Dissimilarity Cumulation Theory in Arc-Connected Spaces

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Abstract

This paper continues the development of the Dissimilarity Cumulation theory and its main psychological application, Universal Fechnerian Scaling (Dzhafarov & Colonius, 2007). In arc-connected spaces the notion of a chain length (the sum of the dissimilarities between the chain's successive elements) can be used to define the notion of a path length, as the limit inferior of the lengths of chains converging to the path in some well-defined sense. The class of converging chains is broader than that of converging inscribed chains. Most of the fundamental results of the metric-based path length theory (additivity, lower semicontinuity, etc.) turn out to hold in the general dissimilarity-based path length theory. This shows that the triangle inequality and symmetry are not essential for these results, provided one goes beyond the traditional scheme of approximating paths by inscribed chains. We introduce the notion of a space with intermediate points which generalizes (and specializes to when the dissimilarity is a metric) the notion of a convex space in the sense of Menger. A space is with intermediate points if for any distinct $\mathbf{a}, \mathbf{b}$ there is a different point $\mathbf{m}$ such that $D\mathbf{am} + D\mathbf{mb} \leq D\mathbf{ab}$ (where D is dissimilarity). In such spaces the metric G induced by D is intrinsic: $G\mathbf{ab}$ coincides with the infimum of lengths of all arcs connecting $\mathbf{a}$ to $\mathbf{b}$. In Universal Fechnerian Scaling D stands for either of the two canonical psychometric increments $\psi\mathbf{ab} - \psi\mathbf{aa}$ and $\psi\mathbf{ba} - \psi\mathbf{aa}$ (ψ denoting discrimination probability). The choice between the two makes no difference for the notions of arc-connectedness, convergence of chains and paths, intermediate points, and other notions of the Dissimilarity Cumulation theory.

KEYWORDS: arc length, convexity, dissimilarity, discrimination probability, Fechnerian Scaling, Menger-convex space, oriented distance, path length, Regular Minimality, same-different judgements, stimulus space.

1. Introduction

This paper continues a systematic development of the *Dissimilarity Cumulation* (DC) theory and its main psychological application, *Universal Fechnerian Scaling* (UFS). Both were introduced in Dzhafarov and Colonius (2007) and represent an extension as well as significant streamlining of a set of ideas previously

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presented under the rubric of *Generalized Fechnerian Scaling* (Dzhafarov, 2001a-b, 2002a-d, 2003a-b, 2004, 2006; Dzhafarov & Colonius, 1999a-b, 2001, 2005a-b, 2006a-c). A familiarity with Dzhafarov and Colonius (2007) may be helpful in understanding the present paper, although the main definitions and findings will be recapitulated below.

Fechnerian Scaling was initially built on the notion of *arc length* for stimulus spaces representable by regions in Euclidean n-space (Dzhafarov & Colonius, 1999a, 2001), such as the CIE or Munsell color systems, a frequency-amplitude space of tones, or a space of weights placed on one's palm. This theory, called *Multidimensional Fechnerian Scaling*, underwent foundational changes in Dzhafarov (2002d, 2003a), on the introduction of the notions of distinct observation areas, psychometric increments of two kinds, and the law of Regular Minimality (see Dzhafarov, 2006; Dzhafarov & Colonius, 2006a). The derivation of Fechnerian ("subjective") distances from the arc length, however, remained unchanged, even though in Dzhafarov and Colonius (2005a) the arc-length-based computations were extended to a broader class of arc-connected spaces.

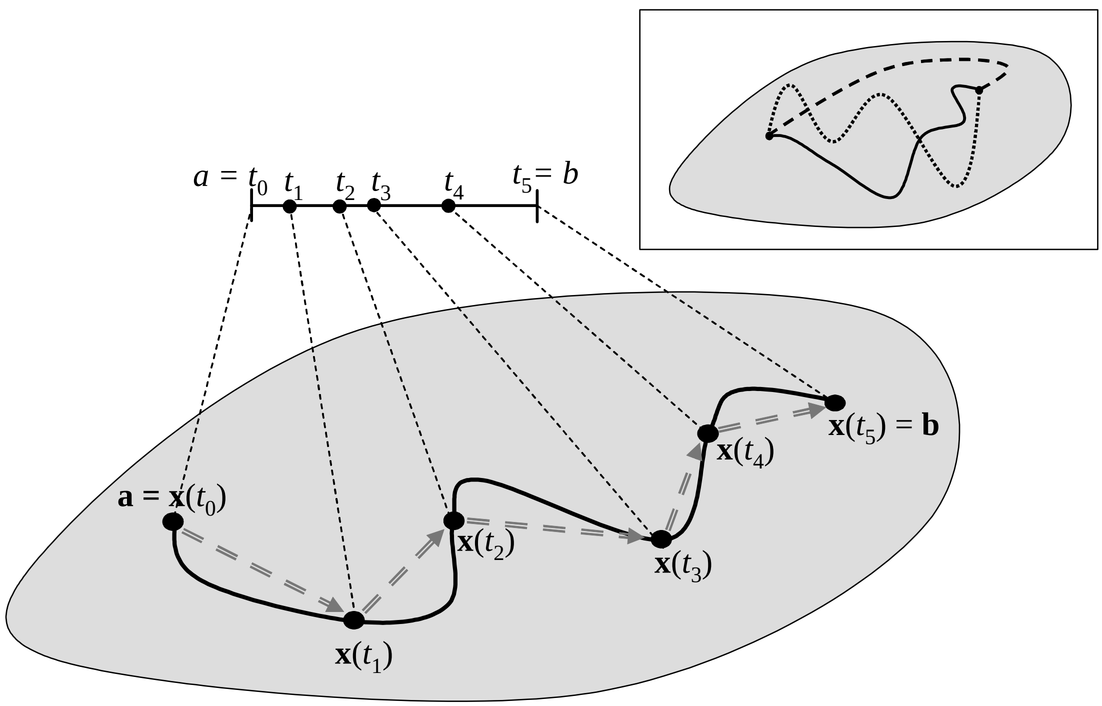

Figure 1. Under several assumptions stipulated in Dzhafarov and Colonius (2005a) about the type of stimulus space and the type of paths connecting **a** to **b**, the *length* of such a path is approximated by the sum of the *dissimilarities* between successive points $\mathbf{x}(t_0), \mathbf{x}(t_1), \ldots, \mathbf{x}(t_k)$ leading from **a** to **b**. As the partition $t_0, t_1, \ldots t_k$ gets denser, the cumulated dissimilarity tends to the length of the path. The inset indicates that one has to compute the lengths for all paths (of the mentioned special type) connecting **a** to **b**, the infimum of these lengths being taken for the *oriented distance* from **a** to **b** (called so because it is not generally symmetrical). The *symmetrical distance* between **a** and **b** is obtained by adding the oriented distances from **a** to **b** and from **b** to **a.** In the theory of Generalized Fechnerian Scaling, the dissimilarity between two elements is defined by *psychometric increments* (as explained in Section 3) computed from probabilities with which two stimuli are judged to be different.

Figure 1 illustrates certain features of these computations, those relevant to the present paper. Paths are continuous mappings of intervals of reals into stimulus spaces. The "special type of paths" mentioned in the legend refers to *smooth* paths, for which the computation of length can be performed by means of Riemann integration, as one does in Finsler geometry and variational calculus (Dzhafarov & Colonius, 1999a, 2001).

In Dzhafarov and Colonius (2005b, 2006b-c) we introduced a different variant of Fechnerian Scaling, aimed at *discrete spaces*, such as a space of color names or other categories (see Dzhafarov & Colonius, 2006a, for the meaning of the discrimination probabilities in this kind of spaces). A space is discrete if for every point $\mathbf{a}$ in it there is an $\alpha > 0$ such that the dissimilarity from $\mathbf{a}$ to any other element of the space does not fall below α. Figure 2 illustrates the computation of Fechnerian distances in discrete spaces.

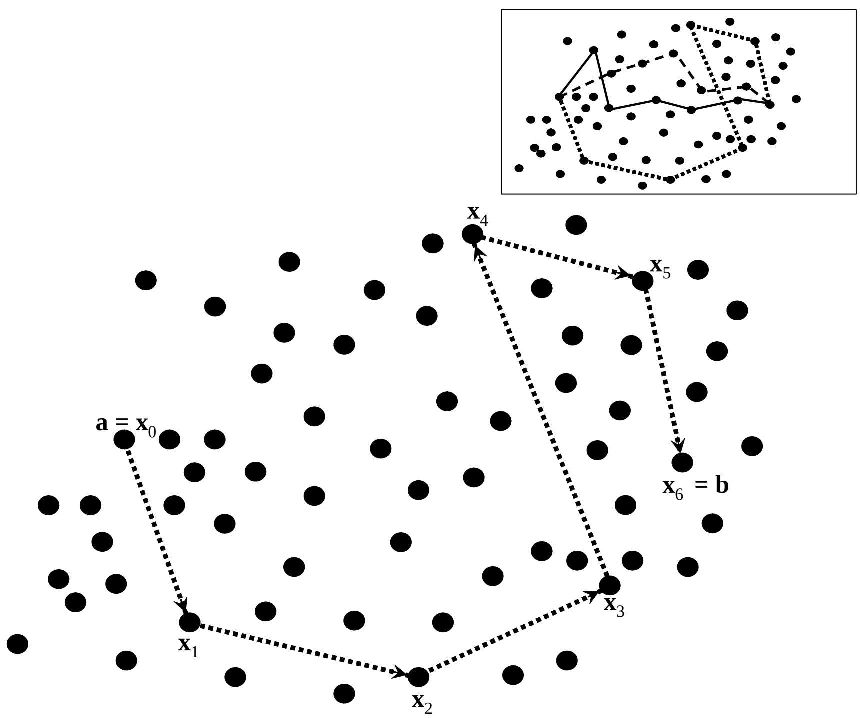

Figure 2. Given a chain of points $\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_k$ leading from $\mathbf{a}$ to $\mathbf{b}$, the dissimilarities between its successive elements are summed (cumulated). In a discrete space, the oriented distance from $\mathbf{a}$ to $\mathbf{b}$ is computed as the infimum of the cumulated dissimilarities over all chains leading from $\mathbf{a}$ to $\mathbf{b}$. The symmetrical distance between $\mathbf{a}$ and $\mathbf{b}$ is computed as in the arc-connected spaces (Fig. 1).

In DC, the new mathematical foundation of Fechnerian Scaling (Dzhafarov & Colonius, 2007), the computation of Fechnerian distances follows the logic of that in discrete spaces but is applied to spaces of entirely arbitrary nature. One considers all possible finite chains of points leading from $\mathbf{a}$ to $\mathbf{b}$ and takes the infimum of their cumulated dissimilarities to be the oriented distance from $\mathbf{a}$ to $\mathbf{b}$. Adding together the oriented distances "to and from" one gets the overall (symmetric) Fechnerian distance. This computation is universally applicable, provided the dissimilarity function is properly defined (i.e., satisfies the four axioms stipulated in Section 2). The notion of a dissimilarity function is a specially constructed generalization of an oriented metric: in particular, it does not have to satisfy the triangle inequality.

It is apparent from this brief description that even if the stimulus space under consideration is arc-connected, the basic logic of the computations in the DC theory does not involve continuous paths. The latter are nevertheless of considerable interest. In the conceptually simplest case, when the dissimilarity D is a metric, oriented or symmetric, for every pair of points $\mathbf{a}, \mathbf{b}$ there may exist a sequence of paths from $\mathbf{a}$ to $\mathbf{b}$ whose lengths converge to the distance from $\mathbf{a}$ to $\mathbf{b}$. In the traditional terminology this would mean that the metric D is an *intrinsic metric* (also called *inner*, or *internal*; see Dzhafarov, 2002b). If D is not a metric, it may happen that any sequence of chains of points leading from $\mathbf{a}$ to $\mathbf{b}$, such that their lengths converge to the distance from $\mathbf{a}$ to $\mathbf{b}$, should have a progressively increasing number of links and gradually vanishing dissimilarities between successive elements (see Fig. 3). Intuitively, this situation suggests substituting continuous paths for finite chains in the computation of metrics from dissimilarities (as discussed in Section 7).

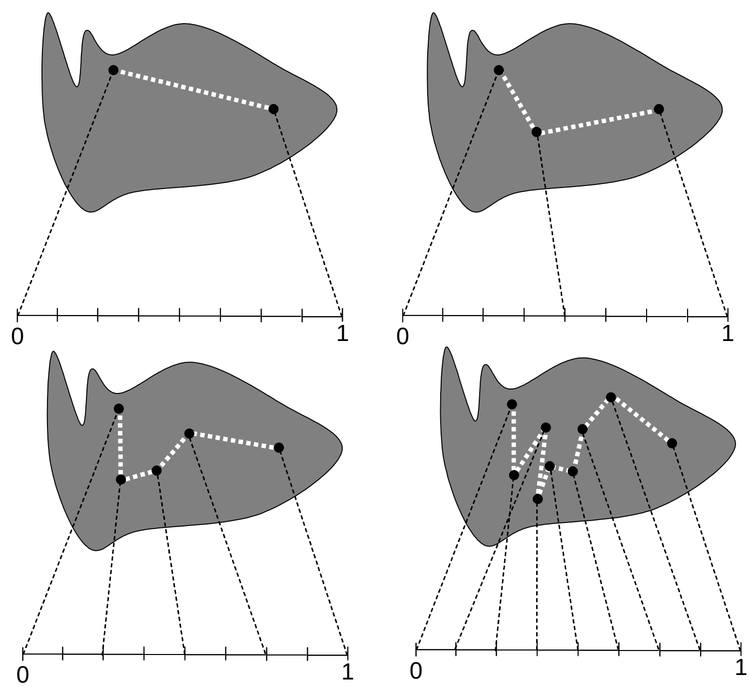

Figure 3. A space in which a sequence of chains from $\mathbf{a}$ to $\mathbf{b}$ whose cumulated dissimilarities converge to the oriented distance from $\mathbf{a}$ to $\mathbf{b}$ has to have the number of elements increasing beyond bounds and the dissimilarity between successive elements converging to zero.

The following two questions arise therefore, to be addressed in the present paper.

1. Can a comprehensive and general theory of path length be constructed based on the notion of dissimilarity rather than that of metric? We answer this question in the affirmative, while imposing no restrictions on paths except for their existence. The basic properties of the metric-based classical theory of path length (as found, e.g., in Blumenthal, 1953; Blumenthal & Menger, 1970; Busemann,

2005) turn out to be preserved in the dissimilarity-based theory, the properties including the lower semicontinuity of length in a sequence of converging paths, the continuity of a path's length along the path, the additivity of path length for concatenated paths, the excess of a path length over the length of an arc it contains, and others. This is a significant finding from a mathematical point of view, as it demonstrates that the triangle inequality and symmetry properties of a metric, prominently used in the proofs of the classical theory, are not essential. To achieve this generalization, however, one has to abandon the scheme of approximating paths by inscribed chains, such as we see in Fig. 1. This scheme has to be substituted for by a more general one in which a path's length is defined as the limit inferior for the length of chains converging to the path in some well specified meaning. This computation properly specializes to the classical one when the dissimilarity is a metric: then (although not only then) the consideration can be confined to the inscribed chains only.[1]

2. What (not overly restrictive) conditions can be imposed on a space endowed with a dissimilarity function for the oriented distance from one point to another to be computable as the infimum of the lengths for all paths connecting these points? Put differently, in what kind of spaces the metrics computed in accordance with the DC theory are intrinsic? We will see that this is the case in *complete spaces with intermediate points*, which generalize the *complete convex spaces* introduced by K. Menger for the classical, metric-based theory (see Blumenthal, 1953; Busemann, 2005; we follow Papadopoulos, 2005, in calling this notion of convexity *Menger convexity*). Again, it is significant from a mathematical point of view that the theory of Menger convexity can be generalized beyond the use of the triangle inequality and symmetry.

A terminological note. The terms DC and UFS are by no means interchangeable. We use the term DC (Dissimilarity Cumulation) to refer to the abstract mathematical theory of dissimilarity functions. UFS (Universal Fechnerian Scaling) is the main psychological application of the DC theory, obtained by positing that psychometric increments of the first and second kind (see Sections 3 and 8) are dissimilarity functions. This paper's primary focus is on the mathematical details of how the general DC theory specializes to arc-connected spaces. The applications of this development to UFS are textually separated. For a detailed account of how UFS motivates the theoretical (and terminological) choices made in DC the reader is referred to Dzhafarov and Colonius (2007). UFS is not the only possible application of DC: as stated in Dzhafarov and Colonius (2007), one may reasonably hypothesize that many "dissimilarity-type" measures, in particular those used in Cluster Analysis and Multidimensional Scaling, are dissimilarity functions too.

[1] Another class of situations where this is possible includes "smooth" paths: these are considered in the follow-up paper (Dzhafarov, 2007).

1.1. Plan of the Paper and Notation

We will begin (in Sections 2 and 3) with a brief and by necessity schematic recapitulation of the main definitions and findings presented in Dzhafarov and Colonius (2007): the reader should refer to that paper for justifications, explanations, and examples.[2] In Sections 4, 5, and 6 we define and establish basic properties of the path length as derived from the notion of dissimilarity. In Section 7 we introduce the spaces with intermediate points and establish the intrinsicality of the induced metrics in such spaces. In Section 8 we translate the notions of DC into the language of UFS.

We will observe the notation conventions adopted in Dzhafarov and Colonius (2007). The target sets in DC (interpreted as stimulus sets in UFS) are denoted by Gothic letters, $\mathfrak{S}$, $\mathfrak{A}$, $\mathfrak{s}$, ...; their elements (interpreted as stimuli in UFS) are denoted by boldface lowercase letters, $\mathbf{a}$, $\mathbf{b}'$, $\mathbf{x}$, $\mathbf{y}_n$, and so on. Functions whose values lie in the target (stimulus) sets are also denoted by boldface lowercase letters, $\mathbf{f}$, $\mathbf{g}^*$, $\mathbf{f}_n$, and so on.

Chains are finite sequences of points (stimuli) presented as strings, $\mathbf{x}_1 \ldots \mathbf{x}_k$, k being referred to as the chain's *cardinality*. Chains are often denoted by uppercase boldface letters, $\mathbf{X}$, $\mathbf{Y}_n$, and so on. The cardinality of $\mathbf{X}$ is denoted by $|\mathbf{X}|$. If $\mathbf{X} = \mathbf{x}_1 \ldots \mathbf{x}_k$, $\mathbf{Y} = \mathbf{y}_1 \ldots \mathbf{y}_l$, then $\mathbf{XY} = \mathbf{x}_1 \ldots \mathbf{x}_k \mathbf{y}_1 \ldots \mathbf{y}_l$, appropriately renumbered. In particular, $\mathbf{aXb}$ is a chain connecting $\mathbf{a}$ to $\mathbf{b}$.

A real-valued function of two or more points (stimuli) is indicated by a symbol for the function followed by a string of points without parentheses: $\psi\mathbf{ab}$, $D\mathbf{abc}$, $D\mathbf{X}_n$, $\Psi^{(\iota)}\mathbf{ab}$, and so on.

If $f\mathbf{ab}$ is defined for some function f, then $f\mathbf{abc}\ldots y\mathbf{z}$ is always understood as

$$f\mathbf{ab} + f\mathbf{bc} + \ldots + f\mathbf{yz}.$$

Thus, for $\mathbf{X} = \mathbf{x}_1 \ldots \mathbf{x}_k$,

$$\begin{aligned} D\mathbf{X} &= \sum_{i=1}^{k-1} D\mathbf{x}_i\mathbf{x}_{i+1}, \\ D\mathbf{aXb} &= D\mathbf{ax}_1 + D\mathbf{x}_k\mathbf{b} + \sum_{i=1}^{k-1} D\mathbf{x}_i\mathbf{x}_{i+1}. \end{aligned}$$

Infinite sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{\mathbf{x}_n\}_{n\in\mathbb{N}}$, $\{\mathbf{X}_n\}_{n\in\mathbb{N}}$, ... are indicated by their generic elements: sequence x_n, sequence $\mathbf{x}_n$, etc. We use the square-bracket notation for intervals of reals (closed, open and half-open): $[a,b]$, $[a,b[$, $]a,b]$, and $]a,b[$.

We also introduce the following convention. If the codomain $\mathfrak{S}$ for a class of functions is fixed, a function $\mathbf{f} : A \mapsto \mathfrak{S}$, where A is some set, can be denoted by $\mathbf{f}|A$. For a subset B of A, the specialization of $\mathbf{f}|A$ to B is denoted by $\mathbf{f}|B$.

[2] For broader psychological context, the reader is also referred to Dzhafarov & Colonius (2006a), and the introductory part of Dzhafarov & Colonius (2005a).

2. Basics of DC

Given an arbitrary set $\mathfrak{S}$, a function

$$D : \mathfrak{S} \times \mathfrak{S} \mapsto \mathbb{R}$$

is called a (uniform) dissimilarity function if it satisfies the following properties:

$\mathcal{D}1$. $\mathbf{a} \neq \mathbf{b} \Longrightarrow D\mathbf{ab} > 0$;

$\mathcal{D}2$. $D\mathbf{aa} = 0$;

$\mathcal{D}3$. (Uniform Continuity) if $D\mathbf{a}_n\mathbf{a}'_n \to 0$ and $D\mathbf{b}_n\mathbf{b}'_n \to 0$, then $D\mathbf{a}'_n\mathbf{b}'_n - D\mathbf{a}_n\mathbf{b}_n \to 0$;

$\mathcal{D}4$. for any sequence of chains $\mathbf{a}_n\mathbf{X}_n\mathbf{b}_n$,

$$D\mathbf{a}_n\mathbf{X}_n\mathbf{b}_n \to 0 \Longrightarrow D\mathbf{a}_n\mathbf{b}_n \to 0.$$

(Refer to Section 1.1 for notation conventions.)

A function

$$M : \mathfrak{S} \times \mathfrak{S} \mapsto \mathbb{R}$$

is an *oriented metric* (or simply *metric*, if confusion is unlikely) if $M\mathbf{ab}$ is nonnegative, vanishing if and only if $\mathbf{a} = \mathbf{b}$, and satisfying the triangle inequality,

$$M\mathbf{ac} \leq M\mathbf{ab} + M\mathbf{bc}.$$

If, in addition, M is symmetric,

$$M\mathbf{ab} = M\mathbf{ba},$$

it is called a *symmetric metric*.[3]

We now list the main mathematical facts established in Dzhafarov and Colonius (2007).

PROPOSITION 1 If M is a symmetric metric, it is a dissimilarity function. If M is an oriented metric, then it is a dissimilarity function if and only if

$$M\mathbf{a}_n\mathbf{a}'_n \to 0 \Longrightarrow M\mathbf{a}'_n\mathbf{a}_n \to 0.$$

All remaining propositions in our list are predicated on the assumption that D is a dissimilarity function on $\mathfrak{S}$.

For any chain $\mathbf{X}$, the cumulated dissimilarity $D\mathbf{X}$ is referred to as the *D-length* of this chain. In particular, $D\mathbf{aXb}$ is the D-length of a chain connecting $\mathbf{a}$ to $\mathbf{b}$. We define

$$\begin{aligned} G\mathbf{ab} &= \inf_{\mathbf{X}} D\mathbf{aXb}, \\ G^*\mathbf{ab} &= G\mathbf{ab} + G\mathbf{ba}. \end{aligned} \tag{1}$$

[3] Traditionally, the term *metric* is used to mean a symmetric metric. Our use of the term is closer to that in Finsler geometry (historically, the initial theoretical framework for Generalized Fechnerian Scaling, see Dzhafarov & Colonius, 1999a-b) where the symmetry requirement is often dropped.

PROPOSITION 2 $G\mathbf{ab}$ is an oriented metric, and $G^*\mathbf{ab}$ is a symmetric metric (also called "overall").

PROPOSITION 3 G is a dissimilarity function. (G^* is a dissimilarity function trivially, as any symmetric metric.)

We define $\mathbf{a}_n \leftrightarrow \mathbf{b}_n$ as meaning $D\mathbf{a}_n\mathbf{b}_n \to 0$.

PROPOSITION 4 The convergence $\leftrightarrow$ is an equivalence relation (i.e., it is reflexive, symmetric, and transitive).

A dissimilarity D and the metric G derived from it induce on $\mathfrak{S}$ one and the same topology and uniformity with respect to which both D and G are (uniformly) continuous.

PROPOSITION 5 Dissimilarity D induces on $\mathfrak{S}$ a topology based on open sets

$$\mathfrak{B}_D(\mathbf{x},\varepsilon) = \{\mathbf{y} \in \mathfrak{S} : D\mathbf{xy} < \varepsilon\}$$

taken for all $\mathbf{x} \in \mathfrak{S}$ and all real $\varepsilon > 0$. (D is continuous with respect to this topology.)

PROPOSITION 6 Dissimilarity D induces on $\mathfrak{S}$ a uniformity based on entourages

$$\mathfrak{U}_D(\varepsilon) = \left\{(\mathbf{x},\mathbf{y}) \in \mathfrak{S}^2 : D\mathbf{xy} < \varepsilon\right\}$$

taken for all real $\varepsilon > 0$. (D is uniformly continuous with respect to this uniformity.)

PROPOSITION 7 $D\mathbf{a}_n\mathbf{b}_n \to 0 \iff G\mathbf{a}_n\mathbf{b}_n \to 0$. The topology (uniformity) induced on $\mathfrak{S}$ by G coincides with the topology (uniformity) induced on $\mathfrak{S}$ by D.

PROPOSITION 8 Space $(\mathfrak{S}, D)$, or equivalently $(\mathfrak{S}, G)$, is topologically a completely normal space: its singletons are closed and any two separated subsets $\mathfrak{A}$ and $\mathfrak{B}$ (i.e., such that $\overline{\mathfrak{A}} \cap \mathfrak{B} = \mathfrak{A} \cap \overline{\mathfrak{B}} = \varnothing$) are contained in two disjoint open subsets. It follows that $(\mathfrak{S}, D)$ is Urysohn and Hausdorff.

PROPOSITION 9 $G\mathbf{ab}$ is uniformly continuous in $(\mathbf{a},\mathbf{b})$, i.e., if $\mathbf{a}'_n \leftrightarrow \mathbf{a}_n$ and $\mathbf{b}'_n \leftrightarrow \mathbf{b}_n$, then $G\mathbf{a}'_n\mathbf{b}'_n - G\mathbf{a}_n\mathbf{b}_n \to 0$.

Note that the uniform continuity of D itself is postulated as Property $\mathcal{D}3$.

For the last proposition in our list recall that according to our notational conventions, if $\mathbf{X} = \mathbf{x}_1 \ldots \mathbf{x}_k$,

$$G\mathbf{aXb} = G\mathbf{ax}_1 + G\mathbf{x}_k\mathbf{b} + \sum_{i=1}^{k-1} G\mathbf{x}_i\mathbf{x}_{i+1}.$$

PROPOSITION 10 $D\mathbf{aX}_n\mathbf{b} \to G\mathbf{ab} \Longrightarrow G\mathbf{aX}_n\mathbf{b} \to G\mathbf{ab}$.

3. Basics of UFS

Here, we recapitulate the basic notions and mathematical facts pertaining to UFS (Dzhafarov & Colonius, 2007), including the motivation for defining the symmetric metric $G^*\mathbf{ab}$ as the sum $G\mathbf{ab}+G\mathbf{ba}$. The reader primarily interested in the abstract mathematical development of DC may skip this section and proceed to Section4.

Two stimuli being compared formally belong to two different sets, $\mathfrak{S}_1$ and $\mathfrak{S}_2$, referred to as *observation areas* (e.g., the stimuli presented on the left and those presented on the right). The *discrimination probability function*

$$\tilde{\psi}\mathbf{xy} = \Pr\left[\mathbf{x} \text{ and } \mathbf{y} \text{ are judged to be different}\right] \tag{2}$$

is therefore

$$\mathfrak{S}_1 \times \mathfrak{S}_2 \mapsto [0,1]\,.$$

UFS is a *purely psychological theory*, in the following sense: $\mathbf{x} \in \mathfrak{S}_1$ is entirely characterized by the function $\mathbf{y} \mapsto \tilde{\psi}\mathbf{xy}$, while $\mathbf{y} \in \mathfrak{S}_2$ is entirely characterized by $\mathbf{x} \mapsto \tilde{\psi}\mathbf{xy}$. In other words, physical descriptions of stimuli are irrelevant, and

$$\begin{array}{l}\text{if } \tilde{\psi}\mathbf{ay} = \tilde{\psi}\mathbf{by}, \text{ for all } \mathbf{y}, \text{ then } \mathbf{a} = \mathbf{b},\\ \text{if } \tilde{\psi}\mathbf{xa} = \tilde{\psi}\mathbf{xb}, \text{ for all } \mathbf{x}, \text{ then } \mathbf{a} = \mathbf{b}.\end{array} \tag{3}$$

The law of Regular Minimality says that

$$\mathbf{h}\left(\mathbf{x}\right) = \arg\min_{\mathbf{y}} \tilde{\psi}\mathbf{xy} \tag{4}$$

and

$$\mathbf{g}\left(\mathbf{y}\right) = \arg\min_{\mathbf{x}} \tilde{\psi}\mathbf{xy} \tag{5}$$

are well-defined functions, and

$$\mathbf{h} \equiv \mathbf{g}^{-1}. \tag{6}$$

This is equivalent to stating the (non-unique) existence of bijections $\mathbf{f}_1 : \mathfrak{S}_1 \mapsto \mathfrak{S}$, $\mathbf{f}_2 : \mathfrak{S}_2 \mapsto \mathfrak{S}$ such that $\mathbf{f}_1\left(\mathbf{x}\right) = \mathbf{f}_2\left(\mathbf{y}\right)$ if and only if $\mathbf{y} = \mathbf{h}\left(\mathbf{x}\right)$, $\mathbf{x} = \mathbf{g}\left(\mathbf{y}\right)$. Under such a mapping the space $\left(\mathfrak{S}_1, \mathfrak{S}_2, \tilde{\psi}\right)$ can be presented in a *canonical form*, $(\mathfrak{S}, \psi)$, where

$$\psi\mathbf{xy} = \tilde{\psi}\mathbf{f}_1^{-1}\left(\mathbf{x}\right)\mathbf{f}_2^{-1}\left(\mathbf{y}\right).$$

The corresponding canonical form of Regular Minimality is

$$\psi\mathbf{xx} < \min\left\{\psi\mathbf{xy},\psi\mathbf{yx}\right\} \tag{7}$$

(equivalently, $\psi\mathbf{xy} > \max\left\{\psi\mathbf{xx},\psi\mathbf{yy}\right\}$), for all distinct $\mathbf{x}, \mathbf{y} \in \mathfrak{S}$.

The *canonical psychometric increments* of the first and second kind are defined as, respectively,

$$\begin{array}{l}\Psi^{(1)}\mathbf{ab} = \psi\mathbf{ab} - \psi\mathbf{aa},\\ \Psi^{(2)}\mathbf{ab} = \psi\mathbf{ba} - \psi\mathbf{aa}.\end{array} \tag{8}$$

It is postulated (as the only other postulate of UFS besides the law of Regular Minimality) that $\Psi^{(1)}$ and $\Psi^{(2)}$ are *dissimilarity functions*. This means that when the DC theory is applied to discrimination probabilities, D can be replaced with either $\Psi^{(1)}$ or $\Psi^{(2)}$.

PROPOSITION 11 $\Psi^{(1)}\mathbf{a}_n\mathbf{b}_n \to 0$ iff $\Psi^{(2)}\mathbf{a}_n\mathbf{b}_n \to 0$ (i.e., both mean $\mathbf{a}_n \leftrightarrow \mathbf{b}_n$).

PROPOSITION 12 The discrimination probability function $\psi\mathbf{ab}$ is uniformly continuous: if $\mathbf{a}'_n \leftrightarrow \mathbf{a}_n$ and $\mathbf{b}'_n \leftrightarrow \mathbf{b}_n$, then $\psi\mathbf{a}'_n\mathbf{b}'_n - \psi\mathbf{a}_n\mathbf{b}_n \to 0$.

We denote

$$\begin{aligned} G_1\mathbf{ab} &= \inf_{\mathbf{X}} \Psi^{(1)}\mathbf{aXb}, \\ G_2\mathbf{ab} &= \inf_{\mathbf{X}} \Psi^{(2)}\mathbf{aXb}. \end{aligned} \tag{9}$$

We call G_1 and G_2, which are oriented metrics by Proposition 2, *Fechnerian metrics* (or oriented Fechnerian metrics). The symmetric metric

$$G^*\mathbf{ab} = G_1\mathbf{ab} + G_1\mathbf{ba} = G_2\mathbf{ab} + G_2\mathbf{ba} \tag{10}$$

is called the *overall Fechnerian metric*. The equality of the two sums is ensured by the following proposition.

PROPOSITION 13 For any $\mathbf{a}, \mathbf{b}$,

$$G_1\mathbf{ab} + G_1\mathbf{ba} = \inf_{(\mathbf{X},\mathbf{Y})} \Psi^{(1)}\mathbf{aXbYa} = \inf_{(\mathbf{X},\mathbf{Y})} \Psi^{(2)}\mathbf{aXbYa} = G_2\mathbf{ab} + G_2\mathbf{ba}.$$

That is, the overall Fechnerian metric $G^*\mathbf{ab}$ is the same for the two kinds of psychometric increments.

We say that f is a *symmetrization scheme* for G_1 and G_2 if it satisfies the following properties:

$$f\left(G_1\mathbf{ab}, G_1\mathbf{ba}\right) = f\left(G_2\mathbf{ab}, G_2\mathbf{ba}\right) \tag{11}$$

and

$$f\left(x, x\right) = kx, \tag{12}$$

where k is some positive number. The previous proposition says that $f(x, y) = x + y$ is a symmetrization scheme. The next proposition says that this is the only possible symmetrization scheme (up to positive scaling). This fact provides the motivation for our definition of $G^*\mathbf{ab}$ as $G\mathbf{ab} + G\mathbf{ba}$ in the general theory of DC.

PROPOSITION 14 The only universally applicable symmetrization scheme (i.e., applicable to all possible spaces satisfying Regular Minimality) is $f(x, y) = \frac{k}{2}(x + y)$.

4. Trails, Paths, and Their Lengths

We return now to the general theory of DC. Recall that Properties $\mathcal{D}1$-$\mathcal{D}4$ are the only assumptions posited for the dissimilarity space $(\mathfrak{S}, D)$, and that these assumptions suffice to unambiguously impose on $(\mathfrak{S}, D)$ topological and uniform structures (Propositions 5 and 6). In particular, since the notion of uniform convergence in the space $(\mathfrak{S}, D)$ is well-defined,

$$\mathbf{a}_n \leftrightarrow \mathbf{b}_n \Longleftrightarrow D\mathbf{a}_n\mathbf{b}_n \to 0,$$

we can meaningfully speak of continuous and uniformly continuous functions from reals into $\mathfrak{S}$.

Let S denote an arbitrary subset of $[a, b]$ with $a, b \in S$. Let $\mathbf{f} : S \mapsto \mathfrak{S}$, or $\mathbf{f}|S$, be some uniformly continuous function with $\mathbf{f}(a) = \mathbf{a}$, $\mathbf{f}(b) = \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are not necessarily distinct. We call such a function a *trail* connecting $\mathbf{a}$ to $\mathbf{b}$. Note that if S is finite, any function $\mathbf{f}|S$ is a trail (see Fig. 4).

If $S = [a, b]$, any continuous function $\mathbf{f}|[a, b]$ is a trail, and such a trail is called a *path*.

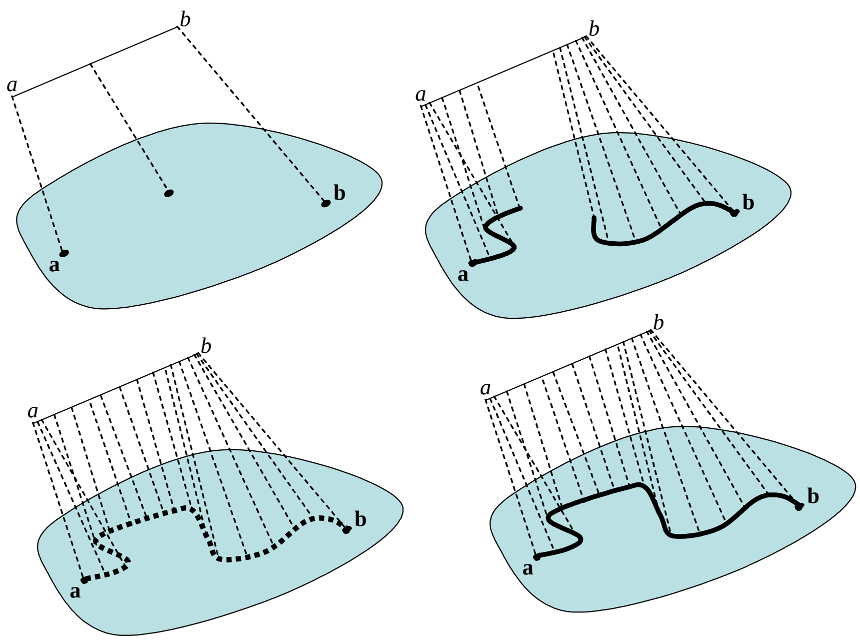

Figure 4. Trails of different kinds. The trail at the bottom right is a path.

Choose an arbitrary *net* on S,

$$\mu = (a = x_0 \leq x_1 \leq \ldots \leq x_k \leq x_{k+1} = b), \tag{13}$$

where all x_i's belong to S (but need not be all pairwise distinct). We call the Hausdorff distance $\delta(\mu, S)$ between S and μ the net's *mesh*. The Hausdorff distance in this case is defined as

$$\delta(\mu, S) = \sup_{y \in S} \min_{x_i \in \mu} \{|x_i - y|\}. \tag{14}$$

Intuitively, as $\delta(\mu_n, S) \to 0$, the nets μ_n provide a progressively better approximation for S.

Given a net $\mu = (x_0, x_1, \ldots, x_k, x_{k+1})$, any chain $\mathbf{X} = \mathbf{x}_0\mathbf{x}_1 \ldots \mathbf{x}_k\mathbf{x}_{k+1}$ (with the elements not necessarily pairwise distinct, and $\mathbf{x}_0$ and $\mathbf{x}_{k+1}$ not necessarily equal to $\mathbf{a}$ and $\mathbf{b}$) can be used to form a *chain-on-net*

$$\mathbf{X}^{\mu} = ((x_0, \mathbf{x}_0), (x_1, \mathbf{x}_1), \ldots, (x_k, \mathbf{x}_k), (x_{k+1}, \mathbf{x}_{k+1})). \tag{15}$$

Denote the class of all such chains-on-nets $\mathbf{X}^{\mu}$ (for all possible pairs of a chain $\mathbf{X}$ and a net μ of the same cardinality) by $\mathcal{M}_S$. If $S = [a, b]$, we also write $\mathcal{M}_a^b$.

Note that a chain-on-net is not a function from $\{x : x \text{ is an element of } \mu\}$ into $\mathfrak{S}$, for it may include pairs $(x_i = x, \mathbf{x}_i)$ and $(x_j = x, \mathbf{x}_j)$ with $\mathbf{x}_i \neq \mathbf{x}_j$. Note also that within a given context $\mathbf{X}^{\mu}$ and $\mathbf{X}^{\nu}$ denote one and the same chain on two nets, whereas $\mathbf{X}^{\mu}$, $\mathbf{Y}^{\mu}$ denote two chains on the same net.

We define the *separation* of the chain-on-net $\mathbf{X}^{\mu} = ((x_0, \mathbf{x}_0), \ldots, (x_{k+1}, \mathbf{x}_{k+1})) \in \mathcal{M}_S$ from a trail $\mathbf{f}|S$ as

$$\sigma_{\mathbf{f}}(\mathbf{X}^{\mu}) = \max_{x_i \in \mu} D\mathbf{f}(x_i)\,\mathbf{x}_i. \tag{16}$$

For a sequence of trails $\mathbf{f}_n|S_n$, any sequence of chains-on-nets $\mathbf{X}_n^{\mu_n} \in \mathcal{M}_{S_n}$ with $\delta(\mu_n, S_n) \to 0$ and $\sigma_{\mathbf{f}_n}\left(\mathbf{X}_n^{\mu_n}\right) \to 0$ will be referred to as a sequence *converging with* $\mathbf{f}_n$. We denote such convergence by $\mathbf{X}_n^{\mu_n} \to \mathbf{f}_n$. In particular, $\mathbf{X}_n^{\mu_n} \to \mathbf{f}$ for a fixed trail $\mathbf{f}|S$ means that $\delta(\mu_n, S) \to 0$ and $\sigma_{\mathbf{f}}\left(\mathbf{X}_n^{\mu_n}\right) \to 0$: in this case we can say that $\mathbf{X}_n^{\mu_n}$ converges *to* $\mathbf{f}$.

It is easy to see that if $S = [a, b]$ (so the trail is a path), then $\delta(\mu_n, S) \to 0$ is equivalent to $\delta\mu_n \to 0$, where

$$\begin{aligned} \mu_n &= \left(a = x_0^n \le x_1^n \le \ldots \le x_{k_n}^n \le x_{k_n+1}^n = b\right), \\ \delta\mu_n &= \max\nolimits_{i=0,1\ldots,k_n}\left(x_{i+1}^n - x_i^n\right). \end{aligned} \tag{17}$$

See Fig. 5 for an illustration.

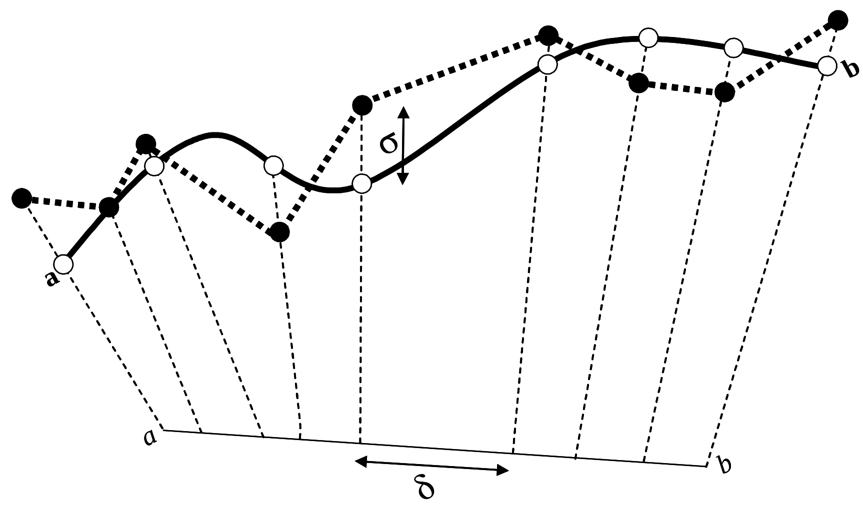

Figure 5. A chain-on-net $\mathbf{X}^{\mu}$ is converging to a path $\mathbf{f}$ as $\sigma = \sigma_{\mathbf{f}}(\mathbf{X}^{\mu}) \to 0$ and $\delta = \delta\mu \to 0$.

We define the *D-length* of $\mathbf{f}|S$ as

$$D\mathbf{f} = \liminf_{\substack{\mathbf{X}^{\mu}\in\mathcal{M}_S \\ \mathbf{X}^{\mu}\to\mathbf{f}}} D\mathbf{X} \;=\; \liminf_{\substack{\mathbf{X}^{\mu}\in\mathcal{M}_S \\ \delta(\mu,S)\to 0 \\ \sigma_{\mathbf{f}}(\mathbf{X}^{\mu})\to 0}} D\mathbf{X} \tag{18}$$

If the domain of $\mathbf{f}$ is to be explicated in $D\mathbf{f}$, we use the notation $D\mathbf{f}\,(S)$ instead of the more correct but less convenient $D\mathbf{f}|S$ or $D\,(\mathbf{f}|S)$. One should keep in mind, however, that the D-length of a trail is not a function of its image $\mathbf{f}\,(S)$ alone but of the function $\mathbf{f} : S \mapsto \mathfrak{S}$. Figure 6 illustrates this point.

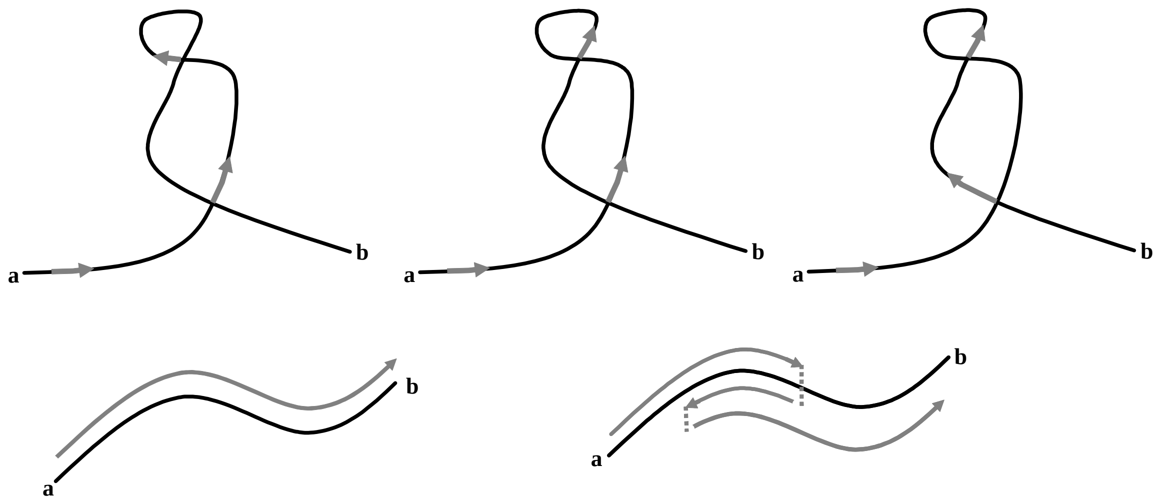

Figure 6. Length is not a function of a path's image, but of how this image is "traversed" (i.e., of how an interval of reals is mapped onto it). The three top panels show three different ways of traversing a curve with loops. The two bottom panels illustrate the fact that even a curve without loops can be traversed in an infinity of different ways.

Given a trail $\mathbf{f}|S$, the class of the chains-on-nets $\mathbf{X}^{\mu}$ such that $\delta\,(\mu, S) < \delta$ and $\sigma_{\mathbf{f}}\,(\mathbf{X}^{\mu}) < \varepsilon$ is nonempty for all positive δ and ε, because this class includes appropriately chosen *inscribed* chains-on-nets

$$((a, \mathbf{a}), (x_1, \mathbf{f}\,(x_1)), \ldots, (x_k, \mathbf{f}\,(x_k)), (b, \mathbf{b})). \tag{19}$$

Here, obviously, $\sigma_{\mathbf{f}}\,(\mathbf{X}^{\mu})$ is identically zero. Note however: even though

$$\liminf_{\substack{\mathbf{X}^{\mu}\in\mathcal{M}_S \\ \delta(\mu,S)\to 0 \\ \sigma_{\mathbf{f}}(\mathbf{X}^{\mu})\to 0}} D\mathbf{X} \;=\; \liminf_{\delta\to 0} \liminf_{\substack{\mathbf{X}^{\mu}\in\mathcal{M}_S \\ \delta(\mu,S)=\delta \\ \sigma_{\mathbf{f}}(\mathbf{X}^{\mu})\to 0}} D\mathbf{X},$$

we generally have

$$\liminf_{\substack{\mathbf{X}^{\mu}\in\mathcal{M}_S \\ \delta(\mu,S)=\delta \\ \sigma_{\mathbf{f}}(\mathbf{X}^{\mu})\to 0}} D\mathbf{X} \;\le\; \liminf_{\substack{\mathbf{X}^{\mu}\in\mathcal{M}_S \\ \delta(\mu,S)=\delta \\ \sigma_{\mathbf{f}}(\mathbf{X}^{\mu})=0}} D\mathbf{X},$$

that is, with our definition of D-length one generally cannot confine one's consideration to the inscribed chains-on-nets only. We illustrate this by an example involving a trail which is a path.

Example. Consider Fig. 7 for which we assume that $\mathfrak{S} = \mathbb{R}^2$ and that, given $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$,

$$D\mathbf{ab} = |a_1 - b_1| + |a_2 - b_2| + \min\{|a_1 - b_1|, |a_2 - b_2|\}.$$

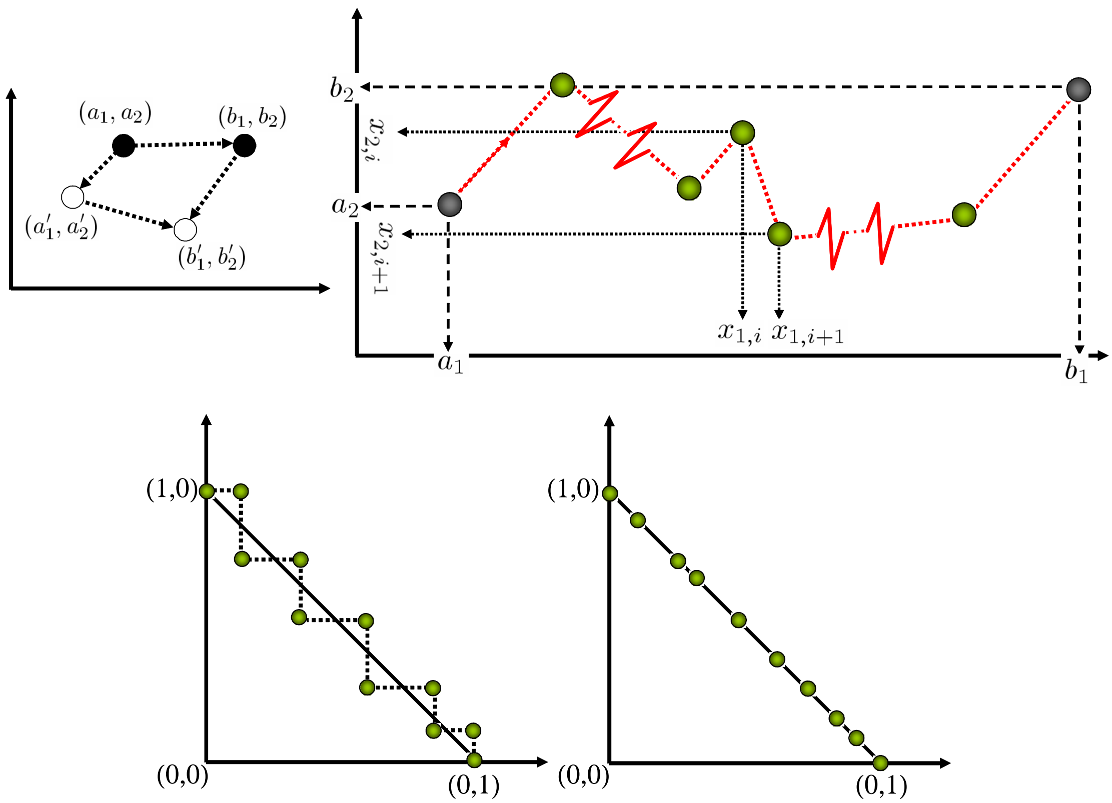

Figure 7. A demonstration that inscribed chains are not sufficient for D-length computations. The space is $\left(\mathbb{R}^2, D\right)$, where D from (a_1, a_2) to (b_1, b_2) is defined as $|a_1 - b_1| + |a_2 - b_2| + \min\{|a_1 - b_1|, |a_2 - b_2|\}$. Top left: $D(a_1, a_2)(a'_1, a'_2)$ and $D(b_1, b_2)(b'_1, b'_2)$ converge to zero if and only if points (a_1, a_2) and (a'_1, a'_2) converge to each other in the Euclidean sense, and the same is true for (b_1, b_2) and (b'_1, b'_2); then $D(a_1, a_2)(b_1, b_2) - D(a'_1, a'_2)(b'_1, b'_2)$ tends to zero (Property $\mathcal{D}3$). Top right: the dissimilarity between two successive elements of a chain is not less than $|x_{1,i} - x_{1,i+1}| + |x_{2,i} - x_{2,i+1}|$, whence the sum of all these dissimilarities cannot fall below $|a_1 - b_1| + |a_2 - b_2|$; this implies Property $\mathcal{D}4$. Bottom left: the staircase chain has the cumulated dissimilarity 2, and 2 is the true D-length of the hypotenuse. Bottom right: the inscribed chain has the cumulated dissimilarity 3.

D trivially satisfies Properties $\mathcal{D}1$-$\mathcal{D}2$. On observing that $D\mathbf{ab}$ is uniformly continuous with respect to the Euclidean norm $\|\cdot\|$, and that

$$\begin{aligned} D\mathbf{a}_n\mathbf{a}'_n &\to 0 \Longleftrightarrow \|\mathbf{a}_n - \mathbf{a}'_n\| \to 0, \\ D\mathbf{b}_n\mathbf{b}'_n &\to 0 \Longleftrightarrow \|\mathbf{b}_n - \mathbf{b}'_n\| \to 0, \end{aligned}$$

it is clear that D satisfies $\mathcal{D}3$. Finally, since for any chain $\mathbf{aXb}$,

$$D\mathbf{aXb} \geq |a_1 - b_1| + |a_2 - b_2|,$$

$\mathcal{D}4$ is satisfied too:

$$D\mathbf{a}_n\mathbf{X}_n\mathbf{b}_n \to 0 \Longrightarrow \|\mathbf{a}_n - \mathbf{b}_n\| \to 0 \Longleftrightarrow D\mathbf{a}_n\mathbf{b}_n \to 0.$$

D therefore is a (symmetric) dissimilarity function.

Consider now the hypotenuse of the isosceles right-angle triangle in the bottom panels of Fig. 7. To make this hypotenuse a path, assume that each point on it is the image of, say, its abscissa value. Two chains are shown, a staircase one and an inscribed one. The nets for both these chains can be chosen, again, as the abscissa values of their elements. Clearly, as the chained elements get progressively denser, the two chains converge to the hypotenuse in the sense of our definition. It is easy to calculate that the D-lengths of the converging chains in both cases remain constant: equal to 2 for all the staircase chains, and to 3 for all the inscribed ones. This shows that the D-length of the hypotenuse cannot be approached by the lengths of the inscribed chains. The value 2 is the true D-length of the hypotenuse because, by the argument used in establishing the property $\mathcal{D}4$, no chain connecting the endpoints of the hypotenuse can have the D-length less than 2. ■

5. Basic Properties of Trail Length

We need a lemma first which shows that in considering chains-on-nets converging to a trail one always can confine one's consideration to only those chains-on-nets which connect the endpoints of the trail.

Lemma 1 If

$$\mathbf{X}_n^{\mu_n} = \left((a, \mathbf{x}_0^n), (x_1^n, \mathbf{x}_1^n), \ldots, \left(x_{k_n}^n, \mathbf{x}_{k_n}^n\right), \left(b, \mathbf{x}_{k_n+1}^n\right)\right) \to \mathbf{f}|S$$

with a trail $\mathbf{f}$ connecting $\mathbf{a}$ to $\mathbf{b}$, and

$$\mathbf{Y}_n^{\mu_n} = \left((a, \mathbf{a}), (x_1^n, \mathbf{x}_1^n), \ldots, \left(x_{k_n}^n, \mathbf{x}_{k_n}^n\right), (b, \mathbf{b})\right)$$

(so all $\mathbf{Y}_n$ connect $\mathbf{a}$ to $\mathbf{b}$), then

$$\mathbf{Y}_n^{\mu_n} \to \mathbf{f}|S$$

and

$$\liminf_{n\to\infty} D\mathbf{X}_n = \liminf_{n\to\infty} D\mathbf{Y}_n.$$

Proof. Clearly,

$$\sigma_{\mathbf{f}}\left(\mathbf{X}_n^{\mu_n}\right) = \max\left\{\sigma_{\mathbf{f}}\left(\mathbf{Y}_n^{\mu_n}\right), D\mathbf{a}\mathbf{x}_0^n, D\mathbf{b}\mathbf{x}_{k_n+1}^n\right\},$$

and $\sigma_{\mathbf{f}}\left(\mathbf{X}_n^{\mu_n}\right) \to 0$ implies $\sigma_{\mathbf{f}}\left(\mathbf{Y}_n^{\mu_n}\right) \to 0$. Since $\delta(\mu, S)$ does not change, we have $\mathbf{Y}_n^{\mu_n} \to \mathbf{f}|S$. Also, $\sigma_{\mathbf{f}}\left(\mathbf{X}_n^{\mu_n}\right) \to 0$ implies $D\mathbf{a}\mathbf{x}_0^n \to 0$, $D\mathbf{b}\mathbf{x}_{k_n+1}^n \to 0$. By the uniform continuity of D then $D\mathbf{a}\mathbf{x}_1^n - D\mathbf{x}_0^n\mathbf{x}_1^n \to 0$ and $D\mathbf{x}_{k_n}^n\mathbf{b} - D\mathbf{x}_{k_n}^n\mathbf{x}_{k_n+1}^n \to 0$. Since

$$D\mathbf{Y}_n = D\mathbf{X}_n + \left(D\mathbf{a}\mathbf{x}_1^n - D\mathbf{x}_0^n\mathbf{x}_1^n\right) + \left(D\mathbf{x}_{k_n}^n\mathbf{b} - D\mathbf{x}_{k_n}^n\mathbf{x}_{k_n+1}^n\right),$$

$D\mathbf{Y}_n - D\mathbf{X}_n \to 0$, and the equality of the lower limits follows. ■

It follows that $D\mathbf{f}$ can be defined as $\liminf D\mathbf{Y}$ taken over all $\mathbf{Y}^\mu \to \mathbf{f}$ with $\mathbf{Y}$ connecting $\mathbf{a}$ to $\mathbf{b}$. That is, given a trail $\mathbf{f}$ connecting $\mathbf{a}$ to $\mathbf{b}$, one can always find a sequence of $\mathbf{X}_n^{\mu_n} \to \mathbf{f}$ with $D\mathbf{X}_n \to D\mathbf{f}$, such that $((a, \mathbf{a}), (b, \mathbf{b})) \subset \mathbf{X}_n^{\mu_n}$.

Theorem 1 For any trail $\mathbf{f}|S$ connecting $\mathbf{a}$ to $\mathbf{b}$,

$$D\mathbf{f} \geq G\mathbf{ab}.$$

Proof. $G\mathbf{ab} = \inf D\mathbf{a}\mathbf{X}\mathbf{b}$ across all possible chains $\mathbf{X}$, so $G\mathbf{ab} \leq \liminf_{n\to\infty} D\mathbf{Y}_n$ for any sequence of chains $\mathbf{Y}_n$ connecting $\mathbf{a}$ to $\mathbf{b}$. But by Lemma 1, $D\mathbf{f} = \liminf_{n\to\infty} D\mathbf{Y}_n$ for at least one such sequence. ■

That is, the D-length of a trail is bounded from below by $G\mathbf{ab}$. There is no upper bound for $D\mathbf{f}$, this quantity need not be finite. Thus, it will be shown below that when $\mathbf{f}$ is a path and D a metric, the notion of $D\mathbf{f}$ essentially coincides with the traditional notion of path length; and examples of paths whose length, in the traditional sense, is infinite, are well-known (see, e.g., Chapter 1 in Papadopoulos, 2005). We call a trail *D-rectifiable* if its D-length is finite.

The next theorem establishes the additivity property for trail length. In its proof we use the operation of concatenation of chains-on-nets. If $\mathbf{X}^\mu = ((x_1, \mathbf{x}_1), \ldots, (x_k, \mathbf{x}_k))$ and $\mathbf{Y}^\nu = ((y_1, \mathbf{y}_1), \ldots, (y_l, \mathbf{y}_l))$, with $x_k \leq y_1$, then

$$\mathbf{X}^\mu \cup \mathbf{Y}^\nu = ((x_1, \mathbf{x}_1), \ldots, (x_k, \mathbf{x}_k), (x_{k+1}, \mathbf{x}_{k+1}), \ldots, (x_{k+l}, \mathbf{x}_{k+l})), \tag{20}$$

where $x_{k+i} = y_i$ and $\mathbf{x}_{k+i} = \mathbf{y}_i$ $(i = 1, \ldots, l)$.

Theorem 2 For any trail $\mathbf{f}|S$ with $a, b \in S \subset [a, b]$, and any point $z \in S$,

$$D\mathbf{f}(S) = D\mathbf{f}([a, z] \cap S) + D\mathbf{f}([z, b] \cap S).$$

Proof. If $z = a$ or $z = b$, the theorem holds trivially, so we assume $z \in \,]a, b[$. We denote $[a, z] \cap S$ and $[z, b] \cap S$ by S_1 and S_2, respectively. Consider two sequences $\mathbf{X}_n^{\alpha_n} \to \mathbf{f}|S_1$ and $\mathbf{Y}_n^{\beta_n} \to \mathbf{f}|S_2$ with $D\mathbf{X}_n \to D\mathbf{f}(S_1)$ and $D\mathbf{Y}_n \to D\mathbf{f}(S_2)$. Let $(z, \mathbf{x}_z^n)$ be the last element of $\mathbf{X}_n^{\alpha_n}$, and $(z, \mathbf{y}_z^n)$ the first element of $\mathbf{Y}_n^{\beta_n}$. Clearly, the sequence of pairwise concatenated chains-on-nets $\mathbf{X}_n^{\alpha_n} \cup \mathbf{Y}_n^{\beta_n}$ converges to $\mathbf{f}$, and

$$D\mathbf{f}(S) \leq \lim_{n\to\infty} D\mathbf{X}_n\mathbf{Y}_n = \lim_{n\to\infty} D\mathbf{X}_n + \lim_{n\to\infty} \mathbf{Y}_n + \lim_{n\to\infty} D\mathbf{x}_z^n\mathbf{y}_z^n = D\mathbf{f}(S_1) + D\mathbf{f}(S_2) + \lim_{n\to\infty} D\mathbf{x}_z^n\mathbf{y}_z^n.$$

Since $\sigma_{\mathbf{f}|S_1}(\mathbf{X}_n^{\alpha_n}) \to 0$ and $\sigma_{\mathbf{f}|S_2}\left(\mathbf{Y}_n^{\beta_n}\right) \to 0$, we have $D\mathbf{f}(z)\,\mathbf{x}_z^n \to 0$, and $D\mathbf{f}(z)\,\mathbf{y}_z^n \to 0$, and then, by the uniform continuity of dissimilarity functions, $D\mathbf{x}_z^n\mathbf{y}_z^n \to 0$. This proves

$$D\mathbf{f}(S) \leq D\mathbf{f}(S_1) + D\mathbf{f}(S_2).$$

(This also proves that $D\mathbf{f}(S_1) < \infty$ and $D\mathbf{f}(S_2) < \infty$ imply $D\mathbf{f}(S) < \infty$.)

To prove that the inequality can be reversed, take any sequence $\mathbf{Z}_n^{\gamma_n} \to \mathbf{f}|S$ such that $D\mathbf{Z}_n \to D\mathbf{f}$. In the net γ_n find $z_{i_n}^n < z \leq z_{i_n+1}^n$ and the corresponding $\mathbf{z}_{i_n}^n$, $\mathbf{z}_{i_n+1}^n$ in the chain $\mathbf{Z}_n$. As $\delta(\gamma_n, S) \to 0$, at least one of the quantities $z_{i_n}^n - z$, $z - z_{i_n+1}^n$ must tend to zero, and we have (both $\mathbf{f}$ and D being uniformly continuous)

$$D\mathbf{f}\left(z_{i_n}^n\right)\mathbf{f}\left(z_{i_n+1}^n\right) - D\mathbf{f}\left(z_{i_n}^n\right)\mathbf{f}(z) - D\mathbf{f}(z)\,\mathbf{f}\left(z_{i_n+1}^n\right) \to 0.$$

Since $\sigma_{\mathbf{f}|S}\left(\mathbf{Z}_n^{\gamma_n}\right) \to 0$, we have, using again the uniform continuity of $\mathbf{f}$ and D,

$$\begin{aligned}
&D\mathbf{z}_{i_n}^n\mathbf{z}_{i_n+1}^n - D\mathbf{f}\left(z_{i_n}^n\right)\mathbf{f}\left(z_{i_n+1}^n\right) \to 0,\\
&D\mathbf{z}_{i_n}^n\mathbf{f}(z) - D\mathbf{f}\left(z_{i_n}^n\right)\mathbf{f}(z) \to 0,\\
&D\mathbf{f}(z)\,\mathbf{z}_{i_n+1}^n - D\mathbf{f}(z)\,\mathbf{f}\left(z_{i_n+1}^n\right) \to 0,
\end{aligned}$$

whence

$$D\mathbf{z}_{i_n}^n\mathbf{z}_{i_n+1}^n - D\mathbf{z}_{i_n}^n\mathbf{f}(z) - D\mathbf{f}(z)\,\mathbf{z}_{i_n+1}^n \to 0.$$

Hence $D\mathbf{Z}_n' \to D\mathbf{f}$, where $\mathbf{Z}_n'$ is $\mathbf{Z}_n$ with $\mathbf{f}(z)$ inserted between $\mathbf{z}_{i_n}^n$ and $\mathbf{z}_{i_n+1}^n$. It is easy to see that

$$\begin{aligned}
\mathbf{U}_n^{\alpha_n} &= \mathbf{Z}_n^{\gamma_n\cap[a,z[} \cup (z, \mathbf{f}(z)) \to \mathbf{f}|S_1,\\
\mathbf{V}_n^{\beta_n} &= (z, \mathbf{f}(z)) \cup \mathbf{Z}_n^{\gamma_n\cap[z,c]} \to \mathbf{f}|S_2,
\end{aligned}$$

whence $\liminf_{n\to\infty} D\mathbf{U}_n \geq D\mathbf{f}(S_1)$ and $\liminf_{n\to\infty} D\mathbf{V}_n \geq D\mathbf{f}(S_2)$. Since

$$\liminf_{n\to\infty} D\mathbf{U}_n + \liminf_{n\to\infty} D\mathbf{V}_n \leq \lim_{n\to 0} D\mathbf{U}_n\mathbf{V}_n = \lim_{n\to 0} D\mathbf{Z}_n' = D\mathbf{f},$$

it follows that $D\mathbf{f}(S_1) + D\mathbf{f}(S_2) \leq D\mathbf{f}(S)$. (This also proves that $D\mathbf{f}(S) < \infty$ implies $D\mathbf{f}(S_1) < \infty$ and $D\mathbf{f}(S_2) < \infty$.) ■

THEOREM 3 $D\mathbf{f}$ for any trail $\mathbf{f}|S$ is nonnegative, and $D\mathbf{f} = 0$ if and only if $\mathbf{f}$ is constant on its domain S (i.e., $\mathbf{f}(S)$ is a singleton).

Proof. The nonnegativity of $D\mathbf{f}$ follows from Theorem 1. If $\mathbf{f}(x) = \mathbf{a}$ for all $x \in S$, any sequence of chains-on-nets $\mathbf{X}_n^{\mu_n}$ with all points of every chain identically equal to $\mathbf{a}$ and with $\delta(\mu_n, S) \to 0$ converges to $\mathbf{f}$, with $D\mathbf{X}_n \equiv 0$. Conversely, if $D\mathbf{f}(S) = 0$, then, by the property of additivity (Theorem 2) and by the nonnegativity of $D\mathbf{f}$, $D\mathbf{f}([x, y] \cap S) = 0$ for any $x, y \in S$, and then, by Theorem 1, $\mathbf{f}(x) = \mathbf{f}(y)$. ■

The quantity

$$\sigma_{\mathbf{f}}(\mathbf{g}) = \sup_{x\in S} D\mathbf{f}(x)\,\mathbf{g}(x) \tag{21}$$

is called the *separation* of trail $\mathbf{g}|S$ from trail $\mathbf{f}|S$. Since D is (uniformly) continuous, if $S = [a, b]$ (or more generally, if S is closed),

$$\sigma_{\mathbf{f}}(\mathbf{g}) = \max_{x \in S} D\mathbf{f}(x)\,\mathbf{g}(x). \tag{22}$$

Two sequences of trails $\mathbf{f}_n|S_n$ and $\mathbf{g}_n|S_n$ are said to be (*uniformly*) *converging* to each other if $\sigma_{\mathbf{f}_n}(\mathbf{g}_n) \to 0$. Due to the symmetry of the convergence in $\mathfrak{S}$ (Proposition 4), this implies $\sigma_{\mathbf{g}_n}(\mathbf{f}_n) \to 0$, so the definition and terminology are well-formed. We symbolize this situation by $\mathbf{f}_n \leftrightarrow \mathbf{g}_n$.[4]

In particular, if $\mathbf{f}|S$ is fixed then a sequence $\mathbf{f}_n|S$ converges to $\mathbf{f}$ if $\sigma_{\mathbf{f}}(\mathbf{f}_n) \to 0$. We present this convergence in symbols as $\mathbf{f}_n \to \mathbf{f}$, even though one could also write $\mathbf{f}_n \leftrightarrow \mathbf{f}$. Note that if $\mathbf{f}_n \to \mathbf{f}$, the endpoints $\mathbf{a}_n = \mathbf{f}_n(a)$ and $\mathbf{b}_n = \mathbf{f}_n(b)$ generally depend on n and differ from, respectively $\mathbf{a} = \mathbf{f}(a)$ and $\mathbf{b} = \mathbf{f}(b)$.

The following very important property is called the *lower semicontinuity* of D-length (as a function of trails). It is obtained as almost immediate consequence of our definition of D-length.

THEOREM 4 For any sequence of trails $\mathbf{f}_n|S \to \mathbf{f}|S$,

$$\liminf_{n\to\infty} D\mathbf{f}_n \geq D\mathbf{f}.$$

Proof. Consider any sequence of chains-on-nets $\mathbf{X}_n^{\mu_n} \in \mathcal{M}_S$ such that $\delta(\mu_n, S) \to 0$, $\sigma_{\mathbf{f}_n}\left(\mathbf{X}_n^{\mu_n}\right) \to 0$, $|D\mathbf{X}_n - D\mathbf{f}_n| \to 0$. It follows from the uniform continuity of D that

$$[\sigma_{\mathbf{f}_n}(\mathbf{X}_n^{\mu_n}) \to 0] \wedge [\sigma_{\mathbf{f}}(\mathbf{f}_n) \to 0] \Longrightarrow \sigma_{\mathbf{f}}(\mathbf{X}_n^{\mu_n}) \to 0.$$

Indeed, for any $\varepsilon > 0$ one can find a $\sigma > 0$ such that if $D\mathbf{f}(z_n)\,\mathbf{f}_n(z_n) < \sigma$ and $D\mathbf{f}_n(z_n)\,\mathbf{z}_n < \sigma$, then $D\mathbf{f}(z_n)\,\mathbf{z}_n < \varepsilon$, where $(z_n, \mathbf{z}_n)$ is an arbitrarily chosen element of $\mathbf{X}_n^{\mu_n}$. Then $\mathbf{X}_n^{\mu_n} \to \mathbf{f}$, whence $\liminf_{n\to\infty} D\mathbf{X}_n \geq D\mathbf{f}$. But $\liminf_{n\to\infty} D\mathbf{X}_n = \liminf_{n\to\infty} D\mathbf{f}_n$. ■

It is worth observing that similar argument cannot be applied to the convergence $\mathbf{f}_n \leftrightarrow \mathbf{g}_n$. The reason for this is that even though the convergence of the type $\mathbf{X}_n^{\mu_n} \to \mathbf{g}_n$ is a well-defined concept, it differs from the convergence $\mathbf{X}_n^{\mu_n} \to \mathbf{f}$ in one important respect: in the latter case $\liminf_{n\to\infty} D\mathbf{X}_n \geq D\mathbf{f}$ by the definition of $D\mathbf{f}$, but in the former case there is no definitive ordering of $\liminf_{n\to\infty} D\mathbf{X}_n$ and $\liminf_{n\to\infty} D\mathbf{g}_n$.

We conclude this section with a useful theorem which says that, given a trail $\mathbf{f}$, one can choose for every $n = 1, 2, \ldots$ an arbitrary inscribed chain-on-net, and then a sequence of $\mathbf{X}_n^{\mu_n} \to \mathbf{f}$ with $D\mathbf{X}_n \to D\mathbf{f}$ can be constructed so that each chain-on-net $\mathbf{X}_n^{\mu_n}$ includes ("passes through") the corresponding inscribed one. Refer to Fig. 8 for a crude illustration (with $\mathbf{f}$ a path).

THEOREM 5 Let $\mathbf{Z}_n^{\nu_n}$ be a sequence of chains-on-nets inscribed in a trail $\mathbf{f}|S$,

$$\mathbf{Z}_n^{\nu_n} = \{(x_i^n, \mathbf{f}(x_i^n))\}_{i=0,\ldots,k_n+1}.$$

[4] This convergence effectively imposes a uniformity (hence also topology) on the space of all trails on a given domain. We will not pursue this topic in this paper.

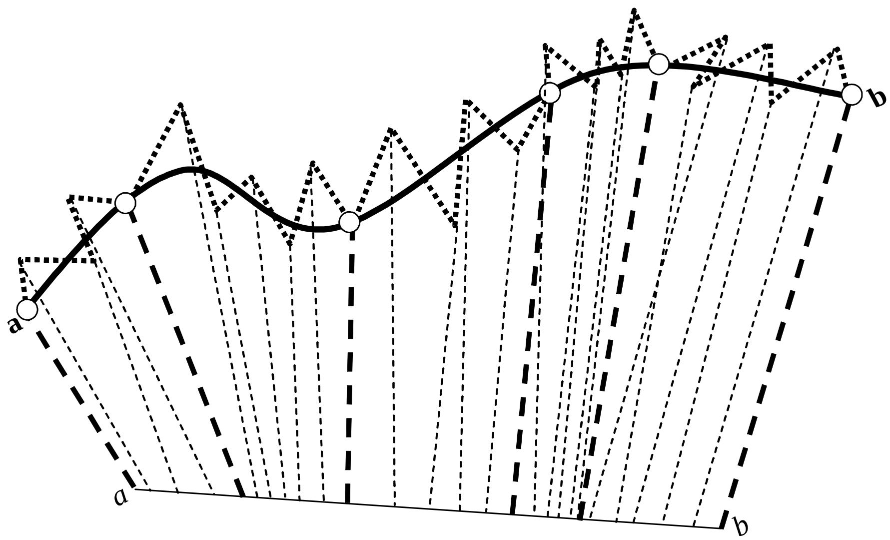

Figure 8. Illustration to Theorem 5. Open circles represent an inscribed chain. The chain shown by the point line "passes through" the inscribed chain and approximates the path.

Then there is a sequence of $\mathbf{X}_n^{\mu_n} \to \mathbf{f}$ with $D\mathbf{X}_n \to D\mathbf{f}$ such that

$$\mathbf{Z}_n^{\nu_n} \subset \mathbf{X}_n^{\mu_n},$$

for every n.

Proof. By the additivity property (Theorem 2),

$$D\mathbf{f}\,(S) = \sum_{i=0}^{k_n} D\mathbf{f}\left(\left[x_i^n, x_{i+1}^n\right] \cap S\right),$$

for every n. By Lemma 1, for each $\mathbf{f}|\left[x_i^n, x_{i+1}^n\right] \cap S$ one can find a chain-on-net $\mathbf{X}_{i,n}^{\mu_{i,n}}$ connecting $(x_i^n, \mathbf{f}\,(x_i^n))$ to $\left(x_{i+1}^n, \mathbf{f}\left(x_{i+1}^n\right)\right)$ and such that

$$\begin{gathered}
\delta\left(\mu_{i,n}, \left[x_i^n, x_{i+1}^n\right] \cap S\right) < \tfrac{1}{n}, \\
\sigma_{\mathbf{f}|\left[x_i^n, x_{i+1}^n\right] \cap S}\left(\mathbf{X}_{i,n}^{\mu_{i,n}}\right) < \tfrac{1}{n}, \\
\left| D\mathbf{X}_{i,n}^{\mu_{i,n}} - D\mathbf{f}\left(\left[x_i^n, x_{i+1}^n\right] \cap S\right)\right| < \tfrac{1}{n(k_n+1)}.
\end{gathered}$$

Putting

$$\mathbf{X}_n^{\mu_n} = \mathbf{X}_{1,n}^{\mu_{1,n}} \cup \ldots \cup \mathbf{X}_{k_n,n}^{\mu_{k_n,n}},$$

it is easy to check that $\mathbf{X}_n^{\mu_n}$ satisfies the statement of the theorem. ■

6. DC in Arc-connected Spaces

A space is called *arc-connected* (or *path-connected*) if any two points in it can be connected by a path. Even though arcs have not yet been introduced in this paper, the terms "arc-connected" and "path-connected" are synonymous, because if two points are connected by a path they are also connected by an arc (see, e.g., Hocking &Young, 1961, pp. 116-117).

Hereafter we will assume that $(\mathfrak{S}, D)$ is an arc-connected space, and we will focus on trails which are paths (except in Section 7 where non-path trails will be invoked as auxiliary constructs). Recall that when dealing with a path we can write $\delta\mu \to 0$ instead of $\delta(\mu, [a, b]) \to 0$, and that the set of all chains-on-nets whose nets connect a to b in $[a, b]$ is denoted by $\mathcal{M}_a^b$.

6.1. Metric Dissimilarities

We begin with the case when the dissimilarity D is a metric (recall that the term is used here in the sense of an *oriented* metric). If D is a metric, the traditional definition of the length of a path $\mathbf{f}$ is

$$D_{ins}\mathbf{f} = \sup D\mathbf{Z}, \tag{23}$$

with the supremum taken over all *inscribed* chains-on-nets $\mathbf{Z}^{\nu}$. This is seemingly different from our definition of $D\mathbf{f}$, but the theorem below shows that $D_{ins}\mathbf{f}$ and $D\mathbf{f}$ coincide.

THEOREM 6 If D is a metric, then

$$D\mathbf{f} = D_{ins}\mathbf{f}$$

for any path $\mathbf{f}$.

Proof. Let $\mathbf{Z}_n^{\nu_n}$ be a sequence of inscribed chains-on-nets with $\delta\nu_n \to 0$ and $D\mathbf{Z}_n \to D_{ins}\mathbf{f}$. Since $\sigma_{\mathbf{f}}(\mathbf{Z}_n^{\nu_n}) \equiv 0$, we have $\mathbf{Z}_n^{\nu_n} \to \mathbf{f}$. By the definition of D-length then, $D\mathbf{f} \le D_{ins}\mathbf{f}$.

According to Theorem 5, one can construct a sequence of $\mathbf{X}_n^{\mu_n} \to \mathbf{f}$ with $D\mathbf{X}_n \to D\mathbf{f}$ such that $\mathbf{Z}_n^{\nu_n} \subset \mathbf{X}_n^{\mu_n}$. For a given n, let $\mathbf{z}, \mathbf{z}'$ be any two successive elements of $\mathbf{Z}_n$ and $\mathbf{z}\mathbf{x}_1 \ldots \mathbf{x}_k\mathbf{z}'$ be the subchain of $\mathbf{X}_n$ connecting them. By the triangle inequality, $D\mathbf{z}\mathbf{x}_1 \ldots \mathbf{x}_k\mathbf{z}' \ge D\mathbf{z}\mathbf{z}'$. It follows that $D\mathbf{X}_n \ge D\mathbf{Z}_n$, whence $D\mathbf{f} \ge D_{ins}\mathbf{f}$. ■

We conclude that in the case of a metric D our dissimilarity-based definition of path length specializes to the classical one (as presented, e.g., in Blumenthal, 1953; Blumenthal & Menger, 1970; Busemann, 2005), except for not assuming that D is symmetric.

The next result is one of the most basic in the classical theory (see, e.g., Busemann, 2005, p. 20). We present the proof in extenso to make sure that it does not require that metric D be symmetric.

THEOREM 7 If D is a metric, $\mathbf{f}$ is any path, and $\mathbf{X}_n^{\mu_n}$ is any sequence of inscribed chains-on-nets with $\delta\mu_n \to 0$, then

$$D\mathbf{X}_n \to D\mathbf{f}.$$

Proof. Assume the contrary: let there be some $\mathbf{X}_n^{\mu_n}$ such that $\delta\mu_n \to 0$ but $D\mathbf{X}_n \not\to D\mathbf{f}$. Since $D\mathbf{f} = \sup D\mathbf{Z}$ across all possible inscribed chains-on-nets, $D\mathbf{X}_n \le D\mathbf{f}$ for all n. Then there is a $\Delta > 0$ and a subsequence of $\mathbf{X}_n^{\mu_n}$ (with no loss of generality, let it be $\mathbf{X}_n^{\mu_n}$ itself) such that $D\mathbf{X}_n \to D\mathbf{f} - \Delta$. Let $\mathbf{Z}^{\nu}$ be a chain-on-net with $D\mathbf{Z} > D\mathbf{f} - \Delta/2$ and

$$\nu = (a = z_0 < z_1 < \ldots < z_l < z_{l+1} = b)\,.$$

For every z_i and every n, choose two successive elements $x_{k_{i,n}}^n, x_{k_{i,n}+1}^n$ of μ_n such that $x_{k_{i,n}}^n \le z_i \le x_{k_{i,n}+1}^n$. For a sufficiently large n, $\delta\mu_n$ will be sufficiently small to ensure that $x_{k_{i,n}}^n$, $x_{k_{i,n}+1}^n$ are uniquely determined by i, and that z_i is the only member of ν falling between them. Denote by $\nu \cup \mu_n$ the nets (ordered sequences) formed by the elements of ν and μ_n. Consider the chains $\mathbf{f}(\nu \cup \mu_n)$, with $\mathbf{f}$ applying to the nets elementwise. Clearly,

$$D\mathbf{f}(\nu \cup \mu_n) = D\mathbf{X}_n + \sum_{i=0}^{l} \left\{ D\mathbf{f}\left(x_{k_{i,n}}^n\right)\mathbf{f}(z_i) + D\mathbf{f}(z_i)\,\mathbf{f}\left(x_{k_{i,n}+1}^n\right) - D\mathbf{f}\left(x_{k_{i,n}}^n\right)\mathbf{f}\left(x_{k_{i,n}+1}^n\right)\right\}.$$

By the uniform continuity of $\mathbf{f}$, the right-hand sum tends to zero, whence

$$D\mathbf{f}(\nu \cup \mu_n) - D\mathbf{X}_n \to 0,$$

implying

$$D\mathbf{f}(\nu \cup \mu_n) \to D\mathbf{f} - \Delta.$$

But by the triangle inequality, for all n,

$$D\mathbf{f}(\nu \cup \mu_n) \ge D\mathbf{f}(\nu) = D\mathbf{Z} > D\mathbf{f} - \Delta/2.$$

This contradiction proves the theorem. ■

Note that the convergence in this theorem is uniform across the inscribed chains-on-nets $\mathbf{X}^{\mu}$: restated in the ε-δ language, the theorem says that, given a path $\mathbf{f}$, for every $\varepsilon > 0$ one can find $\delta > 0$ such that $D\mathbf{X} > D\mathbf{f} - \varepsilon$ whenever $\delta\mu < \delta$. Indeed, otherwise one could find an $\varepsilon > 0$ and a sequence of inscribed chains-on-nets $\mathbf{X}_n^{\mu_n}$ such that $\delta\mu_n \to 0$ but $D\mathbf{X}_n \not\to D\mathbf{f}$.

6.2. G-lengths

Returning to general dissimilarities D, since the metric G induced by D in accordance with

$$G\mathbf{ab} = \inf_{\mathbf{X}} D\mathbf{aXb}$$

is itself a dissimilarity function (Proposition 3), the *G-length* of a path $\mathbf{f} : [a,b] \mapsto \mathfrak{S}$ should be defined as

$$G\mathbf{f} = \liminf_{\substack{\mathbf{X}^{\mu} \in \mathcal{M}_a^b \\ \mathbf{X}^{\mu} \xrightarrow{G} \mathbf{f}}} G\mathbf{X}, \tag{24}$$

where (putting $\mathbf{X} = \mathbf{x}_0\mathbf{x}_1 \ldots \mathbf{x}_k\mathbf{x}_{k+1}$),

$$G\mathbf{X} = \sum_{i=0}^{k} G\mathbf{x}_i\mathbf{x}_{i+1}, \tag{25}$$

and the convergence $\mathbf{X}^{\mu} \xrightarrow{G} \mathbf{f}$ (where μ is the net $a = x_0, x_1, \ldots, x_k, x_{k+1} = b$ corresponding to $\mathbf{X}$) means the conjunction of $\delta\mu \to 0$ and

$$\sigma_{\mathbf{f}}^{*}\left(\mathbf{X}^{\mu}\right) = \max_{i=0,\ldots,k+1} G\mathbf{f}\left(x_i\right)\mathbf{x}_i \to 0.$$

It is easy to see, however, that $\mathbf{X}^{\mu} \xrightarrow{G} \mathbf{f}$ and $\mathbf{X}^{\mu} \to \mathbf{f}$ are interchangeable.

Theorem 8 For any path $\mathbf{f}$,

$$\mathbf{X}^{\mu} \to \mathbf{f} \iff \mathbf{X}^{\mu} \xrightarrow{G} \mathbf{f}.$$

Proof. $\delta\mu \to 0$ has the same meaning in both cases, and

$$\max_{i=0,\ldots,k+1} G\mathbf{f}\left(x_i\right)\mathbf{x}_i \to 0 \iff \max_{i=0,\ldots,k+1} D\mathbf{f}\left(x_i\right)\mathbf{x}_i \to 0$$

due to the uniform equivalence of D and G (Proposition 7). ∎

So we can write unambiguously

$$G\mathbf{f} = \liminf_{\substack{\mathbf{X}^{\mu} \in \mathcal{M}_a^b \\ \mathbf{X}^{\mu} \to \mathbf{f}}} G\mathbf{X}. \tag{26}$$

Since G is a metric, we also have, by Theorem 6,

$$G\mathbf{f} = \sup G\mathbf{Z} \tag{27}$$

with the supremum taken over all inscribed chains-on-nets $\mathbf{Z}^{\nu}$; and by Theorem 7,

$$G\mathbf{f} = \lim_{n\to\infty} G\mathbf{Z}_n \tag{28}$$

for any sequence of inscribed inscribed chains-on-nets $\mathbf{Z}_n^{\nu_n}$ with $\delta\nu_n \to 0$.

What is the relationship between the D-length and G-length of a path? A remarkable and somewhat surprising (at least to the author) fact is that the two are always equal.

Theorem 9 For any path $\mathbf{f}$,

$$D\mathbf{f} = G\mathbf{f}.$$

Proof. That $D\mathbf{f} \geq G\mathbf{f}$ is obvious, as the corresponding inequality holds for any sequence of chains-on-nets converging to $\mathbf{f}$. To prove $D\mathbf{f} \leq G\mathbf{f}$ (where we can assume that $G\mathbf{f}$ is finite, for otherwise the inequality is satisfied trivially), we consider a sequence of inscribed chains-on-nets

$$\mathbf{Z}_n^{\nu_n} = \left(\left(z_0^n = a, \mathbf{z}_0^n = \mathbf{a}\right), \left(z_1^n, \mathbf{z}_1^n\right), \ldots, \left(z_{k_n}^n, \mathbf{z}_{k_n}^n\right), \left(z_{k_n+1}^n = b, \mathbf{z}_{k_n+1}^n = \mathbf{b}\right)\right),$$

such that $\delta\nu_n \to 0$, and $G\mathbf{f}$ is the limit of $G\mathbf{Z}_n$. By the definition of G, for any $\left(z_i^n, \mathbf{z}_i^n\right)$, $\left(z_{i+1}^n, \mathbf{z}_{i+1}^n\right)$ in $\mathbf{Z}_n^{\nu_n}$ one can find a chain $\mathbf{X}_i^n$ such that

$$0 \leq D\mathbf{z}_i^n \mathbf{X}_i^n \mathbf{z}_{i+1}^n - G\mathbf{z}_i^n \mathbf{z}_{i+1}^n \leq \frac{1}{n\left(k_n+1\right)}.$$

Denoting

$$\mathbf{U}_n = \mathbf{z}_0^n \mathbf{X}_0^n \mathbf{z}_1^n \ldots \mathbf{z}_{k_n}^n \mathbf{X}_{k_n}^n \mathbf{z}_{k_n+1}^n,$$

it follows that

$$0 \leq D\mathbf{U}_n - G\mathbf{Z}_n \leq \frac{1}{n},$$

i.e., $D\mathbf{U}_n \to G\mathbf{f}$. For every n and $i = 0, \ldots, k_n$, we pair all elements of $\mathbf{X}_i^n = \mathbf{x}_1^{i,n} \ldots \mathbf{x}_{l_{i,n}}^{i,n}$ in $\mathbf{z}_i^n \mathbf{X}_i^n \mathbf{z}_{i+1}^n$ with z_i^n to create chains-on-nets

$$\mathbf{U}_n^{\mu_n} = \left(\ldots, (z_i^n, \mathbf{z}_i^n), \left(z_i^n, \mathbf{x}_1^{i,n}\right), \ldots, \left(z_i^n, \mathbf{x}_{l_{i,n}}^{i,n}\right)(z_{i+1}^n, \mathbf{z}_{i+1}^n), \ldots\right).$$

Since $D\mathbf{U}_n \to G\mathbf{f}$, to prove $D\mathbf{f} \leq G\mathbf{f}$ it will suffice to show that $\mathbf{U}_n^{\mu_n} \to \mathbf{f}$. The latter is equivalent to $\sigma_{\mathbf{f}}\left(\mathbf{U}_n^{\mu_n}\right) \to 0$, because, clearly, $\delta\mu_n = \delta\nu_n \to 0$. For every n, let $\left(z_{i_n}^n, \mathbf{m}_{i_n}^n\right)$ be an element of $\mathbf{U}_n^{\mu_n}$ such that

$$\sigma_{\mathbf{f}}\left(\mathbf{U}_n^{\mu_n}\right) = D\mathbf{f}\left(z_{i_n}^n\right)\mathbf{m}_{i_n}^n = D\mathbf{z}_{i_n}^n \mathbf{m}_{i_n}^n.$$

$\mathbf{f}$ being uniformly continuous,

$$G\mathbf{z}_{i_n}^n \mathbf{z}_{i_n+1}^n = G\mathbf{f}\left(z_{i_n}^n\right)\mathbf{f}\left(z_{i_n+1}^n\right) \to 0$$

as $\delta\mu_n = \delta\nu_n \to 0$. By the construction of $\mathbf{U}_n$, this implies

$$D\mathbf{z}_{i_n}^n \mathbf{X}_{i_n}^n \mathbf{z}_{i_n+1}^n \to 0,$$

and then, on denoting by $\mathbf{z}_{i_n}^n \mathbf{Y}_{i_n}^n \mathbf{m}_{i_n}^n$ the subchain of $\mathbf{z}_{i_n}^n \mathbf{X}_{i_n}^n$ connecting $\mathbf{z}_{i_n}^n$ to $\mathbf{m}_{i_n}^n$,

$$D\mathbf{z}_{i_n}^n \mathbf{Y}_{i_n}^n \mathbf{m}_{i_n}^n \to 0.$$

The convergence

$$\sigma_{\mathbf{f}}\left(\mathbf{U}_n^{\mu_n}\right) = D\mathbf{z}_{i_n}^n \mathbf{m}_{i_n}^n \to 0$$

now follows by Property $\mathcal{D}4$ of dissimilarity functions. ■

6.3. Basic Properties of D-Length for Paths and Arcs

The properties established in this section parallel the basic properties of path length in the traditional, metric-based theory. All of them can be proved "directly," in terms of dissimilarity D alone. Theorem 9, however, offers a more economic way of dealing with them, based on the following "meta-proposition":

> *A proposition formulated entirely in terms of D-lengths of paths, for an arbitrary dissimilarity function D, is true if and only if the same proposition is true with D being an arbitrary (oriented) metric.*

Unfortunately, it is not possible to simply formulate a proposition, invoke the identity $D\mathbf{f} = G\mathbf{f}$, and refer the reader to the literature, because in the relevant literature known to the author metric is always taken to be symmetric. In some cases the symmetry requirement is critical. Even when it is not, it is often more difficult to convince the reader that an existing proof can be modified (generalized) not to rely on symmetry than to present an explicit proof, with or without reliance on Theorem 9.

The first issue to be considered is the (in)dependence of the D-length of a path on the path's parametrization. In Section 4 we emphasized in relation to Fig. 6 that the D-length of a path is not determined by its image $\mathbf{f}([a, b])$ alone but by the function $\mathbf{f} : [a, b] \mapsto \mathfrak{S}$. Nevertheless two paths $\mathbf{f}|\,[a, b]$ and $\mathbf{g}|\,[c, d]$ with one and the same image do have the same D-length if they are related to each other in a certain way. Specifically, this happens if $\mathbf{f}$ and $\mathbf{g}$ are each others' *reparametrizations*, by which we mean that for some nondecreasing and onto (hence continuous) mapping $\phi : [c, d] \mapsto [a, b]$,

$$\mathbf{g}(x) = \mathbf{f}(\phi(x)), \qquad x \in [c, d]. \tag{29}$$

Note that we use a "symmetrical" terminology (*each other's* reparametrizations) even though the mapping ϕ is not assumed to be invertible (see Fig. 9). If it is invertible, then it is an increasing homeomorphism, and then it is easy to show that $D\mathbf{f} = D\mathbf{g}$. The theorem stated next shows this for the general case.

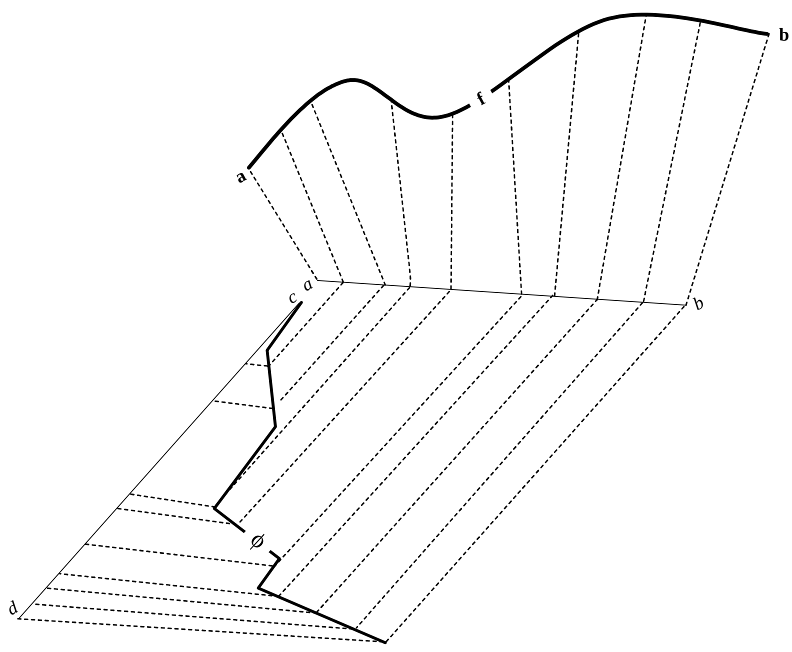

Figure 9. Illustration to Theorem 10. $\mathbf{f}$ is a path with respect to $[a, b]$, and it is also a path with respect to $[c, d]$ mapped onto $[a, b]$ by a nondecreasing function ϕ.

Theorem 10 Given two functions $\mathbf{g}|\,[c, d]$ and $\mathbf{f}|\,[a, b]$ such that

$$\mathbf{g}(x) = \mathbf{f}(\phi(x))$$

for some nondecreasing and onto (hence continuous) mapping

$$\phi : [c,d] \mapsto [a,b],$$

if one of the functions $\mathbf{f}, \mathbf{g}$ is a path then so is the other, and

$$D\mathbf{g} = D\mathbf{f}.$$

Proof. The functions $\mathbf{f}$ and $\mathbf{g}$ are paths if and only if they are continuous (hence uniformly continuous). That the continuity of $\mathbf{f}$ implies the continuity of $\mathbf{g}$ is obvious: $\mathbf{g}$ is the composition of two continuous functions, ϕ and $\mathbf{f}$. The converse is proved by observing that if $\mathfrak{s}$ is a closed subset of $\mathbf{f}([a,b])$, $\mathbf{f}^{-1}(\mathfrak{s})$ is closed in $[a,b]$: indeed, $\mathbf{g}^{-1}(\mathfrak{s})$ is closed (hence compact) in $[c,d]$, and then $\mathbf{f}^{-1}(\mathfrak{s}) = \phi\left(\mathbf{g}^{-1}(\mathfrak{s})\right)$ is compact (hence closed) in $[a,b]$.

The equality $D\mathbf{g} = D\mathbf{f}$ is equivalent to $G\mathbf{g} = G\mathbf{f}$, and we prove the latter by considering the set P_c^d of all nets on $[c,d]$, and the set $P_a^b = \phi\left(P_c^d\right)$ of the nets on $[a,b]$ obtained by applying ϕ to each element of each net in P_c^d. It is clear that P_a^b contains all possible nets on $[a,b]$, and that for any two nets $\mu \in P_c^d$ and $\phi(\mu) \in P_a^b$, the corresponding chains $\mathbf{f}(\mu)$ and $\mathbf{f}(\phi(\mu))$ (where $\mathbf{f}$ is applied elementwise) are identical. It follows that the suprema taken across the G-lengths of the inscribed chains $\mathbf{f}\left(P_c^d\right)$ and $\mathbf{f}\left(P_a^b\right)$ (where, again, $\mathbf{f}$ is applied to each element of each net) should be identical. ■

We establish next the uniform continuity of length traversed along a path (see Fig. 10).

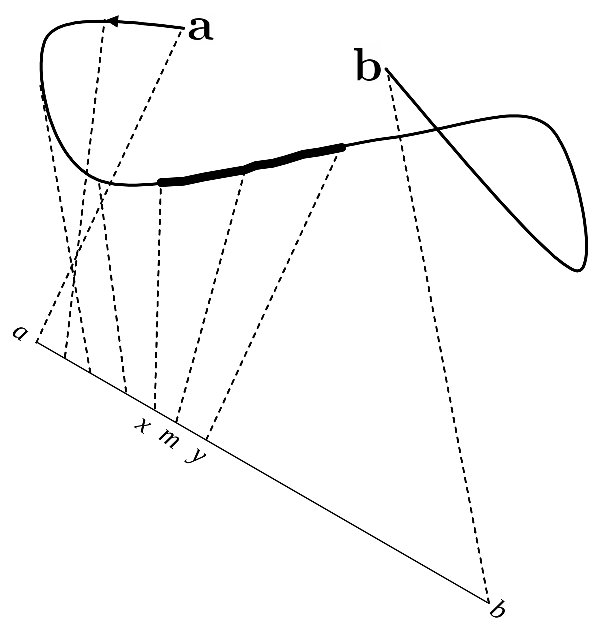

Figure 10. Illustration to Theorem 11. As x and y get closer to m, the length of the corresponding piece of the path gradually vanishes.

THEOREM 11 For any D-rectifiable path $\mathbf{f}|\,[a,b]$ and $[x,y] \subset [a,b]$, $D\mathbf{f}\,([x,y])$ is uniformly continuous in (x,y), nondecreasing in y and nonincreasing in x.

Proof. By the additivity property (Theorem 2), the (nonstrict) increase-in-y and decrease-in-x of $D\mathbf{f}\,([x,y])$ is obvious. To prove the uniform continuity of $D\mathbf{f}\,([x,y]) = G\mathbf{f}\,([x,y])$ it suffices to show that $G\mathbf{f}\,([x,y]) \to 0$ as $y - x \to 0$. We create a net

$$\mu = (a = x_0, x_1, \ldots, x_k = x, y = y_1, \ldots, y_k, y_{k+1} = b)\,.$$

By the uniform continuity of $\mathbf{f}$, and by Theorem 7 (see the remark following its proof), for any $\varepsilon > 0$, $\delta\mu$ can be chosen sufficiently small to ensure both

$$G\mathbf{f}\,(x)\,\mathbf{f}\,(y) < \varepsilon$$

and

$$G\mathbf{X}{+}G\mathbf{f}\,(x)\,\mathbf{f}\,(y) + G\mathbf{Y} > G\mathbf{f}\,([a,b]) - \varepsilon,$$

where $\mathbf{X} = \mathbf{f}\,(x_0, x_1, \ldots, x_k)$, $\mathbf{Y} = \mathbf{f}\,(y_1, \ldots, y_k, y_{k+1})$ (elementwise). At the same time,

$$\begin{aligned} G\mathbf{X} &\leq G\mathbf{f}\,([a,x])\,, \\ G\mathbf{Y} &\leq G\mathbf{f}\,([x,b])\,, \end{aligned}$$

and combining this with the previous inequality,

$$G\mathbf{f}\,([a,x]) + \varepsilon + G\mathbf{f}\,([x,b]) > G\mathbf{f}\,([a,b]) - \varepsilon = G\mathbf{f}\,([a,x]) + G\mathbf{f}\,([x,y]) + G\mathbf{f}\,([x,b]) - \varepsilon.$$

Then

$$G\mathbf{f}\,([x,y]) < 2\varepsilon,$$

and this completes the proof. ■

We observe next that any D-rectifiable path $\mathbf{f}|\,[a,b]$ can be reparametrized into $\mathbf{n}|\,[0,D_0]$, where $D_0 = D\mathbf{f}\,([a,b])$, so that $\mathbf{f}\,(x)$ for any $x \in [a,b]$ corresponds to the D-length $D\mathbf{f}\,([0,x])$ in its new domain $[0,D_0]$. By analogy with the traditional terminology, we call this reparametrization of $\mathbf{f}$ its *natural D-parametrization.*

THEOREM 12 Any path $\mathbf{f}|\,[a,b]$ with $D\mathbf{f}\,([a,b]) = D_0 < \infty$ permits the (unique) natural D-parametrization $\mathbf{n}|\,[0,D_0]$, with

$$\mathbf{f}\,(x) = \mathbf{n}\,(\phi\,(x))\,,$$

for any $x \in [a,b]$, and

$$D\mathbf{n}\,([u,v]) = v - u,$$

for any $0 \leq u \leq v \leq D_0$.

Proof. By Theorem 11, $\phi : x \mapsto D\mathbf{f}([a,x])$ is a nondecreasing continuous mapping with $\phi(a) = 0$ and $\phi(x) = D\mathbf{f}([a,x])$. Hence ϕ maps $[a,x]$ onto $[0,\phi(x)]$. By Theorem 10, $\mathbf{n}|\,[0,\phi(x)]$ defined by $\mathbf{f}(u) = \mathbf{n}(\phi(u))$ is a path with $D\mathbf{n}([0,\phi(x)]) = D\mathbf{f}([a,x]) = \phi(x)$. Since $\phi(x)$ takes on all values on $[0,D_0]$, we have $\mathbf{n} : [0,D_0] \mapsto \mathfrak{S}$ with $D\mathbf{n}([0,v]) = v$. By additivity (Theorem 2), $D\mathbf{n}([u,v]) = D\mathbf{n}([0,v]) - D\mathbf{n}([0,u]) = v-u$. ■

We define an *arc* as a path which can be reparametrized into a homeomorphic path. In other words, $\mathbf{g}|\,[c,d]$ is an arc if one can find a nondecreasing and onto (hence continuous) mapping $\phi : [c,d] \mapsto [a,b]$, such that, for some one-to-one and continuous (hence homeomorphic) function $\mathbf{f} : [a,b] \mapsto \mathfrak{S}$,

$$\mathbf{g}(x) = \mathbf{f}(\phi(x)), \tag{30}$$

for any $x \in [c,d]$. The following lemma provides a simple characterizations of arcs.

LEMMA 2 A path $\mathbf{g}|\,[c,d]$ is an arc if and only if, for any $\mathbf{x} \in \mathbf{g}([c,d])$, $\mathbf{g}^{-1}(\{\mathbf{x}\})$ is an interval (necessarily closed) in $[c,d]$. If $\mathbf{g}$ is a D-rectifiable arc, its natural D-parametrization is a homeomorphism.

Proof. If $\mathbf{f}|\,[a,b]$ is a homeomorphism, and $\mathbf{g}(x) = \mathbf{f}(\phi(x))$ for all $x \in [c,d]$, then

$$\mathbf{g}^{-1}(\{\mathbf{x}\}) = \phi^{-1}\left(\mathbf{f}^{-1}(\{\mathbf{x}\})\right),$$

where $\mathbf{f}^{-1}(\{\mathbf{x}\})$ is a singleton $\{y\}$ in $[a,b]$. Since ϕ is onto and nondecreasing, $\phi(\{y\})$ is either a singleton or a nondegenerate interval on which ϕ is constant (a closed interval, because ϕ is continuous). Conversely, if $\mathbf{g}^{-1}(\{\mathbf{x}\})$ for any $\mathbf{x} \in \mathbf{g}([c,d])$ is an interval (closed, because $\mathbf{g}$ is continuous), then all the nondegenerate intervals $\mathbf{g}^{-1}(\{\mathbf{x}\})$ are pairwise non-intersecting, and their set therefore is at most denumerable (choose in each of these intervals a rational point, establishing thereby a bijection between them and a subset of rationals). Any continuous nondecreasing function $\phi : [c,d] \mapsto [a,b]$ which is constant on each of these intervals will reparametrize $\mathbf{g}$ into a homeomorphism. Such a ϕ can be constructed in a variety of well-known ways.[5] The last statement of the lemma now immediately follows from the fact that $D([a,x])$ is a nondecreasing continuous function constant on each of the nondegenerate intervals $\mathbf{g}^{-1}(\{\mathbf{x}\})$. ■

We show next that any path contains an arc with the same endpoints and a smaller D-length (Fig. 11). This is one case when the classical theory (see, e.g., Blumenthal, 1953, p. 69) does not help us at all, as it critically relies on the symmetry requirement. The result is important, in particular, in the context of

[5] Here is one. Having enumerated the intervals arbitrarily $[u_1,v_1], [u_2,v_2], \ldots$, put $\phi(x) = u_k$ if $x \in [u_k,v_k]$, $k = 1,2,\ldots$. For $x \notin \bigcup_i [u_i,v_i]$, let $x_v = \sup_k \{v_k : v_k < x\}$, $x_u = \inf_k \{u_k : u_k > x\}$. Clearly, $x_v \le x \le x_u$. Put $\phi(x_v) = \lim_{v_k \to x_v -} \phi(v_k) = \lim_{v_k \to x_v -} u_k$. It is easy to check that this quantity either equals u_k, for some k (if there is a maximal $v_k < x$), or it equals x_v (if every left-hand neighborhood of x_v contains an interval $[u_k,v_k]$). Analogously, put $\phi(x_u) = \lim_{u_k \to x_u +} \phi(u_k) = \lim_{u_k \to x_u +} u_k$ (which equals either u_k, for some k, or x_u). Finally, determine $\phi(x)$ by the linear interpolation between $(x_v, \phi(x_v))$ and $(x_u, \phi(x_u))$: $\phi(x) = \phi(x_v) + (\phi(x_u) - \phi(x_v)) \frac{x - x_v}{x_u - x_v}$, with the understanding that $\phi(x) = \phi(x_v) = \phi(x_u)$ if $x_u = x_v$. The function is clearly nondecreasing and continuous.

searching for shortest paths connecting one point to another (see Section 7): in the absence of additional constraints this search can be confined to arcs only.

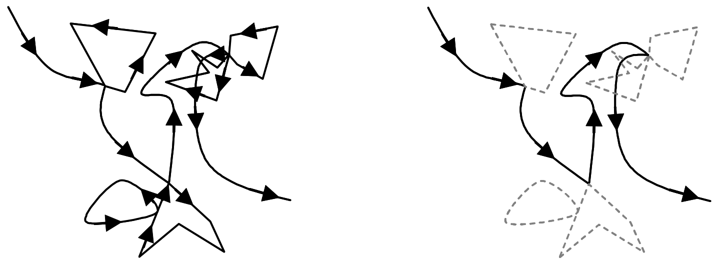

Figure 11. Illustration to Theorem 13. One can remove closed loops from a path and be left with a shorter arc.

THEOREM 13 Let $\mathbf{f}|\,[a,b]$ be a D-rectifiable path connecting $\mathbf{a}$ to $\mathbf{b}$. Then there is an arc $\mathbf{g}|\,[a,b]$ connecting $\mathbf{a}$ to $\mathbf{b}$, such that

$$\mathbf{g}\left([a,b]\right) \subset \mathbf{f}\left([a,b]\right),$$

and

$$D\mathbf{g}\left([a,b]\right) \leq D\mathbf{f}\left([a,b]\right),$$

where the inequality is strict if $\mathbf{f}|\,[a,b]$ is not an arc.

COMMENT If $D\mathbf{f}\left([a,b]\right) = \infty$, we can invoke the topological theorem (Hocking &Young, 1961, pp. 116-117) that guarantees the existence of an arc $\mathbf{g}|\,[a,b]$ with $\{\mathbf{a},\mathbf{b}\} \subset \mathbf{g}\left([a,b]\right) \subset \mathbf{f}\left([a,b]\right)$. The inequality $D\mathbf{g}\left([a,b]\right) \leq D\mathbf{f}\left([a,b]\right) = \infty$ then holds trivially, but the equality, $D\mathbf{g}\left([a,b]\right) = \infty$, may hold even if $\mathbf{f}|\,[a,b]$ is not an arc.

Proof. Let us begin by assuming the existence (to be proved later) of an at most denumerable set

$$L^* = \{[u_1,v_1],[u_2,v_2],\ldots\}$$

of closed intervals in $[a,b]$ with the following properties:

(A) the intervals are nondegenerate, and for any k, $\mathbf{f}\left(u_k\right) = \mathbf{f}\left(v_k\right)$;

(B) if $k \neq l$, $[u_k,v_k] \cap [u_l,v_l] = \varnothing$;

(C) for any distinct x,y outside $\bigcup_i [u_i,v_i]$, $\mathbf{f}\left(x\right) \neq \mathbf{f}\left(y\right)$;

(D) if $x \notin \bigcup_i]u_i,v_i[$ and $x \notin \{u_k,v_k\}$, then $\mathbf{f}\left(x\right) \neq \mathbf{f}\left(u_k\right)$.

Define $\mathbf{g}|\,[a,b]$ as

$$\mathbf{g}\left(x\right) = \begin{cases} \mathbf{f}\left(x\right) & \text{if } x \notin \bigcup_i [u_i,v_i] \\ \mathbf{f}\left(u_k\right) & \text{if } x \in [u_k,v_k], \text{ for some } k. \end{cases}$$

Then, for any $\mathbf{x} \in \mathbf{g}\left([a,b]\right)$, $\mathbf{g}^{-1}\left(\{\mathbf{x}\}\right)$ is either a singleton or one of the intervals $[u_k,v_k]$. By Lemma 2, $\mathbf{g}|\,[a,b]$ is an arc. Consider any sequence of nets μ_n on $[a,b]$ with $\delta\mu_n \to 0$, and modify them as follows:

for any two successive elements x_i^n, x_{i+1}^n in μ_n, if $x_i^n \in \left]u_k, v_k\right[$ and $x_{i+1}^n > v_k$, for some k, insert v_k in μ_n after x_i^n; analogously, if $x_{i+1}^n \in \left]u_k, v_k\right[$ and $x_i^n < u_k$, for some k, insert u_k in μ_n before x_{i+1}^n. Denoting the modified nets by ν_n, clearly, $\delta\nu_n \leq \delta\mu_n$, so $\delta\nu_n \to 0$. Denoting, as before, the inscribed chains corresponding to ν_n by $\mathbf{f}(\nu_n)$ and $\mathbf{g}(\nu_n)$ (elementwise), by Theorem 7,

$$\begin{aligned} G\mathbf{f}(\nu_n) &\to G\mathbf{f}([a,b]), \\ G\mathbf{g}(\nu_n) &\to G\mathbf{g}([a,b]). \end{aligned}$$

It is easy to see that, for any two successive elements y_i^n, y_{i+1}^n in ν_n, the corresponding links $\mathbf{f}(y_i^n)\,\mathbf{f}\left(y_{i+1}^n\right)$ and $\mathbf{g}(y_i^n)\,\mathbf{g}\left(y_{i+1}^n\right)$ are identical, except when $y_i^n, y_{i+1}^n \in [u_k, v_k]$, for some k. In the latter case,

$$G\mathbf{f}(y_i^n)\,\mathbf{f}\left(y_{i+1}^n\right) \geq G\mathbf{g}(y_i^n)\,\mathbf{g}\left(y_{i+1}^n\right) = 0,$$

so we conclude that

$$G\mathbf{f}(\nu_n) \geq G\mathbf{g}(\nu_n),$$

for all n. But then

$$G\mathbf{f}([a,b]) \geq G\mathbf{g}([a,b]).$$

Now, if $\mathbf{f}(x)$ is constant on each of the intervals $[u_k, v_k]$, then $\mathbf{f}$ is an arc, and $\mathbf{g} \equiv \mathbf{f}$. Suppose that $\mathbf{f}(x)$ is not constant in, say, $[u_1, v_1]$. Applying the above reasoning to $\mathbf{f}|\,[a, u_1]$ and $\mathbf{g}|\,[a, u_1]$, we have

$$G\mathbf{f}([a,u_1]) \geq G\mathbf{g}([a,u_1]),$$

and analogously,

$$G\mathbf{f}([v_1,b]) \geq G\mathbf{g}([v_1,b]).$$

But, by Theorem 3, $G\mathbf{f}([u_1, v_1]) > 0$ while $G\mathbf{g}([u_1, v_1]) = 0$. It follows, using the additivity property (Theorem 2), that

$$G\mathbf{f}([a,b]) > G\mathbf{g}([a,b]).$$

whenever $\mathbf{f}|\,[a,b]$ is not an arc.

It remains to show that the set L^* of closed intervals satisfying the properties A-D above does exist.[6] Let us call an open interval $\left]s,t\right[\subset [a,b]$ a *loop interval* if $s < t$ and $\mathbf{f}(s) = \mathbf{f}(t)$. We assume that the set of loop intervals is nonempty (otherwise $\mathbf{f}$ is an arc, and no further considerations are needed). One can always choose an at most denumerable subset L of the loop intervals, such that the intervals in L are pairwise nonoverlapping, and any loop interval outside L overlaps with the union $\bigcup L$ of the intervals in L. To show this, let us call any set of pairwise nonoverlapping loop intervals a *simple set* (of loop intervals). Let $(\Lambda, \subset)$ be the set of all simple sets, partially ordered by inclusion. Since at least one loop interval exists, Λ is nonempty. Let Υ be any subset of Λ such that $(\Upsilon, \subset)$ is a linear order. The union $\bigcup \Upsilon$ of all elements of Υ is clearly a simple set, hence it is an upper bound for $(\Upsilon, \subset)$. By Zorn's Lemma, $(\Lambda, \subset)$ contains a

[6] The remainder of the proof is suggested by D. Dzhafarov.

maximal element – a simple set L that cannot be included in a larger simple set. But this means that any loop interval outside L overlaps with some of the intervals in L.

Let us call this L a *basic set* (of loop intervals). L is not generally uniquely determined by $\mathbf{f}|\,[a,b]$, so let $(\Omega,\prec)$ be the set of all basic sets, partially ordered in the following way: $L \prec L'$ means that every interval in L is strictly contained within an interval of L'. (Note that this implies that the endpoints of every interval in L' lie outside $\bigcup L$.) We already know that Ω is nonempty. Denoting by Θ a subset of Ω such that $(\Theta,\prec)$ is a linear order, consider the set $S = \bigcup_{L\in\Theta} \bigcup L$. Clearly, S is an open set in $[a,b]$, such that, for every $L \in \Theta$, every interval in L is contained within a component (maximal open subinterval) of S. Choosing arbitrary $L \in \Theta$, since every loop interval $]s,t[$ either belongs to L (in which case it is included in a component of S) or is outside L (in which case it overlaps with $\bigcup L$, hence also with S), the set of the components of S is a basic set which is an upper bound for $(\Theta,\prec)$. Invoking Zorn's Lemma again, we establish that $(\Omega,\subset)$ contains a maximal element – a basic set L_* that cannot be included in a larger basic set. This means that no component of L_* can be included in a larger loop interval whose endpoints lie outside $\bigcup L_*$.

Let the set of the loop intervals in L_* be arbitrarily enumerated,

$$L_* = \{]u_1,v_1[\,,]u_2,v_2[\,,\ldots,\}\,.$$

Put

$$L^* = \{[u_1,v_1]\,,[u_2,v_2]\,,\ldots\}\,.$$

The property A holds for L^* trivially. Deny B, and let for some $k \neq l$, $[u_k,v_k] \cap [u_l,v_l] \neq \varnothing$. Let, for definiteness, $u_k \leq u_l$. Since $]u_k,v_k[\,\cap\,]u_l,v_l[\, = \varnothing$ (L_* being a basic set), we must have $u_k < v_k = u_l < v_l$. Then $\mathbf{f}(v_k) = \mathbf{f}(u_l)$, and $]u_k,v_k[$ and $]u_l,v_l[$ being loop intervals, $\mathbf{f}(u_k) = \mathbf{f}(v_l)$. But then $]u_k,v_l[$ is a loop interval with endpoints outside $\bigcup L_*$, such that $]u_k,v_k[\,\subset\,]u_k,v_l[$, which contradicts the maximality of L_* in $(\Omega,\prec)$. The properties C and D are demonstrated by analogous arguments. This completes the proof. ■

6.4. Arclength Metric

In relation to Question 2 posed in Introduction, we are also interested in the following metric induced by the dissimilarity function D. We call it the *arclength metric*, and we denote it A_D. It is defined as

$$A_D\mathbf{ab} = \inf_{\mathbf{f}\in\mathcal{A}_{\mathbf{a}}^{\mathbf{b}}} D\mathbf{f}, \tag{31}$$

where $\mathcal{A}_{\mathbf{a}}^{\mathbf{b}}$ is the class of all *arcs* connecting $\mathbf{a}$ to $\mathbf{b}$.

Note that if we allow the class $\mathcal{A}_{\mathbf{a}}^{\mathbf{b}}$ in this definition to include paths rather than arcs only, the value of $A_D\mathbf{ab}$ will not change: every path $\mathbf{f}$ contains an arc with the same endpoints which is not longer than $\mathbf{f}$ (see Theorem 13).

That A_D is a metric is shown in the theorem below. Strictly speaking, it is not a metric proper but a metric with *extended range*,

$$A_D : \mathfrak{S}\times\mathfrak{S} \mapsto \mathbb{R}^+ \cup \{\infty\}\,.$$

In other words, for some pairs $\mathbf{a}, \mathbf{b}$ the value of $A_D\mathbf{ab}$ may be ∞.

THEOREM 14 A_D is a metric (oriented, with extended range).

Proof. The implication $\mathbf{a} = \mathbf{b} \implies A_D\mathbf{ab} = 0$ is obvious. To show the reverse implication, for any sequence of arcs $\mathbf{f}_n$ (connecting $\mathbf{a}$ to $\mathbf{b}$) with $D\mathbf{f}_n \to 0$ choose a sequence of chains-on-nets $\mathbf{X}_n^{\mu_n}$ (connecting $\mathbf{a}$ to $\mathbf{b}$) with $|D\mathbf{X}_n - D\mathbf{f}_n| \to 0$ to obtain $D\mathbf{X}_n \to 0$ and, since D is dissimilarity, $D\mathbf{ab} = 0$. Finally, the triangle inequality follows from the fact that for any arc $\mathbf{f}$ connecting $\mathbf{a}$ to $\mathbf{b}$ and any arc $\mathbf{g}$ connecting $\mathbf{b}$ to $\mathbf{c}$, $\mathbf{f} \cup \mathbf{g}$ (with the domain of $\mathbf{g}$ shifted, if necessary, to ensure that its lower endpoint coincides with the upper endpoint of the domain of $\mathbf{f}$) is a path connecting $\mathbf{a}$ to $\mathbf{c}$. By Theorem 13, $\mathbf{f} \cup \mathbf{g}$ contains an arc $\mathbf{h}$ connecting $\mathbf{a}$ to $\mathbf{c}$ with $D\mathbf{h} \leq D(\mathbf{f} \cup \mathbf{g}) = D\mathbf{f} + D\mathbf{g}$. ■

The A_D*-length* of a path $\mathbf{f}$ is defined as

$$A_D\mathbf{f} = \liminf_{\substack{\mathbf{X}^\mu \in \mathcal{M}_a^b \\ \mathbf{X}^\mu \to \mathbf{f}}} A_D\mathbf{X}, \tag{32}$$

where (putting $\mathbf{X} = \mathbf{x}_0\mathbf{x}_1 \ldots \mathbf{x}_k\mathbf{x}_{k+1}$)

$$A_D\mathbf{X} = \sum_{i=0}^{k} A_D\mathbf{x}_i\mathbf{x}_{i+1}. \tag{33}$$

Since A_D is a metric (it is easy to check that its extended range makes no difference),

$$A_D\mathbf{f} = \sup A_D\mathbf{Z} = \lim_{n\to\infty} A_D\mathbf{Z}_n, \tag{34}$$

where the supremum is taken over all inscribed chains-on-nets $\mathbf{Z}^\nu$, and the limit is taken for any sequence of inscribed $\mathbf{Z}_n^{\nu_n}$ with $\delta\nu_n \to 0$.

Due to Theorem 9, the relationship between $D\mathbf{f}$, $G\mathbf{f}$, and $A_D\mathbf{f}$ is simple.

THEOREM 15 For any path $\mathbf{f}$,

$$G\mathbf{f} = A_D\mathbf{f} = D\mathbf{f}.$$

Proof. Consider any chain-on-net $\mathbf{Z}^\nu$ inscribed in $\mathbf{f}$, and choose in it two successive elements $(z_i, \mathbf{z}_i)(z_{i+1}, \mathbf{z}_{i+1})$. Clearly,

$$G\mathbf{z}_i\mathbf{z}_{i+1} \leq A_D\mathbf{z}_i\mathbf{z}_{i+1} \leq D\mathbf{f}([z_i, z_{i+1}]).$$

This implies

$$G\mathbf{Z} \leq A_D\mathbf{Z} \leq D\mathbf{f},$$

whence

$$G\mathbf{f} \leq A_D\mathbf{f} \leq D\mathbf{f}.$$

The statement of the theorem now follows from Theorem 9. ■

We see that A_D and G coincide as far as the path length is concerned. At the same time, generally,

$$A_D\mathbf{ab} \geq G\mathbf{ab}. \tag{35}$$

We can now reformulate Question 2 posed in Introduction as that about conditions under which (kinds of spaces in which)

$$A_D\mathbf{ab} = G\mathbf{ab}, \tag{36}$$

for all $\mathbf{a}, \mathbf{b}$. This equality can be taken as the formal definition of the *intrinsicality* of the metric G (induced by dissimilarity D). We take on this question in the next section.

7. Complete Dissimilarity Spaces With Intermediate Points

Refer to Fig. 12. A dissimilarity space $(\mathfrak{S}, D)$ is said to be a space *with intermediate points* if for any distinct $\mathbf{a}, \mathbf{b}$ one can find an $\mathbf{m}$ such that $\mathbf{m} \notin \{\mathbf{a}, \mathbf{b}\}$ and $D\mathbf{amb} \leq D\mathbf{ab}$.

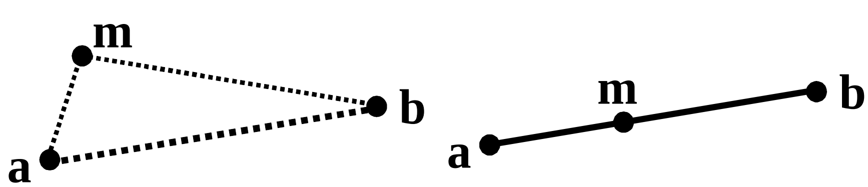

Figure 12. Point $\mathbf{m}$ is intermediate to $\mathbf{a}$ and $\mathbf{b}$ if $D\mathbf{amb} \leq D\mathbf{ab}$. Thus, if D is Euclidean distance (right panel), any $\mathbf{m}$ on the straight line segment connecting $\mathbf{a}$ to $\mathbf{b}$ is intermediate to $\mathbf{a}$ and $\mathbf{b}$.

The notion of a space with intermediate points generalizes the notion of *Menger convexity* (Blumenthal, 1953, p. 41; the term itself is due to Papadopoulos, 2005). If D is a metric, the space is Menger-convex if, for any distinct $\mathbf{a}, \mathbf{b}$, there is a point $\mathbf{m} \notin \{\mathbf{a}, \mathbf{b}\}$ with $D\mathbf{amb} = D\mathbf{ab}$.[7]

Recall that a space is called *complete* if every Cauchy sequence in it converges to a point. Adapted to $(\mathfrak{S}, D)$, the completeness means that given a sequence of points $\mathbf{x}_n$ such that

$$\lim_{\substack{k\to\infty \\ l\to\infty}} D\mathbf{x}_k\mathbf{x}_l = 0,$$

there is a point $\mathbf{x}$ in $\mathfrak{S}$ such that

$$\mathbf{x}_n \leftrightarrow \mathbf{x}.$$

Blumenthal (1953, pp. 41-43) provides a proof attributed to N. Aronszajn that if a Menger-convex space is complete then $\mathbf{a}$ can be connected to $\mathbf{b}$ by a *geodesic arc*, that is, an arc $\mathbf{h}$ with $D\mathbf{h} = D\mathbf{ab}$ (where D is a metric).[8] The main idea of this proof is adopted in the proof of Theorem 16 below. We need some preliminaries first, however.

[7] The original definition is given for symmetric rather than oriented metrics.

[8] This theorem was originally proved by K. Menger, but by different means.

Recall the definition of a trail $\mathbf{f}|S$ for an arbitrary subset S of $[a,b]$ (Section 4). A trail $\mathbf{g}|T$ is called an *extension* of $\mathbf{f}|S$, in symbols $\mathbf{f} \subset \mathbf{g}$, if $S \subset T$ and $\mathbf{f}(x) = \mathbf{g}(x)$ for all $x \in S$.

Lemma 3 If $(\mathfrak{S}, D)$ is complete, any trail $\mathbf{f}|S$ can be uniquely extended into the trail $\mathbf{g}|\overline{S}$ with

$$\mathbf{g}\left(\overline{S}\right) \subset \overline{\mathbf{f}(S)},$$

where the overbar indicates the topological closure operation.

Proof. Let $p \in \overline{S}$, and let $x_n \to p$ $(x_n \in S)$. Since x_n is a Cauchy sequence in $[a,b] \supset S$ and $\mathbf{f}$ is uniformly continuous, we have $D\mathbf{f}(x_n)\mathbf{f}(x_m) \to 0$ as $n \to \infty$ and $m \to \infty$; and since the space $\mathfrak{S}$ is complete, $\mathbf{f}(x_n) \to \mathbf{p}$ for some $\mathbf{p} \in \overline{\mathbf{f}(S)}$. This limit point does not depend on the sequence $x_n \to p$: if S contained a sequence $x'_n \to p$ with $\mathbf{f}(x'_n) \to \mathbf{p}' \neq \mathbf{p}$, we would have had $x_1, x'_1, x_2, x'_2, \ldots \to p$ with $D\mathbf{f}(x_n)\mathbf{f}(x'_n) \to D\mathbf{p}\mathbf{p}' > 0$ which would contradict $|x'_n - x_n| \to 0$. We can therefore put $\mathbf{p} = \mathbf{g}(p)$. Having done so for every $p \in \overline{S}$ we form a continuous mapping $\mathbf{g} : \overline{S} \mapsto \mathfrak{S}$ with $\mathbf{g}(x) = \mathbf{f}(x)$ for every $x \in S$. The uniform continuity of $\mathbf{g}$ is obvious. ■

The trail $\mathbf{g}|\overline{S}$ constructed in this theorem is referred to as the *closure* of the trail $\mathbf{f}|S$ and is denoted by $\overline{\mathbf{f}}$ (or $\overline{\mathbf{f}}|\overline{S}$).

Lemma 4 If $\mathbf{f}|S$ is a trail and $\overline{\mathbf{f}}|\overline{S}$ its closure, then

$$D\mathbf{f}(S) = D\overline{\mathbf{f}}\left(\overline{S}\right).$$

Proof. It is evident that for any chain-on-net $\mathbf{X}^{\mu} = \{(x_i, \mathbf{x}_i)\}_{i=1,\ldots,k}$, $x_i \in \overline{S}$, one can form a chain-on-net $\mathbf{X}^{\nu} = \{(r_i, \mathbf{x}_i)\}_{i=1,\ldots,k}$, $r_i \in S$ (and vice versa) with $\max_i |r_i - x_i|$ chosen so small that $\left|\delta\left(\mu, \overline{S}\right) - \delta(\nu, S)\right|$ is arbitrarily close to zero, and so is the difference $\left|\sigma_{\mathbf{f}|\overline{S}}(\mathbf{X}^{\mu}) - \sigma_{\mathbf{f}|S}(\mathbf{X}^{\nu})\right|$. The latter follows from the uniform continuity of D and $\mathbf{f}$: irrespective of $x, r, \mathbf{x}$, for every $\varepsilon > 0$ one can find a $\rho > 0$ such that

$$D\overline{\mathbf{f}}(x)\mathbf{f}(r) < \rho \Longrightarrow \left|D\overline{\mathbf{f}}(x)\mathbf{x} - D\mathbf{f}(r)\mathbf{x}\right| < \varepsilon;$$

and for every ρ one can find a $\delta > 0$ such that

$$|x - r| < \delta \Longrightarrow D\overline{\mathbf{f}}(x)\mathbf{f}(r) = D\overline{\mathbf{f}}(x)\overline{\mathbf{f}}(r) < \rho.$$

Hence for any sequence $\mathbf{X}_n^{\mu_n} \to \overline{\mathbf{f}}|\overline{S}$ we have $\mathbf{X}_n^{\nu_n} \to \mathbf{f}|S$, and vice versa. Since $D\mathbf{X}_n$ is the same in both sequences, the statement of the theorem follows. ■

A trail $\mathbf{f}|S$ will be called *contractive* if for any $x, y \in S$,

$$D\mathbf{f}([x,y] \cap S) \leq |x - y|. \tag{37}$$

By Theorem 1, any contractive $\mathbf{f}|S$ has the property

$$G\mathbf{f}(x)\mathbf{f}(y) \leq |x - y|, \tag{38}$$

for all $x, y \in S$.

We are ready now to prove the generalization of the Menger-Aronszajn theorem mentioned earlier.

THEOREM 16 In a complete space with intermediate points, any $\mathbf{a}$ can be connected to any $\mathbf{b}$ by an arc $\mathbf{f}$ with

$$D\mathbf{f} \leq D\mathbf{ab}.$$

Proof. Let $D\mathbf{ab} = D_0$. Form the two-element contractive trail $\mathbf{f}_1|\mu_1 = ((0, \mathbf{a}), (D_0, \mathbf{b}))$ and define the sequence of contractive trails $\mathbf{f}_n|\mu_n$ by induction, as follows. Once contractive $\mathbf{f}_1|\mu_1 \subset \ldots \subset \mathbf{f}_n|\mu_n$ have been formed, consider the set $\mathcal{S}_n$ of all contractive trails $\mathbf{f}|\mu \supset \mathbf{f}_n|\mu_n$. (This class is nonempty as it includes $\mathbf{f}_n|\mu_n$.) Subdivide $[0, D_0]$ into successive adjacent intervals $I_1^n, \ldots, I_{2^n}^n$ of length $1/2^n$ each, and for every $\mathbf{f}|\mu \in \mathcal{S}_n$ count the number $\#_n(\mu)$ of the intervals which contain elements of μ. Choose any $\mathbf{f}|\mu \in \mathcal{S}_n$ with the maximal $\#_n(\mu)$ to be $\mathbf{f}_{n+1}|\mu_{n+1}$. On the completion of the induction, we have a sequence of contractive trails $\mathbf{f}_1|\mu_1 \subset \ldots \subset \mathbf{f}_n|\mu_n \subset \ldots$ which define a mapping $\mathbf{h} : M \mapsto \mathfrak{M}$ with

$$M = \bigcup_{i=1}^{\infty} \mu_i, \qquad \mathfrak{M} = \bigcup_{i=1}^{\infty} \mathbf{f}_i(\mu_i).$$

To see that this mapping is uniformly continuous (i.e., it is a trail), consider any sequences x_n, y_n in M with $x_n - y_n \to 0-$. For every x_n, y_n there is a contractive $\mathbf{f}_{k_n}|\mu_{k_n}$ (and all contractive trails with higher indices) in which $x_n, y_n \in \mu_{k_n}$. Then

$$G\mathbf{h}(x_n)\mathbf{h}(y_n) = G\mathbf{f}_{k_n}(x_n)\mathbf{f}_{k_n}(y_n) \leq D\mathbf{f}_{k_n}\left([x_n, y_n] \cap \mu_{k_n}\right) \leq y_n - x_n,$$

whence $\mathbf{h}(x_n) \leftrightarrow \mathbf{h}(y_n)$ as $x_n - y_n \to 0-$. For $x_n - y_n \to 0+$ the consideration is similar. To show that the trail $\mathbf{h}$ is contractive, observe that any $x < y$ in M belong to all μ_n beginning with some value of n. Clearly,

$$\sigma_{\mathbf{h}}(\mathbf{f}_n) \equiv 0,$$
$$\delta([x, y] \cap \mu_n, [x, y] \cap M) \to 0,$$

so

$$\mathbf{f}_n|([x, y] \cap \mu_n) \to \mathbf{h}|([x, y] \cap M).$$

By the lower semicontinuity property (Theorem 4),

$$D\mathbf{h}([x, y] \cap M) \leq \liminf_{n \to \infty} D\mathbf{f}_n([x, y] \cap \mu_n) \leq y - x.$$

By Lemma 3 we extend $\mathbf{h}|M$ into the trail $\overline{\mathbf{h}}|\overline{M}$ and observe that it is contractive too, for, by Lemma 4,

$$D\overline{\mathbf{h}}\left([x, y] \cap \overline{M}\right) = D\mathbf{h}([x, y] \cap M) \leq y - x.$$

We conclude that $\overline{\mathbf{h}}|\overline{M} \in \mathcal{S}_n$ for all n. To prove that $\overline{\mathbf{h}}$ is a path we have to show that $\overline{M} = [0, D_0]$. Assume the contrary: $[0, D_0]$ contains a $q \notin \overline{M}$. Since $\overline{M}$ is closed, $q \in [q', q'']$ such that $\overline{M} \cap [q', q''] = \{q', q''\}$. Then, denoting $\mathbf{q}' = \overline{\mathbf{h}}(q')$, $\mathbf{q}'' = \overline{\mathbf{h}}(q'')$,

$$D\mathbf{q}'\mathbf{q}'' = D\overline{\mathbf{h}}\left(\overline{M} \cap [q', q'']\right) \leq q'' - q'.$$

Define now a pair $(m, \mathbf{m})$ as follows. If $\mathbf{q}' = \mathbf{q}''$, put $\mathbf{m} = \mathbf{q}'$ and $m = \frac{1}{2}(q' + q'')$. If $\mathbf{q}' \neq \mathbf{q}''$, by the definition of the space with intermediate points there is a point $\mathbf{m} \notin \{\mathbf{q}', \mathbf{q}''\}$ such that $D\mathbf{q}'\mathbf{m}\mathbf{q}'' \leq D\mathbf{q}'\mathbf{q}''$. Let $m = q' + \frac{D\mathbf{q}'\mathbf{m}}{D\mathbf{q}'\mathbf{q}''}(q'' - q')$. It is easy to check that $D\mathbf{q}'\mathbf{m} \leq m - q'$ and $D\mathbf{m}\mathbf{q}'' \leq q'' - m$. Then the trail

$$\mathbf{h}^* = \left(\overline{\mathbf{h}}|\overline{M} \cap [0, q']\right) \cup (m, \mathbf{m}) \cup \left(\overline{\mathbf{h}}|\overline{M} \cap [q'', D_0]\right)$$

is contractive, so $\mathbf{h}^* \in \mathcal{S}_n$ for all n. But for a sufficiently large n, one of the intervals $I_1^n, \ldots, I_{2^n}^n$ used in the construction of $\mathbf{f}_1|\mu_1 \subset \ldots \subset \mathbf{f}_n|\mu_n \subset \ldots$ will contain m while itself contained in $]q', q''[$. This means that $\#_n(\mu)$ for any $\mathbf{f}|\mu \in \mathcal{S}_n$ is at least by 1 less than $\#_n\left(\overline{M} \cup \{m\}\right)$ for $\mathbf{h}^*$. This contradicts the criterion used for choosing $\mathbf{f}_{n+1}|\mu_{n+1}$ and proves $\overline{M} = [0, D_0]$.

It remains to observe that $D\overline{\mathbf{h}}([0, D_0]) \leq D_0$ and that by Theorem 13, if $\overline{\mathbf{h}}$ is not an arc it can be made one by removing its loops and ending up with an even smaller D-length. ■

An important consequence and the central point of this section is that $G\mathbf{ab}$ in this space can be viewed as the infimum of lengths of all arcs connecting $\mathbf{a}$ to $\mathbf{b}$. Thus, in a complete space with intermediate points the metric G induced by D is intrinsic.

Recall that

$$A_D\mathbf{ab} = \inf_{\mathbf{f} \in \mathcal{A}_\mathbf{a}^\mathbf{b}} D\mathbf{f},$$

where $\mathcal{A}_\mathbf{a}^\mathbf{b}$ is the class of all arcs connecting $\mathbf{a}$ to $\mathbf{b}$.

THEOREM 17 In a complete space with intermediate points,

$$G\mathbf{ab} = A_D\mathbf{ab}.$$

Proof. This is a corollary to Theorem 16. For any sequence of chains-on-nets $\mathbf{X}_n$ connecting $\mathbf{a}$ to $\mathbf{b}$ with $D\mathbf{X}_n \to G\mathbf{ab}$, each chain can be replaced with an at least as D-short a path $\mathbf{f}_n$, by replacing each link $\mathbf{x}_{i_n}\mathbf{x}_{i_n+1}$ in each chain $\mathbf{X}_n$ with an appropriately chosen arc. Then each $\mathbf{f}_n$ can be transformed into an arc by Theorem 13. This establishes $G\mathbf{ab} \geq A_D\mathbf{ab}$, and the equality follows from Theorem 1. ■

8. UFS in Arc-Connected Spaces

The application of the DC theory for arc-connected spaces to discrimination probabilities is straightforward: simply substitute $\Psi^{(1)}$ or $\Psi^{(2)}$ for D. It is important, however, to establish that the main notions of DC have the same meaning, and that the main computations yield the same values, whether one uses $\Psi^{(1)}$ or $\Psi^{(2)}$ (see the section on the logic of Fechnerian Scaling in Dzhafarov & Colonius, 2007).

THEOREM 18 If $\mathbf{f}$ is a trail, path, or arc in $\left(\mathfrak{S}, \Psi^{(1)}\right)$ then it has the same designation in $\left(\mathfrak{S}, \Psi^{(2)}\right)$. The meaning of convergence $\mathbf{X}_n^{\mu_n} \to \mathbf{f}$ or $\mathbf{f}_n \to \mathbf{f}$ is also the same in $\left(\mathfrak{S}, \Psi^{(1)}\right)$ and $\left(\mathfrak{S}, \Psi^{(2)}\right)$.

Proof. Follows from the equivalence of the topologies and uniformities induced by $\Psi^{(1)}$ and $\Psi^{(2)}$. ■

The notion of the space with intermediate points is also precisely the same for $\Psi^{(1)}$ and $\Psi^{(2)}$.

THEOREM 19 $(\mathfrak{S}, \Psi^{(1)})$ is a space with intermediate points if and only if so is $(\mathfrak{S}, \Psi^{(2)})$. $(\mathfrak{S}, \Psi^{(1)})$ is complete if and only if so is $(\mathfrak{S}, \Psi^{(2)})$.

Proof. For distinct $\mathbf{a}, \mathbf{b}$, the existence of an $\mathbf{m} \notin \{\mathbf{a}, \mathbf{b}\}$ such that $\Psi^{(1)}\mathbf{amb} \leq \Psi^{(1)}\mathbf{ab}$ translates into

$$(\psi\mathbf{am} - \psi\mathbf{aa}) + (\psi\mathbf{mb} - \psi\mathbf{mm}) \leq (\psi\mathbf{ab} - \psi\mathbf{aa}),$$

or

$$\psi\mathbf{am}+\psi\mathbf{mb} - \psi\mathbf{mm} \leq \psi\mathbf{ab}.$$

But the latter is equivalent to

$$(\psi\mathbf{am} - \psi\mathbf{mm}) + (\psi\mathbf{mb} - \psi\mathbf{bb}) \leq (\psi\mathbf{ab} - \psi\mathbf{bb}),$$

or

$$\Psi^{(2)}\mathbf{bma} \leq \Psi^{(1)}\mathbf{ba}$$

Since the choice of $\mathbf{a}, \mathbf{b}$ is arbitrary, the first statement is proved. The second statement follows from the equivalence of the uniformities induced by $\Psi^{(1)}$ and $\Psi^{(2)}$. ■

To continue we need to deal with chains, trails, and chains-on-nets "traversed in the opposite direction." For a chain $\mathbf{X} = \mathbf{x}_1 \ldots \mathbf{x}_k$, the reverse chain $\widetilde{\mathbf{X}} = \mathbf{x}'_1 \ldots \mathbf{x}'_k$ is defined by $\mathbf{x}'_i = \mathbf{x}_{k+1-i}$ $(i = 1, \ldots, k)$. For a trail $\mathbf{f} : S \mapsto \mathfrak{S}$ $(S \subset [a, b])$, the reverse trail $\widetilde{\mathbf{f}} : \widetilde{S} \mapsto \mathfrak{S}$ is defined by $\widetilde{S} = \{x : (a + b - x) \in S\}$ and $\widetilde{\mathbf{f}}(x) = \mathbf{f}(a + b - x)$. If $\mathbf{f}$ is a path $[a, b] \mapsto \mathfrak{S}$ then $\widetilde{\mathbf{f}}$ is also a path $[a, b] \mapsto \mathfrak{S}$, with $\widetilde{\mathbf{f}}(x) = \mathbf{f}(a + b - x)$.

For a chain-on-net $\mathbf{X}^\mu = (\mathbf{x}_1, x_1) \ldots (\mathbf{x}_k, x_k)$, with $x_1 = a$, $x_k = b$, the reverse chain-on-net $\widetilde{\mathbf{X}^\mu} = (\mathbf{x}'_1, x'_1) \ldots (\mathbf{x}'_k, x'_k)$ is defined by $\mathbf{x}'_i = \mathbf{x}_{k+1-i}$ and $x'_i = a + b - x$.

THEOREM 20 For any chain $\mathbf{X}$ connecting $\mathbf{a}$ to $\mathbf{b}$,

$$\Psi^{(1)}\mathbf{X} - \Psi^{(2)}\widetilde{\mathbf{X}} = \psi\mathbf{bb} - \psi\mathbf{aa}.$$

Proof. For $\mathbf{X} = \mathbf{a}\mathbf{x}_1 \ldots \mathbf{x}_k\mathbf{b}$, putting, as always, $\mathbf{x}_0 = \mathbf{a}$, $\mathbf{x}_{k+1} = \mathbf{b}$,

$$\Psi^{(1)}\mathbf{X} = \sum_{i=0}^{k} (\psi\mathbf{x}_i\mathbf{x}_{i+1} - \psi\mathbf{x}_i\mathbf{x}_i) = \sum_{i=0}^{k} \psi\mathbf{x}_i\mathbf{x}_{i+1} - \sum_{i=0}^{k} \psi\mathbf{x}_i\mathbf{x}_i$$

and

$$\Psi^{(2)}\widetilde{\mathbf{X}} = \sum_{i=0}^{k} \left(\psi\mathbf{x}'_{i+1}\mathbf{x}'_i - \psi\mathbf{x}'_i\mathbf{x}'_i\right) = \sum_{i=0}^{k} \psi\mathbf{x}_i\mathbf{x}_{i+1} - \sum_{i=0}^{k-1} \psi\mathbf{x}_{i+1}\mathbf{x}_{i+1},$$

whence the result obtains by subtraction. ■

Theorem 21 If $\mathbf{f}$ is a path connecting $\mathbf{a}$ to $\mathbf{b}$, then

$$\Psi^{(1)}\mathbf{f} - \Psi^{(2)}\widetilde{\mathbf{f}} = \psi\mathbf{bb} - \psi\mathbf{aa}.$$

Proof. It is evident that $\mathbf{X}_n^{\mu_n} \to \mathbf{f}$ if and only if $\widetilde{\mathbf{X}_n^{\mu_n}} \to \widetilde{\mathbf{f}}$. Assume, with no loss of generality (Lemma 1), that all $\mathbf{X}_n$ connect $\mathbf{a}$ to $\mathbf{b}$. By Theorem 20, $\Psi^{(1)}\mathbf{X}_n - \Psi^{(2)}\widetilde{\mathbf{X}_n} = \psi\mathbf{bb} - \psi\mathbf{aa}$, whence the result obtains immediately. ■

Theorem 22 For any paths $\mathbf{f}, \mathbf{g}$ connecting $\mathbf{a}$ to $\mathbf{b}$,

$$\Psi^{(1)}\mathbf{f} + \Psi^{(1)}\tilde{\mathbf{g}} = \Psi^{(2)}\widetilde{\mathbf{f}} + \Psi^{(2)}\mathbf{g}.$$

Proof. A corollary to Theorem 21. ■

Note that with appropriately chosen parametrization, $\mathbf{f} \cup \tilde{\mathbf{g}}$ can be viewed as a *closed path* containing $\mathbf{a}$ and $\mathbf{b}$; and $\widetilde{\mathbf{f}} \cup \mathbf{g}$ can be viewed as the same closed path but traversed in the opposite direction. The statement of Theorem 22 then can also be presented as

$$\Psi^{(1)}\left(\mathbf{f} \cup \tilde{\mathbf{g}}\right) = \Psi^{(2)}\left(\widetilde{\mathbf{f}} \cup \mathbf{g}\right). \tag{39}$$

Recall that we denote by A_D the arclength (oriented) metric induced by D, and that we use asterisks to designate symmetric, or overall distances obtained by adding together the oriented distances "to and from." Thus,

$$\begin{aligned} A^*_{\Psi^{(1)}}\mathbf{ab} &= A_{\Psi^{(1)}}\mathbf{ab} + A_{\Psi^{(1)}}\mathbf{ba}, \\ A^*_{\Psi^{(2)}}\mathbf{ab} &= A_{\Psi^{(2)}}\mathbf{ab} + A_{\Psi^{(2)}}\mathbf{ba}. \end{aligned} \tag{40}$$

Theorem 23 For any $\mathbf{a}, \mathbf{b}$,

$$A_{\Psi^{(1)}}\mathbf{ab} - A_{\Psi^{(2)}}\mathbf{ba} = \psi\mathbf{bb} - \psi\mathbf{aa}$$

and

$$A^*_{\Psi^{(1)}}\mathbf{ab} = A^*_{\Psi^{(2)}}\mathbf{ba}$$

Proof. The first statement is a corollary to Theorem 21. Adding the equations

$$\begin{aligned} A_{\Psi^{(1)}}\mathbf{ab} - A_{\Psi^{(2)}}\mathbf{ba} &= \psi\mathbf{bb} - \psi\mathbf{aa} \\ A_{\Psi^{(1)}}\mathbf{ba} - A_{\Psi^{(2)}}\mathbf{ab} &= \psi\mathbf{aa} - \psi\mathbf{bb} \end{aligned}$$

we obtain the second statement. ■

In the previously published versions of Fechnerian Scaling (beginning with Dzhafarov, 2002d) the equation $A^*_{\Psi^{(1)}}\mathbf{ab} = A^*_{\Psi^{(2)}}\mathbf{ba}$ has been referred to as the "second main theorem of Fechnerian Scaling."

9. Conclusion

We have established that the DC theory can be specialized to arc-connected spaces with no additional constraints imposed either on these spaces or on the type of paths. In this respect the new theory proposed in Dzhafarov and Colonius (2007) has a definite advantage in generality over the one presented in Dzhafarov and Colonius (2005a). We have shown that the path length can be defined in terms of a dissimilarity function as the limit inferior of the lengths of appropriately chosen chains converging to paths. Unlike in the classical metric-based theory of path length, the converging chains generally are not confined to inscribed chains only: the vertices of the converging chains are allowed to "jitter and meander" around the path they are converging to. Given this difference, however, most of the basic results of the metric-based theory are shown to hold true in the dissimilarity-based theory.

The dissimilarity-based length theory properly specializes to the classical one when the dissimilarity in question is itself a metric (in fact without assuming that this metric is symmetric). In this case the limit inferior over all converging chains coincides with that computed over the inscribed chains only. It is also the case that the length of any path computed by means of a dissimilarity function remains the same if the dissimilarity function is replaced with the metric it induces.

We have introduced a class of spaces in which the metric induced by the dissimilarity function defined on these spaces are intrinsic: which means that the distance between two given points can be computed as the infimum of the lengths of all arcs connecting these points. We call them spaces with intermediate points, the concept generalizing that of the metric-based theory's Menger convexity (see Section 7).

All of this shows that the properties $\mathcal{D}3$ and $\mathcal{D}4$ of a dissimilarity function (see Section 2) rather than the symmetry and triangle inequality of a metric are essential in dealing with the notions of path length and intrinsic metrics.

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