] = \mathbf{D}_{\blacklozenge}\}$$

for the pairwise comparisons. These are presented in Fig. 8. The pairwise comparisons are too numerous to show them in the same format as the triple comparisons in Figs. 3–6. As a result, we only present the results averaged across all different quadruples of RT distributions. The main finding contained in Fig. 8 (in addition to what is already known from the analysis of the triple comparisons) is that it is generally easier to discriminate addition from minimum than to discriminate addition from maximum. This result was consistently found across a wide variety of parameters for the component distributions, but there were some exceptions: out of 72 different RT quadruples (less some cases excluded as indicated in Appendices 1–4) we found a few examples where plus was about equally discriminable from both minimum and maximum, and one case, with addition as the true decomposition rule and with parameters $c_{A1} = 2.0$, $c_{A2} = 3.0$, $c_{B1} = 2.0$, $c_{B2} = 3.0$, (under s.-independence) where a small but reliable tendency existed for addition and minimum to be less discriminable than addition and maximum. With no exceptions, however, minimum and maximum are more discriminable from each other than either of them is from addition. As shown in the introductory section, this result should be treated in the same way as the p.p.s.-interdependence superiority effect considered earlier—as an empirical finding pertaining to a broad but limited class of RT distributions.

CONCLUSION

The most important information conveyed by the results of our Monte-Carlo study is that the sample sizes required to identify the true decomposition rule among the three operations of traditional interest, addition, minimum, and maximum, are generally within the reach of a realistic, though sometimes large, experiment. This means that if a family of RTs indeed has one of the corresponding architectures (serial, parallel-OR, and parallel-AND), then one can reliably establish which one. If none of these architectures is present, then this also can be reliably established, by means of the procedure developed in Dzhafarov and Cortese (1996).

These determinations are especially easy to make under the assumption of p.p.s.-interdependence, and given the conceptual simplicity of this form of stochastic relationship, as well as the existence of some empirical support for it (Dzhafarov, 1992; Dzhafarov & Rouder, 1996), analysis of the decomposability of RTs into p.p.s.-interdependent components seems well worth the effort. Under the more traditional assumption of s.-independence, the determination of the true decomposition rule (or the determination that none of the three applies) may be more onerous a task; but only at the lowest values of the effectiveness of the index factors the identification probabilities in our study do not reach high levels (say, 0.75) at realistic sample sizes (less than 5000 observations per treatment in a 2×2 factorial design). As effectiveness of the index factors approaches zero, an experimenter may have no other option but to alter the levels of the factors in an attempt to produce greater differences between the empirical RT distributions. Ideally, the effectiveness measure should be computable directly from the quadruples of RT samples, and not from the population distribution functions as in our study. Sample-level versions of (7) and (8) can, of course, be readily proposed, but their reliability is yet to be determined in future research.

APPENDICES

Appendix 1.

Supremal distances between the component distribution functions and overall effectiveness of index factors for stochastically independent components: variation in the scale parameter *b*; the shape parameter $c = 2$.

	Scale parameter b				Distance				Effectiveness
	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{B}_1$	$\mathbf{B}_2$	d_{21}^{11}	d_{12}^{22}	d_{12}^{11}	d_{21}^{22}	
Plus	1.0	1.3	1.0	1.3	.15	.13	.15	.13	.08
	1.0	1.6	1.0	1.6	.27	.22	.27	.22	.24
	1.3	1.6	1.3	1.6	.12	.10	.12	.10	.05
	1.0	1.6	1.0	1.3	.27	.24	.15	.11	.13
	1.0	1.6	1.3	1.6	.24	.22	.13	.10	.11
	1.0	1.3	1.3	1.6	.15	.11	.13	.12	.07
Max	1.0	1.3	1.0	1.3	.15	.12	.15	.12	.07
	1.0	1.6	1.0	1.6	.28	.18	.28	.18	.21
	1.3	1.6	1.3	1.6	.12	.10	.12	.10	.05
	1.0	1.6	1.0	1.3	.28	.23	.15	.09	.12
	1.0	1.6	1.3	1.6	.23	.18	.14	.10	.10
	1.0	1.3	1.3	1.6	.15	.09	.14	.12	.06
Min	1.0	1.3	1.0	1.3	.08	.11	.08	.11	.04
	1.0	1.6	1.0	1.6	.13	.21	.13	.21	.12
	1.3	1.6	1.3	1.6	.07	.09	.07	.09	.03
	1.0	1.6	1.0	1.3	.13	.18	.08	.13	.07
	1.0	1.6	1.3	1.6	.18	.21	.05	.08	.05
	1.0	1.3	1.3	1.6	.11	.13	.05	.07	.03

Appendix 2.

Supremal distances between the component distribution functions and overall effectiveness of index factors for stochastically independent components: variation in the shape parameter c; the scale parameter $b = 1$.

	Shape parameter c				Distance				Effectiveness
	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{B}_1$	$\mathbf{B}_2$	d_{21}^{11}	d_{12}^{22}	d_{12}^{11}	d_{21}^{22}	
Plus	1.0	2.0	1.0	2.0	.08	.10	.08	.10	.03
	1.0	3.0	1.0	3.0	.12	.17	.12	.17	.08
	2.0	3.0	2.0	3.0	.05	.06	.05	.06	.01
	1.0	3.0	1.0	2.0	.12	.14	.08	.11	.05
	1.0	3.0	2.0	3.0	.08	.17	.04	.06	.03
	1.0	2.0	2.0	3.0	.10	.11	.04	.05	.02
Max	1.0	2.0	1.0	2.0	.11	.12	.11	.12	.05
	1.0	3.0	1.0	3.0	.15	.18	.15	.18	.11
	2.0	3.0	2.0	3.0	.07	.07	.07	.07	.02
	1.0	3.0	1.0	2.0	.15	.17	.11	.13	.08
	1.0	3.0	2.0	3.0	.17	.15	.16	.07	.07
	1.0	2.0	2.0	3.0	.12	.13	.06	.07	.03
Min	1.0	2.0	1.0	2.0	.13	.16	.13	.16	.08
	1.0	3.0	1.0	3.0	.18	.25	.18	.25	.18
	2.0	3.0	2.0	2.0	.08	.09	.08	.09	.03
	1.0	3.0	1.0	2.0	.18	.23	.13	.17	.12
	1.0	3.0	2.0	3.0	.23	.25	.06	.09	.07
	1.0	2.0	2.0	3.0	.16	.17	.06	.08	.05

Appendix 3.

Supremal distances between the component distribution functions and overall effectiveness of index factors for perfectly positively stochastically interdependent components: variation in the scale parameter b; the shape parameter $c = 2$.

	Scale parameter b				Distance				Effectiveness
	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{B}_1$	$\mathbf{B}_2$	d_{21}^{11}	d_{12}^{22}	d_{12}^{11}	d_{21}^{22}	
Plus	1.0	1.3	1.0	1.3	.10	.09	.10	.09	.09
	1.0	1.6	1.0	1.6	.19	.15	.19	.15	.19
	1.3	1.6	1.3	1.6	.08	.07	.08	.07	.08
	1.0	1.6	1.0	1.3	.19	.17	.10	.08	.10
	1.0	1.6	1.3	1.6	.17	.15	.09	.07	.10
	1.0	1.3	1.3	1.6	.09	.08	.09	.08	.09
Max	1.0	1.3	1.0	1.3	.19	.00	.19	.00	.19
	1.0	1.6	1.0	1.6	.33	.00	.33	.00	.33
	1.3	1.6	1.3	1.6	.15	.00	.15	.00	.15
	1.0	1.6	1.0	1.3	.33	.15	.19	.00	.19
	1.0	1.6	1.3	1.6	(Duplicates the case for parameters 1.3, 1.6, 1.3, 1.6)				
	1.0	1.3	1.3	1.6	(Excluded case: Factor B ineffective)				
Min	1.0	1.3	1.0	1.3	.00	.19	.00	.19	.19
	1.0	1.6	1.0	1.6	.00	.33	.00	.33	.33
	1.3	1.6	1.3	1.6	.00	.15	.00	.05	.15
	1.0	1.6	1.0	1.3	(Duplicates the case for parameters 1.0, 1.3, 1.0, 1.3)				
	1.0	1.6	1.3	1.6	.19	.33	.00	.15	.15
	1.0	1.3	1.3	1.6	(Excluded case: Factor A ineffective)				

Appendix 4.

Supremal distances between the component distribution functions and overall effectiveness of index factors for perfectly positively stochastically interdependent components: variation in the shape parameter c; the scale parameter $b = 1$.

	Scale parameter b				Distance				Effectiveness
	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{B}_1$	$\mathbf{B}_2$	d_{21}^{11}	d_{12}^{22}	d_{12}^{11}	d_{21}^{22}	
Plus	1.0	2.0	1.0	2.0	.10	.10	.10	.10	.10
	1.0	3.0	1.0	3.0	.15	.16	.15	.16	.16
	2.0	3.0	2.0	3.0	.05	.06	.05	.06	.06
	1.0	3.0	1.0	2.0	.15	.15	.10	.11	.11
	1.0	3.0	2.0	3.0	.15	.16	.06	.06	.06
	1.0	2.0	2.0	3.0	.10	.11	.06	.05	.06
Max	1.0	2.0	1.0	2.0	.18	.13	.18	.13	.17
	1.0	3.0	1.0	3.0	.28	.19	.28	.19	.28
	2.0	3.0	2.0	3.0	.11	.08	.11	.08	.11
	1.0	3.0	1.0	2.0	.28	.13	.18	.13	.18
	1.0	3.0	2.0	3.0	.13	.19	.11	.08	.11
	1.0	2.0	2.0	3.0	(Excluded case: Cross-over rearrangements)				
Min	1.0	2.0	1.0	2.0	.13	.18	.13	.18	.18
	1.0	3.0	1.0	3.0	.19	.28	.19	.28	.28
	2.0	3.0	2.0	3.0	.08	.11	.08	.11	.11
	1.0	3.0	1.0	2.0	.19	.18	.13	.18	.18
	1.0	3.0	2.0	3.0	.18	.28	.08	.11	.11
1.0	2.0	2.0	3.0	(Excluded case: Cross-over rearrangements)					

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REFERENCES

Dzhafarov, E. N. (1992). The structure of simple reaction time to step-function signals. *Journal of Mathematical Psychology*, **36**, 235–268.

Dzhafarov, E. N. (1993). Grice-representability of response time distribution families. *Psychometrika*, **58**, 281–314.

Dzhafarov, E. N., & Cortese, J. M. (1996). Empirical recovery of response time decomposition rules, I: Sample-level Decomposition tests. *Journal of Mathematical Psychology* **40**, 185–202.

Dzhafarov, E. N., & Rouder, J. N. (1996). Empirical discriminability of two models for stochastic relationship between additive components of response time. *Journal of Mathematical Psychology*, **40**, 48–63.

Dzhafarov, E. N., & Schweickert, R. (1995). Decompositions of response times: An almost general theory. *Journal of Mathematical Psychology*, **39**, 285–314.

Hohle, R. H. (1965). Inferred components of reaction times as functions of foreperiod duration. *Journal of Experimental Psychology*, **69**, 381–386.

Luce, R. D. (1986). *Response times*: *Their role in inferring elementary mental organization*. New York: Oxford Univ. Press.

Maloney, L. T., & Wandel, B. A. (1984). A model of a single visual channel's response to weak test lights. *Vision Research*, **24**, 633–640.

Massaro, D. W., & Cowan, N. (1993). Information processing models: Microscopes of the mind. In L. W. Porter & M. R. Rosenzweig (Eds.), *Annual review of psychology*. Palo Alto: Annual Reviews, Inc.

McGill, W. J. (1963). Stochastic latency mechanisms. In R. D. Luce, R. R. Bush, & E. Galanter (Eds.), *Handbook of Mathematical Psychology*, Vol. 1, pp. 309–360. New York: Wiley.

Nozawa, G. (1992). *Stochastic models of human information processing*. Unpublished doctoral dissertation, Dartmouth College, Hanover, NH.

Ratcliff, R. A. (1978). A theory of memory retrieval. *Psychological Review*, **85**, 59–108.

Roberts, S., & Sternberg, S. (1992). The meaning of additive reaction-time effects: Tests of three alternatives. In D. E. Meyer and S. Kornblum (Eds.), *Attention and performance XIV*, pp. 611–654. Cambridge, MA: MIT Press.

Schweickert, R. (1978). A critical path generalization of the additive factor method: Analysis of a Stroop task. *Journal of Mathematical Psychology*, **18**, 105–139.

Schweickert, R. (1992). Information, time, and the structure of mental events: A twenty-five year review. In D. E. Meyer and S. Kornblum (Eds.), *Attention and performance XIV*, pp. 535–566. Cambridge, MA: MIT Press.

Townsend, J. T. (1990). Serial vs. parallel processing: Sometimes they look like tweeledum and tweedledee but they can (and should) be distinguished. *Psychological Science*, **1**, 46–54.

Townsend, J. T., & Ashby, F. G. (1983). *Stochastic modeling of elementary psychological processes*. Cambridge, England: Cambridge Univ. Press.

Townsend, J. T., & Nozawa, G. (1988). *Strong evidence for parallel processing with simple dot stimuli*. Paper presented at Twenty-Ninth Annual Meeting of the Psychonomic Society, Chicago, IL.

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