On the Reverse Problem of Fechnerian Scaling

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Abstract

Fechnerian Scaling imposes metrics on two sets of stimuli related to each other by a discrimination function subject to Regular Minimality. The two sets of stimuli usually represent the same set of stimulus values presented to an observer in two distinct observation areas. A discrimination function associates with every pair of stimuli a nonnegative number interpretable as the degree or probability with which, "from the observer's point of view," the two stimuli differ from each other, overall or in a specified respect. Regular Minimality is a principle according to which the relation "to be the best match for" across the two stimulus areas has certain uniqueness and symmetry properties. Fechnerian distances are computed by means of a dissimilarity cumulation procedure: discrimination values are first converted into an appropriately defined dissimilarity function, the sums of the latter's values are computed for all finite chains of stimuli connecting a given pair of stimuli, and the infimum of such sums (cumulative dissimilarities) is taken to be the distance from one element of this pair to the other. The Fechnerian distance between two stimuli in one observation area is the same as the Fechnerian distance between the corresponding (best matching) stimuli in the other observation area. This chapter deals with the reverse problem of Fechnerian Scaling: under which conditions one can compute the discrimination function values given the Fechnerian distances and the discrimination values between the best matching stimuli.

1 Background

Some familiarity with the modern theory of (generalized) Fechnerian Scaling is desirable but not necessary for reading this chapter: the background information needed will be recapitulated. The reader interested in the latest published version of the theory is referred to Dzhafarov and Colonius (2007) and Dzhafarov (2008a, 2008b, 2010). For historical details and the origins of the adjective "Fechnerian" the reader can consult Dzhafarov (2001) and Dzhafarov and Colonius (2011).

A prototypical example of an experiment to which Fechnerian Scaling pertains is this: an observer is presented various pairs of stimuli (sounds, color patches, drawings, photographs of faces) and asked to indicate for every pair whether the two stimuli are the same or different (possibly, in a specified respect, as in "do these photographs depict the same person?"). The assumption is that each pair $(\mathbf{a}, \mathbf{b})$ is associated with the probability

$$\psi(\mathbf{a}, \mathbf{b}) = \Pr[\mathbf{a} \text{ and } \mathbf{b} \text{ are judged to be different}].$$

Every stimulus is characterized by its *value* (e.g., the shape of a line drawing) and its *observation area* (or *stimulus area*), usually a spatiotemporal location, serving to distinguish the two stimuli to be compared (e.g., two line drawings can be presented in distinct locations, one to the left of the other, or successively, one before the other). The pairs $(\mathbf{a}, \mathbf{b})$ therefore are defined as

$$(\mathbf{a}, \mathbf{b}) = \left\{ \begin{array}{c} \text{value of } \mathbf{a} \text{ in observation area 1,} \\ \text{value of } \mathbf{b} \text{ in observation area 2} \end{array} \right\},$$

so that $(\mathbf{a}, \mathbf{b}) \neq (\mathbf{b}, \mathbf{a})$.

Assuming, as we do throughout this chapter, that the two observation areas are fixed, $\mathbf{a}$ and $\mathbf{b}$ in $(\mathbf{a}, \mathbf{b})$ belong to different sets even if their values are identical. We denote these sets $\mathcal{A}$ and $\mathcal{B}$, so that the *discrimination probability function* ψ is

$$\psi : \mathcal{A} \times \mathcal{B} \to [0, 1] .$$

This definition can be immediately generalized. First, we can replace $[0, 1]$ with the set of all nonnegative reals,

$$\psi : \mathcal{A} \times \mathcal{B} \to \mathbb{R}^{+}.$$

This allows us to include discrimination functions which are not probabilities, such as expected values of numerical estimates of dissimilarity given in response to pairs of stimuli. Although the present treatment is confined to probabilistic ψ, its extension to arbitrary (bounded or unbounded) functions is straightforward. Second, we can interpret $\mathcal{A}$ and $\mathcal{B}$ as being arbitrary sets, not necessarily

$$\begin{array}{c} \mathcal{A} = \mathcal{V} \times \{1\} , \\ \mathcal{B} = \mathcal{V} \times \{2\} , \end{array}$$

with $\mathcal{V}$ being a common set of stimulus values. Thus, one can consider

$$\begin{array}{c} \mathcal{A} = \mathcal{V}_1 \times \{1\} , \\ \mathcal{B} = \mathcal{V}_2 \times \{2\} , \end{array}$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ are different subsets of a set $\mathcal{V}$ of possible stimulus values. This may be convenient if the matching pairs, as defined below, involve "constant error," that is, if $\mathcal{V}_2$ is the set of matches for the elements of $\mathcal{V}_1$ and $\mathcal{V}_2 \neq \mathcal{V}_1$. We can even consider the possibility that $\mathcal{A}$ and $\mathcal{B}$ are sets of different nature, with the relation "are the same" being defined in special ways. For instance, $\mathcal{A}$ may be a set of examinees and $\mathcal{B}$ the set of tests, with the relation "**a** and **b** are the same" meaning that the problem **b** is neither too difficult nor too easy for the examinee **a** (in the opinion of a judge, or as computed from performance data). For other non-traditional examples of pairwise comparisons, see Dzhafarov and Colonius (2006).

Without loss of generality, let us assume that no two distinct stimuli in $\mathcal{A}$ or in $\mathcal{B}$ are *equivalent*, in the following sense: if $\mathbf{a}_1 \neq \mathbf{a}_2$ in $\mathcal{A}$, then

$$\psi (\mathbf{a}_1, \mathbf{b}) \neq \psi (\mathbf{a}_2, \mathbf{b})$$

for some $\mathbf{b} \in \mathcal{B}$; and if $\mathbf{b}_1 \neq \mathbf{b}_2$ in $\mathcal{B}$, then

$$\psi (\mathbf{a}, \mathbf{b}_1) \neq \psi (\mathbf{a}, \mathbf{b}_2)$$

for some $\mathbf{a} \in \mathcal{A}$. (If this is not the case, then $\mathcal{A}$ and $\mathcal{B}$ can always be "reduced" to the requisite form.) We say that $\mathbf{a} \in \mathcal{A}$ *is matched by* $\mathbf{b} \in \mathcal{B}$ and write $\mathbf{a}\mathrm{M}\mathbf{b}$ if

$$\psi (\mathbf{a}, \mathbf{b}) = \min_{\mathbf{y} \in \mathcal{B}} \psi (\mathbf{a}, \mathbf{y}) .$$

Analogously, if

$$\psi(\mathbf{a},\mathbf{b}) = \min_{\mathbf{x}\in\mathcal{A}} \psi(\mathbf{x},\mathbf{b}),$$

we say that $\mathbf{b} \in \mathcal{B}$ *is matched by* $\mathbf{a} \in \mathcal{A}$ and write $\mathbf{b}\mathrm{M}\mathbf{a}$. The space $(\mathcal{A},\mathcal{B},\psi)$, or the discrimination function ψ itself, is said to satisfy *Regular Minimality* (Dzhafarov, 2002b; Dzhafarov & Colonius, 2006; Kujala & Dzhafarov, 2008) if

$(\mathcal{RM}1)$ for every $\mathbf{a} \in\mathcal{A}$ there is one and only one $\mathbf{b} \in\mathcal{B}$ such that $\mathbf{a}\mathrm{M}\mathbf{b}$; $(\mathcal{RM}2)$ for every $\mathbf{b} \in\mathcal{B}$ there is one and only one $\mathbf{a} \in\mathcal{A}$ such that $\mathbf{b}\mathrm{M}\mathbf{a}$; $(\mathcal{RM}3)$ $\mathbf{a}\mathrm{M}\mathbf{b}$ if and only if $\mathbf{b}\mathrm{M}\mathbf{a}$, for all $(\mathbf{a},\mathbf{b}) \in \mathcal{A}\times\mathcal{B}$.

If this is the case, we can relabel the stimuli in $\mathcal{A}$ and $\mathcal{B}$ so that any two matching stimuli receive the same label. Formally, if Regular Minimality holds, then one can find (non-uniquely) bijective functions

$$\mathbf{h} : \mathcal{A} \to \mathfrak{S}$$

and

$$\mathbf{g} : \mathcal{B} \to \mathfrak{S}$$

such that

$$\mathbf{a}\mathrm{M}\mathbf{b} \Longleftrightarrow \mathbf{h}(\mathbf{a}) = \mathbf{g}(\mathbf{b}).$$

Any such mapping $(\mathbf{h},\mathbf{g})$ is called a *canonical transformation* of the space $(\mathcal{A},\mathcal{B},\psi)$, and it creates a *canonical discrimination space* $(\mathfrak{S},\psi^*)$, with the function

$$\psi^* : \mathfrak{S}\times\mathfrak{S} \to \mathbb{R}^+$$

defined by

$$\psi^*(\mathbf{a},\mathbf{b}) = \psi\left(\mathbf{h}^{-1}(\mathbf{a}),\mathbf{g}^{-1}(\mathbf{b})\right).$$

Clearly, the function ψ^* satisfies Regular Minimality in the simplest (canonical) form: for any distinct $\mathbf{a},\mathbf{b} \in \mathfrak{S}$,

$$\psi^*(\mathbf{a},\mathbf{a}) < \min\{\psi^*(\mathbf{a},\mathbf{b}),\psi^*(\mathbf{b},\mathbf{a})\}.$$

Although the canonical discrimination space $(\mathfrak{S},\psi^*)$ is not uniquely determined by $(\mathcal{A},\mathcal{B},\psi)$, it can be viewed as being essentially unique, in the following sense. Any two canonical spaces, $(\mathfrak{S}_1,\psi_1^*)$ and $(\mathfrak{S}_2,\psi_2^*)$, are related to each other by a bijective transformation $\mathbf{t} : \mathfrak{S}_1 \to \mathfrak{S}_2$ such that

$$\psi_1^*(\mathbf{a},\mathbf{b}) = \psi_2^*(\mathbf{t}(\mathbf{a}),\mathbf{t}(\mathbf{b})).$$

In other words, the canonical transformation is unique up to trivial renaming of the stimuli.

Henceforth we will deal with canonical discrimination spaces $(\mathfrak{S},\psi^*)$ only. We drop the asterisk for the canonical discrimination function and write ψ in place of ψ^*. We also use the notational conventions adopted in most of the author's previous publications on generalized Fechnerian Scaling:

1. for any binary function $f : \mathfrak{S}\times\mathfrak{S} \to \mathbb{R}$, we write $f\mathbf{ab}$ instead of $f(\mathbf{a},\mathbf{b})$ (in particular, the discrimination function is written as $\psi\mathbf{ab}$);

2. if a binary function f is followed by a string (or *chain*) $\mathbf{X} = \mathbf{x}_1 \ldots, \mathbf{x}_n$ of more than one point, then

$$f\mathbf{X} = f\mathbf{x}_1 \ldots \mathbf{x}_n = \sum_{i=1}^{n-1} f\mathbf{x}_i\mathbf{x}_{i+1};$$

3. for a chain $\mathbf{X} = \mathbf{x}_1 \ldots, \mathbf{x}_n$ with $n = 1$ or $n = 0$ the expression $f\mathbf{X}$ is set equal to zero;

4. any two chains of points $\mathbf{X}$ and $\mathbf{Y}$ can be concatenated into a chain $\mathbf{XY}$ (e.g., if $\mathbf{X} = \mathbf{x}_1 \ldots, \mathbf{x}_n$, then $\mathbf{aXb}$ is the chain $\mathbf{ax}_1 \ldots, \mathbf{x}_n\mathbf{b}$).

A *dissimilarity function* $D : \mathfrak{S} \times \mathfrak{S} \to \mathbb{R}$ is defined by the following properties: for any $\mathbf{a}, \mathbf{b} \in \mathfrak{S}$, any sequences $\mathbf{a}_n, \mathbf{a}'_n, \mathbf{b}_n, \mathbf{b}'_n$ in $\mathfrak{S}$, and any sequence of chains $\mathbf{X}_n$ with elements in $\mathfrak{S}$ ($n = 1, 2, \ldots$),

($\mathcal{D}1$) $D\mathbf{ab} \geq 0$;
($\mathcal{D}2$) $D\mathbf{ab} = 0$ if and only if $\mathbf{a} = \mathbf{b}$;
($\mathcal{D}3$) if $\max\{D\mathbf{a}_n\mathbf{a}'_n, D\mathbf{b}_n\mathbf{b}'_n\} \to 0$ then $D\mathbf{a}_n\mathbf{b}_n - D\mathbf{a}'_n\mathbf{b}'_n \to 0$;
($\mathcal{D}4$) if $D\mathbf{a}_n\mathbf{X}_n\mathbf{b}_n \to 0$ then $D\mathbf{a}_n\mathbf{b}_n \to 0$.

Given any chain $\mathbf{X} = \mathbf{x}_1 \ldots, \mathbf{x}_n$, the quantity

$$D\mathbf{X} = \sum_{i=1}^{n-1} D\mathbf{x}_i\mathbf{x}_{i+1}$$

is called the *cumulative dissimilarity* for this chain.

In Fechnerian Scaling the role of dissimilarity functions is played by the *psychometric increments* of the first and second kind, defined as, respectively,

$$\Psi^{(1)}\mathbf{ab} = \psi\mathbf{ab} - \psi\mathbf{aa},$$
$$\Psi^{(2)}\mathbf{ab} = \psi\mathbf{ba} - \psi\mathbf{aa}.$$

(It is clearly unnecessary to consider separately the versions $\psi\mathbf{ba} - \psi\mathbf{bb} = \Psi^{(1)}\mathbf{ba}$ and $\psi\mathbf{ab} - \psi\mathbf{bb} = \Psi^{(2)}\mathbf{ba}$.) In other words, a canonical discrimination space $(\mathfrak{S}, \psi)$ induces a *double-dissimilarity space* $\left(\mathfrak{S}, \Psi^{(1)}, \Psi^{(2)}\right)$.

We use the notation

$$\mathbf{a}_n \leftrightarrow \mathbf{b}_n$$

(as $n \to \infty$) to designate any of the pairwise equivalent convergences

$$\Psi^{(1)}\mathbf{a}_n\mathbf{b}_n \to 0,$$
$$\Psi^{(1)}\mathbf{b}_n\mathbf{a}_n \to 0,$$
$$\Psi^{(2)}\mathbf{a}_n\mathbf{b}_n \to 0,$$
$$\Psi^{(2)}\mathbf{b}_n\mathbf{a}_n \to 0.$$

The discrimination function ψ is uniformly continuous with respect to the uniformity induced by this convergence:

$$\left.\begin{array}{l} \mathbf{a}'_n \leftrightarrow \mathbf{a}_n \\ \mathbf{b}'_n \leftrightarrow \mathbf{b}_n \end{array}\right\} \Longrightarrow \psi\mathbf{a}'_n\mathbf{b}'_n - \psi\mathbf{a}_n\mathbf{b}_n \to 0.$$

In particular, $\mathbf{a}_n \leftrightarrow \mathbf{b}_n$ implies

$$\psi \mathbf{b}_n \mathbf{b}_n - \psi \mathbf{a}_n \mathbf{a}_n \to 0.$$

A *metric* (or *distance function*) $G : \mathfrak{S} \times \mathfrak{S} \to \mathbb{R}$ can be defined as a dissimilarity function satisfying the triangle inequality: for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathfrak{S}$,

$$G\mathbf{ab} + G\mathbf{bc} \geq G\mathbf{ac}.$$

This definition differs from the classical Frechét's definition in that it does not require global symmetry,

$$G\mathbf{ab} = G\mathbf{ba}.$$

However, it is more specific than the notion of a *quasimetric* (defined by dropping from the classical definition of a metric the global symmetry requirement). Namely, since G is a dissimilarity it has the following *symmetry-in-the-small* property: for all sequences $\mathbf{a}_n, \mathbf{b}_n$ in $\mathfrak{S}$,

$$G\mathbf{a}_n \mathbf{b}_n \to 0 \iff G\mathbf{b}_n \mathbf{a}_n \to 0.$$

In Fechnerian Scaling, the *Fechnerian metric* G_1 induced by the dissimilarity function $\Psi^{(1)}$ is defined as

$$G_1\mathbf{ab} = \inf_{\mathbf{X}} \Psi^{(1)}\mathbf{aXb}.$$

Analogously,

$$G_2\mathbf{ab} = \inf_{\mathbf{X}} \Psi^{(2)}\mathbf{aXb}$$

is the Fechnerian metric induced by the dissimilarity function $\Psi^{(2)}$. Both G_1 and G_2 are well-defined metrics, and

$$\left.\begin{array}{c} G_1\mathbf{ab} + G_1\mathbf{ba} \\ \shortparallel \\ G_2\mathbf{ab} + G_2\mathbf{ba} \end{array}\right\} = G\mathbf{ab}.$$

This sum, $G\mathbf{ab}$, is a conventional (symmetric) metric. It is referred to as the *overall Fechnerian metric*.

The asymmetric (*"oriented"*) metrics G_1 and G_2 are also related to each other by the identities

$$\left.\begin{array}{c} G_1\mathbf{ab} - G_2\mathbf{ba} \\ \shortparallel \\ G_2\mathbf{ab} - G_1\mathbf{ba} \end{array}\right\} = \psi\mathbf{bb} - \psi\mathbf{aa}.$$

This follows from the procedure of computing G_1 and G_2 from $\Psi^{(1)}$ and $\Psi^{(2)}$ and from the immediately verifiable identities

$$\left.\begin{array}{c} \Psi^{(1)}\mathbf{ab} - \Psi^{(2)}\mathbf{ba} \\ \shortparallel \\ \Psi^{(2)}\mathbf{ab} - \Psi^{(1)}\mathbf{ba} \end{array}\right\} = \psi\mathbf{bb} - \psi\mathbf{aa}.$$

For any sequences $\mathbf{a}_n$ and $\mathbf{b}_n$ in $\mathfrak{S}$, we have, as $n \to \infty$,

$$\mathbf{a}_n \leftrightarrow \mathbf{b}_n \iff G_1\mathbf{a}_n\mathbf{b}_n \to 0 \iff G_2\mathbf{a}_n\mathbf{b}_n \to 0.$$

It follows that G_1 and G_2 are uniformly continuous with respect to the uniformity induced by $\leftrightarrow$.

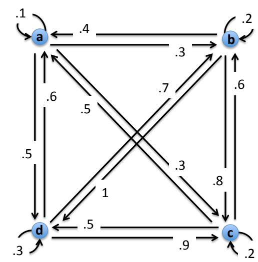

Figure 1: Stimulus space of Example 1. For each stimulus pair $(\mathbf{x}, \mathbf{y})$, the number attached to the arrow from $\mathbf{x}$ to $\mathbf{y}$ is the value of $\psi\mathbf{xy}$.

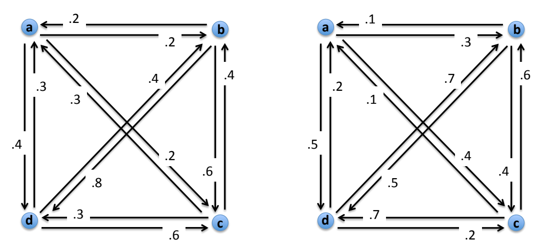

Figure 2: Psychometric increments of the first and second kind computed for the stimulus space shown in Figure 1. For each stimulus pair $(\mathbf{x}, \mathbf{y})$, the number attached to the arrow from $\mathbf{x}$ to $\mathbf{y}$ is the value of $\Psi^{(1)}\mathbf{xy}$ (on the left) or $\Psi^{(2)}\mathbf{xy}$ (on the right).

Example 1. Let $\mathfrak{S}$ be the set of four stimuli, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$,[1] with the values of ψ shown in Figure 1. Fechnerian computations are illustrated in Figures 2-5. Thus, the number 0.4 attached to the arrow from **d** to **b** in Figure 2, left, is

$$\Psi^{(1)}\mathbf{db} = \psi\mathbf{db} - \psi\mathbf{dd} = 0.7 - 0.3.$$

The number 0.7 attached to the corresponding arrow on the right is

$$\Psi^{(2)}\mathbf{db} = \psi\mathbf{bd} - \psi\mathbf{dd} = 1 - 0.3.$$

For any finite stimulus set, the psychometric increments $\Psi^{(1)}$ and $\Psi^{(2)}$ are dissimilarity functions (i.e., satisfy the properties $\mathcal{D}1$-$\mathcal{D}4$ above) if and only if ψ satisfies Regular Minimality. The Fechnerian distances are shown in Figure 3. For instance, the number 0.2 attached to the arrow from **a** to **b** on the left is computed by forming all possible chains leading from **a** to **b**, calculating their cumulative dissimilarities and choosing the smallest. Omitting chains containing loops, as we obviously do not need them in searching for the minimum, we get the list

chain	cumulative $\Psi^{(1)}$
ab	0.2
acb	0.2+0.4
adb	0.4+0.4
acdb	0.2+0.3+0.4
abcb	0.4+0.6+0.4

in which the direct link **ab** is clearly a *geodesic* (a shortest path). We conclude therefore that $G_1\mathbf{ab} = 0.2$. Note that geodesics generally are not unique if they exist (they have to exist in finite stimulus sets but not generally). The analogous calculations for, say, the stimuli **c** and **d** on the right yield

chain	cumulative $\Psi^{(2)}$
cd	0.7
cad	0.1+0.5
cbd	0.6+0.5
cabd	0.1+0.3+0.5
cbad	0.6+0.1+0.5

.

Here, the geodesic is **cad** and we conclude that $G_2\mathbf{cd} = 0.6$. This geodesic is shown in Figure 4, right, together with the two other "indirect" geodesic paths, those consisting of more than two stimuli. Figure 5 presents the values of the overall Fechnerian distance, obtained as

$$G_1\mathbf{xy} + G_1\mathbf{yx} = G_2\mathbf{xy} + G_2\mathbf{yx}$$

[1] As a rule we use symbols **a** and **b** (interchangeably with **x** and **y**, respectively) to generically refer to stimuli in, respectively, the first and second observation areas. Thus, in an expression like $\Psi^{(1)}\mathbf{ab} > 0$, **a** and **b** are variables, arbitrary members of $\mathfrak{S}$. However, in some of our examples **a** and **b** are used to designate specific stimuli, together with other specific stimuli (here, **c** and **d**). The two uses of **a** and **b** should be easily distinguishable by the context.

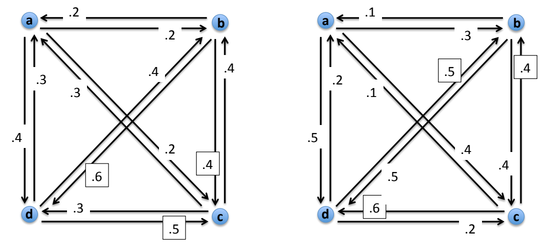

Figure 3: Fechnerian distances of the first and second kind computed from the psychometric increments in Figure 2. The number attached to the arrow from $\mathbf{x}$ to $\mathbf{y}$ is the value of $G_1\mathbf{xy}$ (on the left) or $G_2\mathbf{xy}$ (on the right). The framed numbers indicate Fechnerian distances that are smaller than the corresponding psychometric increments: the geodesics for them are shown in Figure 4.

for every stimulus pair $(\mathbf{x}, \mathbf{y})$. Thus, for $(\mathbf{x}, \mathbf{y}) = (\mathbf{c}, \mathbf{d})$, we have

$$G\mathbf{cd} = G\mathbf{dc} = \begin{cases} G_1\mathbf{cd} + G_1\mathbf{dc} = 0.3 + 0.5 \\ \qquad\qquad\qquad \shortparallel \\ G_2\mathbf{cd} + G_2\mathbf{dc} = 0.6 + 0.2. \end{cases}$$

Note that $G\mathbf{cd}$ can be viewed as the cumulative dissimilarity

$$\Psi^{(1)}\mathbf{cdac} = \Psi^{(2)}\mathbf{cadc}$$

for the *geodesic loop* **cdac** obtained by concatenating the geodesic paths from **c** to **d** and back; the loop should be read in the opposite directions for $\Psi^{(1)}$ and $\Psi^{(2)}$.

2 Problem

We have seen that a canonical discrimination space $(\mathfrak{S}, \psi)$ induces a *double-metric* space $(\mathfrak{S}, G_1, G_2)$ from which one can form the space $(\mathfrak{S}, G)$ with a conventional, symmetric metric G. It is easy to see that these computations cannot generally be reversed. The following example shows that $(\mathfrak{S}, G)$ does not allow one to reconstruct $(\mathfrak{S}, \psi)$ uniquely.

Example 2. Let $(\mathfrak{S}, \psi)$ induce $(\mathfrak{S}, G_1, G_2)$ and $(\mathfrak{S}, G)$, and let $\psi\mathbf{aa}$ be some nonconstant function of $\mathbf{a}$. Denote by $\omega_{\mathbf{a}}$ an arbitrary function such that

$$\omega_{\mathbf{a}} \not\equiv \psi\mathbf{aa}$$

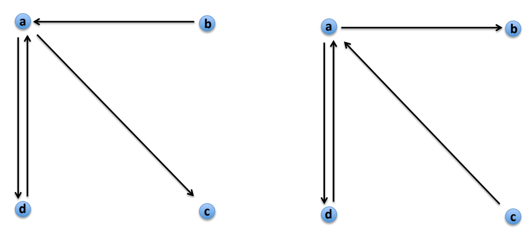

Figure 4: Geodesic paths of the first and second kind corresponding to the framed values of the Fechnerian distances in Figure 3. Each geodesic consists of three stimuli connected by two consecutive arrows.

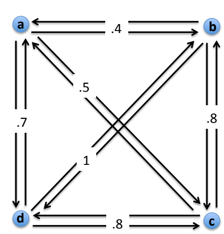

Figure 5: The overall (symmetric) Fechnerian distances computed from the Fechnerian distances shown in Figure 3.

(that is, $\omega_{\mathbf{a}}$ and $\psi\mathbf{aa}$ are not identical),

$$0 \le \omega_{\mathbf{a}} \le \min\left\{\ 1 - \sup_{\mathbf{b}} \Psi^{(1)}\mathbf{ab}\ \right\},$$

and

$$\omega_{\mathbf{b}} - \omega_{\mathbf{a}} \le \Psi^{(1)}\mathbf{ab}.$$

All three inequalities can always be achieved, for instance, by putting $\omega_{\mathbf{a}} \equiv 0$ (a nonconstant $\psi\mathbf{aa}$ has to be positive at some $\mathbf{a}$, and then $\ 1 - \sup_{\mathbf{b}} \Psi^{(1)}\mathbf{ab} > 0\ $). The function

$$\overline{\psi}\mathbf{ab} = \Psi^{(1)}\mathbf{ab} + \omega_{\mathbf{a}}$$

is clearly bounded by 1 from above. It satisfies Regular Minimality because

$$\overline{\psi}\mathbf{aa} = \omega_{\mathbf{a}} \le \overline{\psi}\mathbf{ab}$$

and

$$\omega_{\mathbf{a}} \le \overline{\psi}\mathbf{ba} = \Psi^{(1)}\mathbf{ba} + \omega_{\mathbf{b}}.$$

The function $\overline{\psi}$ induces precisely the same metric space $(\mathfrak{S}, G)$ as the original function ψ. Indeed, from the definition of $\overline{\psi}$ it follows that

$$\overline{\Psi}^{(1)}\mathbf{ab} = \Psi^{(1)}\mathbf{ab},$$

where $\overline{\Psi}^{(1)}$ is the psychometric increment of the first kind computed from $\overline{\psi}$. This means that $\overline{G}_1$ computed from $\overline{\Psi}^{(1)}$ coincides with G_1. But then also

$$\overline{G}_1\mathbf{ab} + \overline{G}_1\mathbf{ba} = G_1\mathbf{ab} + G_1\mathbf{ba}.$$

Figures 6 and 7 illustrate the procedure just described on the stimulus space of Example 1. The left panel of Figure 7 coincides with that of Figure 3 because the psychometric increments of the first kind remain the same. By adding the numbers attached to opposite arrows one can verify that although the Fechnerian distances of the second kind do change, they yield the same overall Fechnerian distances.

Is it possible then that ψ (or at least the psychometric increments $\Psi^{(1)}$ and $\Psi^{(2)}$) can be reconstructed if one knows both G_1 and G_2? The next example shows that this is not the case.

Example 3. Let $\mathfrak{S}$ be a countable set of stimuli enumerated $\mathbf{s}_1, \mathbf{s}_2, \ldots$, and let

$$\psi\mathbf{s}_i\mathbf{s}_j = \begin{cases} 0 & \text{if} \quad i = j \\ 1/3 & \text{if} \quad |i-j| = 1 \\ 2/3 + \gamma_i & \text{if} \quad i - j = 2 \\ 2/3 + \gamma_i' & \text{if} \quad j - i = 2 \\ 1 & \text{if} \quad |i-j| \ge 3, \end{cases}$$

where $0 \le \gamma_i, \gamma_i' < 1/3$. This is an example of a uniformly discrete stimulus space, considered in Section 4.3. The psychometric increments

$$\Psi^{(1)} \equiv \Psi^{(2)} \equiv \psi$$

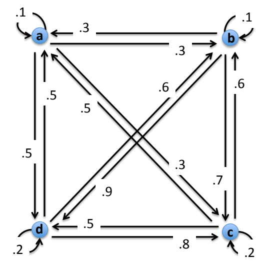

Figure 6: Stimulus space of Example 1 modified in accordance with Example 2.

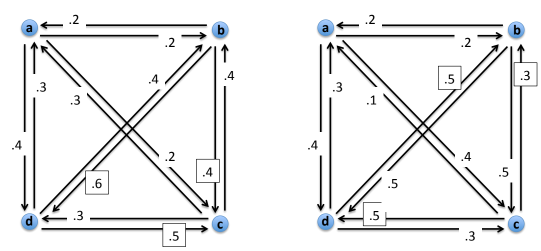

Figure 7: Fechnerian distances of the first and second kind computed from the psychometric increments in Figure 6.

in this case are dissimilarity functions because ψ satisfies Regular Minimality. For every chain containing $\mathbf{s}_i$ and $\mathbf{s}_{i+2}$ as successive elements,

$$\mathbf{X}\mathbf{s}_i\mathbf{s}_{i+2}\mathbf{Y} \text{ or } \mathbf{X}\mathbf{s}_{i+2}\mathbf{s}_i\mathbf{Y},$$

the cumulative dissimilarity

$$\Psi^{(\iota)}\mathbf{X}\mathbf{s}_i\mathbf{s}_{i+2}\mathbf{Y} \text{ or } \Psi^{(\iota)}\mathbf{X}\mathbf{s}_{i+2}\mathbf{s}_i\mathbf{Y}$$

where ι stands for 1 or 2, cannot increase if one replaces this chain with

$$\mathbf{X}\mathbf{s}_i\mathbf{s}_{i+1}\mathbf{s}_{i+2}\mathbf{Y} \text{ or } \mathbf{X}\mathbf{s}_{i+2}\mathbf{s}_{i+1}\mathbf{s}_i\mathbf{Y},$$

respectively:

$$\Psi^{(\iota)}\mathbf{X}\mathbf{s}_i\mathbf{s}_{i+2}\mathbf{Y} - \boldsymbol{\Psi}^{(\iota)}\mathbf{X}\mathbf{s}_i\mathbf{s}_{i+1}\mathbf{s}_{i+2}\mathbf{Y} = \gamma_i$$

and

$$\Psi^{(\iota)}\mathbf{X}\mathbf{s}_{i+2}\mathbf{s}_i\mathbf{Y} - \boldsymbol{\Psi}^{(\iota)}\mathbf{X}\mathbf{s}_{i+2}\mathbf{s}_{i+1}\mathbf{s}_i\mathbf{Y} = \gamma_i'.$$

Hence in computing G_ι as the infimum (here, minimum) of cumulative dissimilarities across a set of chains, one can confine one's consideration to chains in which for any two successive elements $\mathbf{s}_i$ and $\mathbf{s}_j$, either $|i-j|=1$ or $|i-j|\geq 3$. This means that G_ι cannot depend on the functions γ_i and γ_i'. In fact, as one can easily verify,

$$G_1\mathbf{s}_i\mathbf{s}_j = G_2\mathbf{s}_i\mathbf{s}_j = \begin{cases} |j-i|/3 & \text{if} \quad |i-j|<3 \\ 1 & \text{if} \quad |i-j|\geq 3, \end{cases}$$

irrespective of the functions γ_i and γ_i'. The values of $\psi\mathbf{s}_i\mathbf{s}_{i+2}$ and $\psi\mathbf{s}_{i+2}\mathbf{s}_i$ can be arbitrarily chosen between $2/3$ and 1.

The next example demonstrates the same point, that $(\mathfrak{S}, G_1, G_2)$ determines neither ψ nor the psychometric increments uniquely, for a continuous stimulus space.

Example 4. Let $\mathfrak{S}$ be the interval $[0,1]$, and let

$$\psi\mathbf{ab} = \begin{cases} \frac{2b-a}{4} + \frac{|a-b|^p}{2} & \text{if} \quad a\leq b, \\ \\ \frac{3a-2b}{4} + \frac{|a-b|^p}{4} & \text{if} \quad a>b, \end{cases}$$

where a and b are the numerical values of stimuli $\mathbf{a}$ and $\mathbf{b}$, respectively, and $p>1$. We have here

$$\Psi^{(1)}\mathbf{ab} = \begin{cases} \frac{b-a}{2} + \frac{|a-b|^p}{2} & \text{if} \quad a\leq b, \\ \\ \frac{a-b}{2} + \frac{|a-b|^p}{4} & \text{if} \quad a>b, \end{cases}$$

and

$$\Psi^{(2)}\mathbf{ab} = \begin{cases} \frac{3(b-a)}{4} + \frac{|a-b|^p}{4} & \text{if} \quad a<b, \\ \\ \frac{a-b}{4} + \frac{|a-b|^p}{2} & \text{if} \quad a\geq b. \end{cases}$$

We omit a demonstration that these psychometric increments are dissimilarity functions: it can be done using the methods presented in Dzhafarov (2010). For any $a<m<b$ in $[0,1]$,

$$\Psi^{(1)}\mathbf{ab} > \Psi^{(1)}\mathbf{amb}$$

and

$$\Psi^{(2)}\mathbf{ab} > \Psi^{(2)}\mathbf{amb}.$$

This is an example of a space with intermediate points, considered in Dzhafarov (2008a). The "inverse triangle inequalities" imply that, for any $\mathbf{a}$ and $\mathbf{b}$, the cumulative dissimilarities $\Psi^{(1)}\mathbf{aXb}$ and $\Psi^{(2)}\mathbf{aXb}$ decrease as one progressively refines the chain $\mathbf{X} = \mathbf{x}_1 \ldots \mathbf{x}_k$ whose elements' values partition the interval between a and b (i.e., for $a < b$, $a < x_1 < \ldots < x_k < b$, and analogously for $a > b$). As $n \to \infty$ and the maximal gap in $\left(a, x_1^{(n)}, \ldots, x_{k_n}^{(n)}, b\right)$ for chains $\mathbf{X}_n = \mathbf{x}_1^n \ldots \mathbf{x}_{k_n}^n$ tends to zero,

$$\Psi^{(1)}\mathbf{aX}_n\mathbf{b} \to G_1\mathbf{ab} = \frac{|a-b|}{2}$$

and

$$\Psi^{(2)}\mathbf{aX}_n\mathbf{b} \to G_2\mathbf{ab} = \begin{cases} \frac{3(b-a)}{4} & \text{if} \quad a < b, \\ \\ \frac{a-b}{4} & \text{if} \quad a \geq b. \end{cases}$$

As these values do not depend on p, the function ψ cannot be reconstructed even if one knows all values of $G_1\mathbf{ab}$ and $G_2\mathbf{ab}$.

The question arises: can one impose on the function ψ certain constraints under which ψ can be uniquely restored from $(\mathfrak{S}, G_1, G_2)$ and the set of "self-discrimination" probabilities[2]

$$\{\omega_{\mathbf{a}} = \psi\mathbf{aa} : \mathbf{a} \in \mathfrak{S}\}?$$

On this level of generality the problem is too difficult, however. Its formulation does not exclude the possibility that $\psi\mathbf{ab}$ for a given pair of stimuli $(\mathbf{a}, \mathbf{b})$ is determined by the values of G_1, G_2 and ω on some subset of pairs in $\mathfrak{S} \times \mathfrak{S}$, if not the entire Cartesian product. The problem we pose in this paper is more restricted: under what conditions can one compute $\psi\mathbf{ab}$ from the quantities

$$G_1\mathbf{ab}, G_2\mathbf{ab}, G_1\mathbf{ba}, G_2\mathbf{ba}, \omega_{\mathbf{a}}, \omega_{\mathbf{b}}?$$

We refer to this as the *reverse problem of Fechnerian Scaling (in the restricted sense)*. The so-called "Fechner's problem" (as formulated in Luce & Galanter, 1963) is closely related to the reverse problem in the restricted sense but is left outside the scope of this chapter. The reader interested in the issue is referred to Falmagne (1985) and Dzhafarov (2002a).

3 General Considerations

The formulation of the reverse problem immediately suggests the following representation for $\psi\mathbf{ab}$. The reverse problem has a solution if and only if ψ can be presented in either of the two equivalent forms,

$$\psi\mathbf{ab} = \omega_{\mathbf{a}} + G_1\mathbf{ab} + R\left(G_1\mathbf{ab}, G_2\mathbf{ab}, G_1\mathbf{ba}, G_2\mathbf{ba}, \omega_{\mathbf{a}}, \omega_{\mathbf{b}}\right)$$

or

$$\psi\mathbf{ab} = \omega_{\mathbf{b}} + G_2\mathbf{ba} + R\left(G_1\mathbf{ab}, G_2\mathbf{ab}, G_1\mathbf{ba}, G_2\mathbf{ba}, \omega_{\mathbf{a}}, \omega_{\mathbf{b}}\right),$$

where R is uniformly continuous, nonnegative, and vanishing at $\mathbf{a} = \mathbf{b}$.

[2] One should keep in mind that due to a canonical transformation of the space, the first and the second $\mathbf{a}$ in $\psi\mathbf{aa}$ may be stimuli physically different in value, and even if not, they always have different observation areas. Therefore the "self" in "self-discrimination" is a convenient but potentially misleading prefix.

Proof. It is obvious that any function of

$$G_1\mathbf{ab}, G_2\mathbf{ab}, G_1\mathbf{ba}, G_2\mathbf{ba}, \omega_{\mathbf{a}}, \omega_{\mathbf{b}}$$

can be presented in either of the two forms. That they are equivalent follows from

$$\omega_{\mathbf{a}} + G_1\mathbf{ab} = \omega_{\mathbf{b}} + G_2\mathbf{ba},$$

which is a consequence of

$$\left.\begin{array}{c} G_1\mathbf{ab} - G_2\mathbf{ba} \\ \shortparallel \\ G_2\mathbf{ab} - G_1\mathbf{ba} \end{array}\right\} = \omega_{\mathbf{b}} - \omega_{\mathbf{a}}.$$

Since

$$(\psi\mathbf{ab} - \omega_{\mathbf{a}}) - G_1\mathbf{ab} = \Psi^{(1)}\mathbf{ab} - G_1\mathbf{ab}$$

is the difference of uniformly continuous functions, R is uniformly continuous. R is nonnegative because

$$\psi\mathbf{ab} - \omega_{\mathbf{a}} = \Psi^{(1)}\mathbf{ab} \geq G_1\mathbf{ab}$$

and

$$\psi\mathbf{ab} - \omega_{\mathbf{b}} = \Psi^{(2)}\mathbf{ba} \geq G_2\mathbf{ba}.$$

R vanished at $\mathbf{a} = \mathbf{b}$ because

$$\psi\mathbf{ab} = \omega_{\mathbf{a}} = \omega_{\mathbf{b}}$$

and

$$G_1\mathbf{ab} = G_2\mathbf{ba} = 0.$$

This proves the "only if" part of the theorem. The "if" part is obvious. □

The examples in the previous section show that the representation given in this theorem does not have to exist. Moreover, despite its formulation in the form of a necessary and sufficient condition, it is not obvious whether a function ψ satisfying this condition can in fact be constructed: the condition in question relates ψ to G_1 and G_2, which are themselves computed from ψ. The situation is remedied by the following examples.

Example 5. Let $\mathfrak{S}$ be the interval $[0,1]$, and

$$\psi\mathbf{ab} = \frac{a}{6} + \frac{|a-b|}{3} + \frac{95}{216}(a-b)^2(a+b).$$

The function is easily checked to be between 0 and 1 and satisfy Regular Minimality. The psychometric increments of the first kind are

$$\Psi^{(1)}\mathbf{ab} = \frac{1}{3}|a-b| + \frac{1}{2}\left(a^2 - b^2\right)(a-b),$$

and they can be shown to form a dissimilarity function (we skip this demonstration). Since, for any $a, b \in [0,1]$ and m between a and b,

$$\Psi^{(1)}\mathbf{amb} < \Psi^{(1)}\mathbf{ab},$$

we use the same argument as in Example 4 to arrive at

$$G_1\mathbf{ab} = \frac{|a-b|}{3}.$$

One can check now that

$$\begin{aligned}
&\tfrac{a}{6} + \tfrac{|a-b|}{3} + \tfrac{95}{216}(a-b)^2(a+b) \\
&= \tfrac{a}{6} + \tfrac{|a-b|}{3} + \left(\tfrac{2}{3}|a-b| + \left(\tfrac{a}{6} - \tfrac{b}{6}\right)\right)\left(\left(\tfrac{b}{6} - \tfrac{a}{6}\right) + \tfrac{2}{3}|a-b|\right)\left(\tfrac{a}{6} + \tfrac{b}{6}\right) \\
&= \omega_{\mathbf{a}} + G_1\mathbf{ab} + (2G_1\mathbf{ab} + (\omega_{\mathbf{a}} - \omega_{\mathbf{b}}))\left((\omega_{\mathbf{b}} - \omega_{\mathbf{a}}) + 2G_1\mathbf{ba}\right)(\omega_{\mathbf{a}} + \omega_{\mathbf{b}}).
\end{aligned}$$

That is, ψ can be presented in a form required in Theorem 3, with

$$R = \omega_{\mathbf{a}} + G_1\mathbf{ab} + (2G_1\mathbf{ab} + (\omega_{\mathbf{a}} - \omega_{\mathbf{b}}))\left((\omega_{\mathbf{b}} - \omega_{\mathbf{a}}) + 2G_1\mathbf{ba}\right)(\omega_{\mathbf{a}} + \omega_{\mathbf{b}}).$$

This can be rewritten to involve G_1 and G_2 symmetrically: using the identities

$$2G_1\mathbf{ab} + (\omega_{\mathbf{a}} - \omega_{\mathbf{b}}) = G_1\mathbf{ab} + G_2\mathbf{ba}$$

and

$$(\omega_{\mathbf{b}} - \omega_{\mathbf{a}}) + 2G_1\mathbf{ba} = G_1\mathbf{ba} + G_2\mathbf{ab},$$

we get

$$\psi\mathbf{ab} = \begin{cases} \omega_{\mathbf{a}} + G_1\mathbf{ab} + (G_1\mathbf{ab} + G_2\mathbf{ba})(G_2\mathbf{ab} + G_1\mathbf{ba})(\omega_{\mathbf{a}} + \omega_{\mathbf{b}}) \\ \shortparallel \\ \omega_{\mathbf{b}} + G_2\mathbf{ba} + (G_2\mathbf{ba} + G_1\mathbf{ab})(G_1\mathbf{ba} + G_2\mathbf{ab})(\omega_{\mathbf{b}} + \omega_{\mathbf{a}}). \end{cases}$$

This representation is of interest in view of the symmetry considerations to be invoked below.

The next example provides another demonstration of the same nature, arguably the simplest possible because it corresponds to $R \equiv 0$.

Example 6. Consider the space shown in Figure 8. One can check that, for any pair of stimuli $\mathbf{x}, \mathbf{y} \in \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$ and any chain $\mathbf{X}$ with elements in $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$,

$$\Psi^{(1)}\mathbf{xy} \leq \Psi^{(1)}\mathbf{xXy}$$

and

$$\Psi^{(2)}\mathbf{xy} \leq \Psi^{(2)}\mathbf{xXy}.$$

This means that the psychometric increments shown in Figure 9 coincide with the corresponding Fechnerian distances, and we have

$$\psi\mathbf{ab} = \begin{cases} \omega_{\mathbf{a}} + G_1\mathbf{ab} \\ \shortparallel \\ \omega_{\mathbf{b}} + G_2\mathbf{ba}, \end{cases}$$

which are special cases of the representations in Theorem 3.

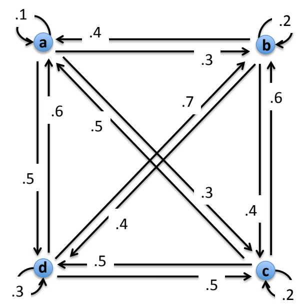

Figure 8: Stimulus space of Example 6.

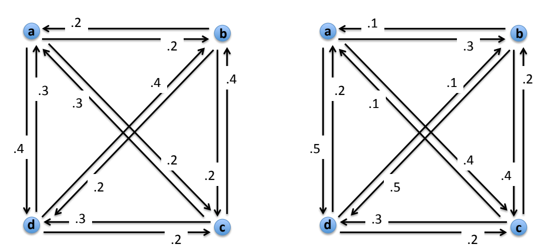

Figure 9: Psychometric increments (in this case coinciding with Fechnerian distances) of the first and second kind computed for the stimulus space shown in Figure 8.

Theorem 3 encompasses a wealth of possible special cases. One can restrict this class by considering stimulus spaces with special properties (as we do in the next section) or by imposing certain symmetry constraints on the function R directly. The latter can be done as follows.

First, we can eliminate from the expression

$$R\left(G_1\mathbf{ab}, G_2\mathbf{ab}, G_1\mathbf{ba}, G_2\mathbf{ba}, \omega_\mathbf{a}, \omega_\mathbf{b}\right)$$

quantities determinable from other quantities. Knowing $G_1\mathbf{ab}$, $G_1\mathbf{ba}$, $\omega_\mathbf{b}$ $\omega_\mathbf{a}$ one can compute $G_2\mathbf{ab}$, $G_2\mathbf{ba}$ as

$$G_2\mathbf{ba} = G_1\mathbf{ab} - (\omega_\mathbf{b} - \omega_\mathbf{a})$$

and

$$G_2\mathbf{ab} = G_1\mathbf{ba} + (\omega_\mathbf{b} - \omega_\mathbf{a})$$

(these identities were used in Example 5). This leads to

$$R\left(G_1\mathbf{ab}, G_2\mathbf{ab}, G_1\mathbf{ba}, G_2\mathbf{ba}, \omega_\mathbf{a}, \omega_\mathbf{b}\right) = \left\{ \begin{array}{c} R_1\left(G_1\mathbf{ab}, G_1\mathbf{ba}, \omega_\mathbf{a}, \omega_\mathbf{b}\right) \\ \shortparallel \\ R_2\left(G_2\mathbf{ba}, G_2\mathbf{ab}, \omega_\mathbf{b}, \omega_\mathbf{a}\right). \end{array} \right.$$

Now, it may sometimes be natural to posit (e.g., in psychophysical applications, when two observation areas contain the same set of stimulus values) that the two observation areas are interchangeable, and so are the stimuli $\mathbf{a}$ and $\mathbf{b}$. More precisely, one can assume that R_1 and R_2 remain invariant

(1) if one exchanges $\mathbf{a}$ and $\mathbf{b}$ in all their arguments; and

(2) if one replaces G_1 with G_2 or vice versa.

Since the arguments of R_2 can be obtained from those of R_1 by successively applying these two rules, we have

$$R_1 \equiv R_2 \equiv R^*.$$

The assumption in question can now be formulated by saying that any asymmetry between the two observation areas is only in the first two summands of the four equivalent representations for ψ:

$$\psi\mathbf{ab} = \left\{ \begin{array}{c} \omega_\mathbf{a} + G_1\mathbf{ab} + R^*\left(G_1\mathbf{ab}, G_1\mathbf{ba}, \omega_\mathbf{a}, \omega_\mathbf{b}\right) \\ \omega_\mathbf{a} + G_1\mathbf{ab} + R^*\left(G_1\mathbf{ba}, G_1\mathbf{ab}, \omega_\mathbf{b}, \omega_\mathbf{a}\right) \\ \omega_\mathbf{b} + G_2\mathbf{ba} + R^*\left(G_2\mathbf{ba}, G_2\mathbf{ab}, \omega_\mathbf{b}, \omega_\mathbf{a}\right) \\ \omega_\mathbf{b} + G_2\mathbf{ba} + R^*\left(G_2\mathbf{ab}, G_2\mathbf{ba}, \omega_\mathbf{a}, \omega_\mathbf{b}\right). \end{array} \right.$$

Let us call such a function R^* *symmetric*. If a reverse problem has a solution with a symmetric function R^*, then

$$G_1\mathbf{ab} - G_2\mathbf{ab} = \psi\mathbf{ab} - \psi\mathbf{ba},$$

$$G_1\mathbf{ab} - G_1\mathbf{ba} = \Psi^{(1)}\mathbf{ab} - \Psi^{(1)}\mathbf{ba},$$

$$G_2\mathbf{ab} - G_2\mathbf{ba} = \Psi^{(2)}\mathbf{ab} - \Psi^{(2)}\mathbf{ba}.$$

Proof. The first of these identities is obtained by subtracting

$$\psi\mathbf{ba} = \omega_{\mathbf{a}} + G_2\mathbf{ab} + R\left(G_2\mathbf{ab}, G_2\mathbf{ba}, \omega_{\mathbf{a}}, \omega_{\mathbf{b}}\right)$$

from

$$\psi\mathbf{ab} = \omega_{\mathbf{a}} + G_1\mathbf{ab} + R\left(G_1\mathbf{ab}, G_1\mathbf{ba}, \omega_{\mathbf{a}}, \omega_{\mathbf{b}}\right).$$

The representations for the differences of the psychometric increments follow then from the identities

$$G_2\mathbf{ba} = G_1\mathbf{ab} - \left(\omega_{\mathbf{b}} - \omega_{\mathbf{a}}\right)$$

and

$$G_2\mathbf{ab} = G_1\mathbf{ba} + \left(\omega_{\mathbf{b}} - \omega_{\mathbf{a}}\right).$$

□

Example 7. The following functions satisfy the symmetry requirement for R^*:

$$\psi\mathbf{ab} = \begin{cases} \omega_{\mathbf{a}} + G_1\mathbf{ab} + \left(G_1\mathbf{ab} + G_2\mathbf{ba}\right)\left(G_2\mathbf{ab} + G_1\mathbf{ba}\right)\left(\omega_{\mathbf{a}} + \omega_{\mathbf{b}}\right) \\ \shortparallel \\ \omega_{\mathbf{b}} + G_2\mathbf{ba} + \left(G_2\mathbf{ba} + G_1\mathbf{ab}\right)\left(G_1\mathbf{ba} + G_2\mathbf{ab}\right)\left(\omega_{\mathbf{b}} + \omega_{\mathbf{a}}\right) \end{cases}$$

and

$$\psi\mathbf{ab} = \begin{cases} \omega_{\mathbf{a}} + G_1\mathbf{ab} + f\left(G\mathbf{ab}, S\left(\omega_{\mathbf{a}}, \omega_{\mathbf{b}}\right)\right) \\ \shortparallel \\ \omega_{\mathbf{b}} + G_2\mathbf{ba} + f\left(G\mathbf{ab}, S\left(\omega_{\mathbf{a}}, \omega_{\mathbf{b}}\right)\right), \end{cases}$$

where S is some commutative function (which may, as a special case, be identically constant, say, zero).

4 Special Stimulus Spaces

4.1 Directly linked spaces

Let us say that a point $\mathbf{a}$ in a stimulus space $(\mathfrak{S}, \psi)$ is *directly* 1*-linked* to point $\mathbf{b}$ (or directly linked to it in the first observation area) if

$$\Psi^{(1)}\mathbf{ab} = G_1\mathbf{ab}.$$

Analogously, a point $\mathbf{a}$ is *directly* 2*-linked* to point $\mathbf{b}$ (or directly linked to it in the second observation area) if

$$\Psi^{(2)}\mathbf{ab} = G_2\mathbf{ab}.$$

A point $\mathbf{a}$ is directly 1-linked to a point $\mathbf{b}$ if and only if $\mathbf{b}$ is directly 2-linked to $\mathbf{a}$.

Proof. The equality

$$\Psi^{(1)}\mathbf{ab} = G_1\mathbf{ab}$$

means that, for every chain $\mathbf{X} = \mathbf{x}_1 \ldots \mathbf{x}_k$,

$$\Psi^{(1)}\mathbf{aXb} = \Psi^{(1)}\mathbf{ax}_1 + \sum_{i=1}^{k-1} \Psi^{(1)}\mathbf{x}_i\mathbf{x}_{i+1} + \Psi^{(1)}\mathbf{x}_k\mathbf{b} \geq \Psi^{(1)}\mathbf{ab}.$$

This can be written as

$$\left(\psi \mathbf{a}\mathbf{x}_1 - \omega_{\mathbf{a}}\right) + \sum_{i=1}^{k-1} \left(\psi \mathbf{x}_i\mathbf{x}_{i+1} - \omega_{\mathbf{x}_i}\right) + \left(\psi \mathbf{x}_k\mathbf{b} - \omega_{\mathbf{x}_k}\right) \geq \left(\psi \mathbf{a}\mathbf{b} - \omega_{\mathbf{a}}\right).$$

Replacing $\omega_{\mathbf{a}}$ with $\omega_{\mathbf{b}}$ on both sides and rearranging the ω-terms as

$$\begin{array}{ccccccc}
\omega_{\mathbf{b}} & \omega_{\mathbf{x}_1} & \cdots & \omega_{\mathbf{x}_i} & \cdots & \omega_{\mathbf{x}_{k-1}} & \omega_{\mathbf{x}_k} \\
\downarrow & \downarrow & \cdots & \downarrow & \cdots & \downarrow & \downarrow \\
\omega_{\mathbf{x}_1} & \omega_{\mathbf{x}_2} & \cdots & \omega_{\mathbf{x}_{i+1}} & \cdots & \omega_{\mathbf{x}_k} & \omega_{\mathbf{b}},
\end{array}$$

we get

$$\left(\psi \mathbf{a}\mathbf{x}_1 - \omega_{\mathbf{x}_1}\right) + \sum_{i=1}^{k-1} \left(\psi \mathbf{x}_i\mathbf{x}_{i+1} - \omega_{\mathbf{x}_{i+1}}\right) + \left(\psi \mathbf{x}_k\mathbf{b} - \omega_{\mathbf{b}}\right) \geq \left(\psi \mathbf{a}\mathbf{b} - \omega_{\mathbf{b}}\right).$$

In other words, for every chain $\mathbf{Y} = \mathbf{x}_k \ldots \mathbf{x}_1$,

$$\Psi^{(2)}\mathbf{b}\mathbf{Y}\mathbf{a} = \Psi^{(2)}\mathbf{b}\mathbf{x}_k + \sum_{i=1}^{k-1} \Psi^{(2)}\mathbf{x}_{i+1}\mathbf{x}_i + \Psi^{(2)}\mathbf{x}_1\mathbf{a} \geq \Psi^{(1)}\mathbf{b}\mathbf{a}.$$

But this means

$$\Psi^{(2)}\mathbf{b}\mathbf{a} = G_2\mathbf{b}\mathbf{a},$$

proving the theorem. □

Points $\mathbf{a}$ *and* $\mathbf{b}$ *are directly 1-linked to each other if and only if they are directly 2-linked to each other. In a space with any two points directly 1-linked any two points are directly 2-linked (and vice versa).* The space referred to in this corollary is called a *directly linked* space. The reverse problem for such a space has its simplest possible solution:

$$\psi \mathbf{a}\mathbf{b} = \begin{cases} G_1\mathbf{a}\mathbf{b} + \omega_{\mathbf{a}} \\ \quad\shortparallel \\ G_2\mathbf{b}\mathbf{a} + \omega_{\mathbf{b}}. \end{cases}$$

Example 8. Consider a space $(\mathfrak{S}, \psi)$ such that

$$\psi \mathbf{a}\mathbf{b} = M\mathbf{a}\mathbf{b} + r_1(\mathbf{a}) + r_2(\mathbf{b}),$$

where M is a symmetric metric and r_1, r_2 some nonnegative functions. Then

$$\Psi^{(1)}\mathbf{a}\mathbf{b} = M\mathbf{a}\mathbf{b} + r_2(\mathbf{b}) - r_2(\mathbf{a})$$

and

$$\Psi^{(2)}\mathbf{a}\mathbf{b} = M\mathbf{a}\mathbf{b} + r_1(\mathbf{a}) - r_1(\mathbf{b}).$$

To ensure Regular Minimality, we posit

$$|r_1(\mathbf{a}) - r_1(\mathbf{b})| < M\mathbf{a}\mathbf{b}$$

and

$$|r_2(\mathbf{a}) - r_2(\mathbf{b})| < M\mathbf{a}\mathbf{b}.$$

We verify that for any $\mathbf{a},\mathbf{b},\mathbf{m}$ in $\mathfrak{S}$,

$$\left.\begin{array}{l}\Psi^{(1)}\mathbf{am}=M\mathbf{am}+r_2(\mathbf{m})-r_2(\mathbf{a})\\+\\\Psi^{(1)}\mathbf{mb}=M\mathbf{mb}+r_2(\mathbf{b})-r_2(\mathbf{m})\end{array}\right.\geq M\mathbf{ab}+r_2(\mathbf{b})-r_2(\mathbf{a})=\Psi^{(1)}\mathbf{ab},$$

proving thereby

$$\Psi^{(1)}\mathbf{ab}=G_1\mathbf{ab}.$$

Analogously we prove

$$\Psi^{(2)}\mathbf{ab}=G_2\mathbf{ab}.$$

Since

$$\left.\begin{array}{c}G_1\mathbf{ab}+G_1\mathbf{ba}\\ \shortparallel\\ G_2\mathbf{ab}+G_2\mathbf{ba}\end{array}\right\}=\left.\begin{array}{c}\Psi^{(1)}\mathbf{ab}+\Psi^{(1)}\mathbf{ba}\\ \shortparallel\\ \Psi^{(1)}\mathbf{ab}+\Psi^{(1)}\mathbf{ba}\end{array}\right\}=2M\mathbf{ab},$$

we conclude that

$$M\mathbf{ab}=\frac{1}{2}G\mathbf{ab},$$

so that the definition of ψ can be given as

$$\psi\mathbf{ab}=\frac{1}{2}G\mathbf{ab}+r_1(\mathbf{a})+r_2(\mathbf{b}).$$

This is one possible form of presenting the "quadrilateral dissimilarity model" (Dzhafarov & Colonius, 2006).

The statement of the following theorem is obvious and given without proof. We denote (referring to Theorem 3)

$$R\left(G_1\mathbf{ab},G_2\mathbf{ab},G_1\mathbf{ba},G_2\mathbf{ba},\omega_{\mathbf{a}},\omega_{\mathbf{b}}\right)=R_{\mathbf{ab}}.$$

If the reverse problem has a solution, then point $\mathbf{a}$ is directly 1-linked to point $\mathbf{b}$ (and $\mathbf{b}$ is directly 2-linked to point $\mathbf{a}$) if and only if $R_{\mathbf{ab}}=0$. This observation agrees with Theorem 4.1: both direct linkages are equivalent to $R_{\mathbf{ab}}=0$. Example 6 shows that a directly linked space can be easily constructed.

4.2 Spaces with metric-in-the-small dissimilarities

A dissimilarity function D is said to be *metric-in-the-small* if, whenever $\mathbf{a}_n\leftrightarrow\mathbf{b}_n$ with $\mathbf{a}_n\neq\mathbf{b}_n$,

$$\frac{D\mathbf{a}_n\mathbf{b}_n}{G\mathbf{a}_n\mathbf{b}_n}\to 1.$$

The convergence is from the right because

$$D\mathbf{a}_n\mathbf{b}_n\geq G\mathbf{a}_n\mathbf{b}_n.$$

Applying this definition to psychometric increments,

$$\left.\begin{array}{c}\mathbf{a}_n\leftrightarrow\mathbf{b}_n\\ \&\\ \mathbf{a}_n\neq\mathbf{b}_n\end{array}\right\}\Longrightarrow\frac{\Psi^{(1)}\mathbf{a}_n\mathbf{b}_n}{G_1\mathbf{a}_n\mathbf{b}_n}\to 1$$

and

$$\left.\begin{array}{c}\mathbf{a}_n \leftrightarrow \mathbf{b}_n \\ \& \\ \mathbf{a}_n \neq \mathbf{b}_n\end{array}\right\} \Longrightarrow \frac{\Psi^{(2)}\mathbf{a}_n\mathbf{b}_n}{G_2\mathbf{a}_n\mathbf{b}_n} \to 1.$$

Clearly, these convergences imply

$$\frac{\Psi^{(\iota)}\mathbf{a}_n\mathbf{b}_n + \Psi^{(\iota)}\mathbf{b}_n\mathbf{a}_n}{G\mathbf{a}_n\mathbf{b}_n} \to 1.$$

Recall that $\mathbf{a}_n \leftrightarrow \mathbf{b}_n$ means any of the equivalent convergences

$$\begin{array}{l}\Psi^{(\iota)}\mathbf{a}_n\mathbf{b}_n \to 0, \\ \Psi^{(\iota)}\mathbf{b}_n\mathbf{a}_n \to 0,\end{array}$$

where ι stands for 1 or 2. If the reverse problem has a solution, then the dissimilarities $\Psi^{(1)}$ and $\Psi^{(2)}$ are metric-in-the-small if and only if $R(x, y, u, v, a, b)$ is of a higher degree of infinitesimality than either of the arguments x and v.

Proof. Rewrite the expressions

$$\begin{array}{rcl}\psi\mathbf{a}_n\mathbf{b}_n & = & \omega_{\mathbf{a}_n} + G_1\mathbf{a}_n\mathbf{b}_n + R_{\mathbf{a}_n\mathbf{b}_n} \\ & = & \omega_{\mathbf{b}_n} + G_2\mathbf{b}_n\mathbf{a}_n + R_{\mathbf{a}_n\mathbf{b}_n}\end{array}$$

as

$$\frac{\Psi^{(1)}\mathbf{a}_n\mathbf{b}_n}{G_1\mathbf{a}_n\mathbf{b}_n} = 1 + \frac{R\left(G_1\mathbf{a}_n\mathbf{b}_n, G_2\mathbf{a}_n\mathbf{b}_n, G_1\mathbf{b}_n\mathbf{a}_n, G_2\mathbf{b}_n\mathbf{a}_n, \omega_{\mathbf{a}_n}, \omega_{\mathbf{b}_n}\right)}{G_1\mathbf{a}_n\mathbf{b}_n}$$

and

$$\frac{\Psi^{(2)}\mathbf{b}_n\mathbf{a}_n}{G_2\mathbf{b}_n\mathbf{a}_n} = 1 + \frac{R\left(G_1\mathbf{a}_n\mathbf{b}_n, G_2\mathbf{a}_n\mathbf{b}_n, G_1\mathbf{b}_n\mathbf{a}_n, G_2\mathbf{b}_n\mathbf{a}_n, \omega_{\mathbf{a}_n}, \omega_{\mathbf{b}_n}\right)}{G_2\mathbf{b}_n\mathbf{a}_n}.$$

We know that, as $\mathbf{a}_n \leftrightarrow \mathbf{b}_n$, both

$$G_1\mathbf{a}_n\mathbf{b}_n \to 0$$

and

$$G_2\mathbf{b}_n\mathbf{a}_n \to 0.$$

The left-hand sides tend to 1 if and only if the ratios on the right tend to zero, proving the theorem. □

If we make use of the symmetry constraint of Section 3 and present the function R in the above theorem as

$$R^*\left(G_1\mathbf{ab}, G_1\mathbf{ba}, \omega_{\mathbf{a}}, \omega_{\mathbf{b}}\right) = R^*\left(G_2\mathbf{ba}, G_2\mathbf{ab}, \omega_{\mathbf{b}}, \omega_{\mathbf{a}}\right),$$

then the condition of the higher-order infinitesimality acquires a simpler form. *If the reverse problem has a solution with a symmetric R^*, then $\Psi^{(1)}$ is metric-in-the-small if and only if so is $\Psi^{(2)}$ and if and only if $R^*(x, y, a, b)$ is of a higher degree of infinitesimality than both x and y.*

Example 9. In particular, if R^* can be presented as in the second function of Example 7,

$$R^* = f\left(G\mathbf{ab}, S\left(\omega_{\mathbf{a}}, \omega_{\mathbf{b}}\right)\right),$$

the condition of the higher-order infinitesimality reduces to

$$\frac{R^*\left(x, a\right)}{x} \to 0$$

as $x \to 0$ $(x \neq 0)$. Such functions as

$$\psi\mathbf{ab} = \begin{cases} \omega_{\mathbf{a}} + G_1\mathbf{ab} + \frac{|\omega_{\mathbf{a}} - \omega_{\mathbf{b}}|}{\omega_{\mathbf{a}} + \omega_{\mathbf{b}}}\left(G\mathbf{ab}\right)^2 \\ \qquad\qquad \shortparallel \\ \omega_{\mathbf{b}} + G_2\mathbf{ba} + \frac{|\omega_{\mathbf{a}} - \omega_{\mathbf{b}}|}{\omega_{\mathbf{a}} + \omega_{\mathbf{b}}}\left(G\mathbf{ab}\right)^2 \end{cases}$$

and

$$\psi\mathbf{ab} = \begin{cases} \omega_{\mathbf{a}} + G_1\mathbf{ab} + k_2\left(G\mathbf{ab}\right)^2 + k_3\left(G\mathbf{ab}\right)^3 + \ldots + k_r\left(G\mathbf{ab}\right)^r \\ \qquad\qquad\qquad \shortparallel \\ \omega_{\mathbf{b}} + G_2\mathbf{ba} + k_2\left(G\mathbf{ab}\right)^2 + k_3\left(G\mathbf{ab}\right)^3 + \ldots + k_r\left(G\mathbf{ab}\right)^r \end{cases}$$

provide examples.

4.3 Uniformly discrete spaces

A space $(\mathfrak{S}, \psi)$ is *uniformly discrete* if

$$\inf_{\mathbf{x} \neq \mathbf{y}} \Psi^{(1)}\mathbf{xy} > 0.$$

This condition is equivalent to

$$\inf_{\mathbf{x} \neq \mathbf{y}} \Psi^{(2)}\mathbf{xy} > 0$$

because, as we know,

$$\Psi^{(1)}\mathbf{x}_n\mathbf{y}_n \to 0 \Longleftrightarrow \Psi^{(2)}\mathbf{x}_n\mathbf{y}_n \to 0.$$

Any finite space is uniformly discrete.

In the following we will tacitly assume, with no loss of generality, that all chains $\mathbf{X} = \mathbf{x}_1 \ldots \mathbf{x}_k$ considered are *non-wasteful*, in the following sense: for no $i = 1, \ldots, k-1$, $\mathbf{x}_i = \mathbf{x}_{i+1}$.

A chain $\mathbf{X} = \mathbf{x}_1 \ldots \mathbf{x}_k$ is called 1-*basic* if, for any $1 \leq i < k$, $\mathbf{x}_i$ is directly 1-linked to $\mathbf{x}_{i+1}$. A 2-basic chain is defined analogously.

The class of all 1-basic (2-basic) chains connecting $\mathbf{a}$ to $\mathbf{b}$ and containing k elements, not counting $\mathbf{a}$ and $\mathbf{b}$, is denoted by $\mathcal{C}_k^1\mathbf{ab}$ (respectively, $\mathcal{C}_k^2\mathbf{ab}$). Clearly, $k = 0, 1, \ldots$. For any $\mathbf{a}, \mathbf{b}$ in a uniformly discrete space one can find k_1 and k_2 such that

$$G_1\mathbf{ab} = \inf_{\mathbf{aXb} \in \mathcal{C}_{k_1}^1\mathbf{ab}} \Psi^{(1)}\mathbf{aXb}$$

and

$$G_2\mathbf{ab} = \inf_{\mathbf{aXb} \in \mathcal{C}_{k_2}^2\mathbf{ab}} \Psi^{(2)}\mathbf{aXb}.$$

Proof. Let ι stand for 1 or 2. Consider all chains $\mathbf{X}^{(k)}$ containing $k = 0, 1, ...,$ elements, and define

$$G_\iota^{(k)}\mathbf{ab} = \inf_{\mathbf{X}^{(k)}} \Psi^{(\iota)}\mathbf{aX}^{(k)}\mathbf{b}.$$

Denoting

$$s_\iota = \inf_{\mathbf{x}\neq\mathbf{y}} \Psi^{(\iota)}\mathbf{xy} > 0,$$

we have

$$\Psi^{(\iota)}\mathbf{aX}^{(k)}\mathbf{b} \geq (k+1)\, s_\iota,$$

hence also

$$G_\iota^{(k)}\mathbf{ab} \geq (k+1)\, s_\iota.$$

Therefore, for some $K > 0$, we have to have

$$G_\iota\mathbf{ab} < G_\iota^{(K)}\mathbf{ab}.$$

Consider a number k with the following property: in some sequence of chains $\mathbf{aX}_n\mathbf{b}$ such that

$$\Psi^{(\iota)}\mathbf{aX}_n\mathbf{b} \to G_\iota\mathbf{ab}$$

the chains with k elements occur infinitely often. Denote by k_0 the largest number with this property (which exists because $k < K$). By construction, there is a sequence of chains $\mathbf{aX}_n^{(k_0)}\mathbf{b}$ such that

$$\Psi^{(\iota)}\mathbf{aX}_n^{(k_0)}\mathbf{b} \to G_\iota\mathbf{ab},$$

but there is no sequence of chains $\mathbf{aX}_n^{(>k_0)}\mathbf{b}$ in which each $\mathbf{X}_n^{(>k_0)}$ has more than k_0 elements such that

$$\Psi^{(\iota)}\mathbf{aX}_n^{(>k_0)}\mathbf{b} \to G_\iota\mathbf{ab}.$$

We will show that all but a finite number of these chains $\mathbf{aX}_n^{(k_0)}\mathbf{b}$ are ι-basic. Suppose this is not the case. Then one can choose a sequence of chains $\mathbf{aX}_n^{(k_0)}\mathbf{b}$ all of which are not ι-basic, with

$$\Psi^{(\iota)}\mathbf{aX}_n^{(k_0)}\mathbf{b} \to G_\iota\mathbf{ab}.$$

Let $\mathbf{x}_{i_n,n}\mathbf{x}_{i_n+1,n}$ be a link in each of these chains with $\mathbf{x}_{i_n,n}$ not directly ι-linked to $\mathbf{x}_{i_n+1,n}$ (where i_n may be 0 or k_0, with $\mathbf{x}_{0,n} = \mathbf{a}$ and $\mathbf{x}_{k_0+1,n} = \mathbf{b}$). Then one can find nonempty chains $\mathbf{Y}_n$ such that

$$\Psi^{(\iota)}\mathbf{ax}_{1,n}...\mathbf{x}_{i_n,n}\mathbf{Y}_n\mathbf{x}_{i_n+1,n}...\mathbf{x}_{k_0,n}\mathbf{b} < \Psi^{(\iota)}\mathbf{aX}_n^{(k_0)}\mathbf{b},$$

and since the convergence of $\Psi^{(\iota)}\mathbf{aX}_n^{(k_0)}\mathbf{b}$ to $G_\iota\mathbf{ab}$ is from the right,

$$\Psi^{(\iota)}\mathbf{ax}_{1,n}...\mathbf{x}_{i_n,n}\mathbf{Y}_n\mathbf{x}_{i_n+1,n}...\mathbf{x}_{k_0,n}\mathbf{b} \to G_\iota\mathbf{ab}.$$

Clearly, $\mathbf{Y}_n$ must contain an element

$$\mathbf{m}_n \notin \{\mathbf{x}_{i_n,n}, \mathbf{x}_{i_n+1,n}\},$$

whence the chains

$$\mathbf{x}_{1,n}...\mathbf{x}_{i_n}\mathbf{Y}_n\mathbf{x}_{i_n+1}...\mathbf{x}_{k_0,n}$$

contain more than k_0 elements. But this contradicts the definition of k_0. □

With ι standing for 1 or 2, in a uniformly discrete space any point can be directly ι-linked to some other point, and any two points can be connected by an ι-basic chain. To formulate another immediate consequence of Theorem 4.3, we need a new concept. A *base* for G_1 is a subset

$$\mathfrak{D} \subseteq \mathfrak{S} \times \mathfrak{S}$$

such that if

$$G_1\mathbf{ab} = G_1^*\mathbf{ab}$$

for all $(\mathbf{a}, \mathbf{b}) \in \mathfrak{D}$, then

$$G_1\mathbf{ab} = G_1^*\mathbf{ab}$$

for all $(\mathbf{a}, \mathbf{b}) \in \mathfrak{S} \times \mathfrak{S}$. The definition of a base for G_2 is analogous. *With ι standing for 1 or 2, in a uniformly discrete space the set of all directly ι-linked ordered pairs of points forms a base for G_ι.* We conclude this chapter by considering the reverse problem for a special case of uniformly discrete spaces, those *with geodesics*. All finite spaces fall within this category.

A uniformly discrete space is said to be with geodesics if, for any points $\mathbf{a}$ and $\mathbf{b}$ in it,

$$G_1\mathbf{ab} = \min_{\mathbf{X}} \Psi^{(1)}\mathbf{aXb}$$

and

$$G_2\mathbf{ab} = \min_{\mathbf{X}} \Psi^{(2)}\mathbf{aXb}.$$

In other words, the requirement is that for any $\mathbf{a}$ and $\mathbf{b}$ one be able to find chains $\mathbf{X}_1$ and $\mathbf{X}_2$ such that

$$G_1\mathbf{ab} = \Psi^{(1)}\mathbf{aX}_1\mathbf{b}$$

and

$$G_2\mathbf{ab} = \Psi^{(2)}\mathbf{aX}_2\mathbf{b}.$$

The chains $\mathbf{aX}_1\mathbf{b}$ and $\mathbf{aX}_2\mathbf{b}$ in this definition are referred to as *geodesics of the first and second kind*, respectively. That geodesics need not exist in all uniformly discrete spaces is shown by the following example.

Example 10. Let $\mathfrak{S}$ consist of $\mathbf{a}$,$\mathbf{b}$, and $\mathbf{c}_1, \mathbf{c}_2, \ldots$. Let $\Psi^{(1)}$ be a symmetric function with the following values: for $i, j \in \{1, 2, \ldots\}$,

$$\begin{aligned}
&\Psi^{(1)}\mathbf{ab} = 2,\\
&\Psi^{(1)}\mathbf{ac}_i = 1 + {}^1\!/_i,\\
&\Psi^{(1)}\mathbf{bc}_i = {}^1\!/_2,\\
&\Psi^{(1)}\mathbf{c}_i\mathbf{c}_j = |i - j|\,.
\end{aligned}$$

The space is uniformly discrete because

$$\inf_{\mathbf{x} \neq \mathbf{y}} \Psi^{(1)}\mathbf{xy} = \frac{1}{2}.$$

For the sequence of chains $\mathbf{ac}_n\mathbf{b}$, as $n \to \infty$,

$$\Psi^{(1)}\mathbf{ac}_n\mathbf{b} = \frac{3}{2} + \frac{1}{n} \to \frac{3}{2},$$

and it is easy to see that

$$G_1\mathbf{ab} = \frac{3}{2}.$$

There is, however, no chain $\mathbf{X}$ such that

$$\Psi^{(1)}\mathbf{aXb} = \frac{3}{2}.$$

Therefore, this uniformly discrete space is not with geodesics.

In a uniformly discrete space with geodesics, we say that $\mathbf{a}$ is *strongly 1-linked* (*strongly 2-linked*) to $\mathbf{b}$ if $\mathbf{ab}$ is the only geodesic of the first (respectively, second) kind connecting $\mathbf{a}$ to $\mathbf{b}$. Clearly, strong ι-linkage implies direct ι-linkage ($\iota = 1, 2$).

A chain $\mathbf{X} = \mathbf{x}_1 ... \mathbf{x}_k$ is called *strongly ι-basic* if, for any $1 \le i < k$, $\mathbf{x}_i$ is strongly ι-linked to $\mathbf{x}_{i+1}$ ($\iota = 1, 2$). With ι standing for 1 or 2, in a uniformly discrete space with geodesics,

(i) any two points can be connected by a strongly ι-basic geodesic chain;

(ii) any point can be strongly ι-linked to some other point; and

(iii) the set of all strongly ι-linked ordered pairs of points forms a base for G_ι.

Proof. Assume (i) is not true. Then, for some $\mathbf{a}$ and $\mathbf{b}$, all geodesics $\mathbf{aXb}$ are not strongly ι-basic. Choose a geodesic

$$\mathbf{ax}_1 \ldots \mathbf{x}_k\mathbf{b}$$

with the largest number of elements k. It should exist by the same argument as in Theorem 4.3:

$$\Psi^{(\iota)}\mathbf{ax}_1 \ldots \mathbf{x}_k\mathbf{b} \ge (k+1) \inf_{\mathbf{x} \ne \mathbf{y}} \Psi^{(\iota)}\mathbf{xy} > 0.$$

Let $\mathbf{x}_i\mathbf{x}_{i+1}$ be a link in $\mathbf{aXb}$ with $\mathbf{x}_i$ not strongly ι-linked to $\mathbf{x}_{i+1}$. Then there exists a geodesic chain $\mathbf{x}_i\mathbf{Y}\mathbf{x}_{i+1}$ with a nonempty $\mathbf{Y}$, whence

$$\mathbf{ax}_1 \ldots \mathbf{x}_i\mathbf{Y}\mathbf{x}_{i+1} \ldots \mathbf{x}_k\mathbf{b}$$

is a geodesic chain from $\mathbf{a}$ to $\mathbf{b}$ with more than k elements. This contradiction proves (i). The statements (ii) and (iii) are immediate corollaries. □

With ι standing for 1 or 2, in a uniformly discrete space with geodesics, a point $\mathbf{a}$ is strongly ι-linked to a point $\mathbf{b}$ if and only if, for any point $\mathbf{m}$ distinct from $\mathbf{a}$ and $\mathbf{b}$,

$$G_\iota\mathbf{amb} > G_i\mathbf{ab}.$$

Proof. We prove the equivalence of the negations of the two statements:

(1) "$\mathbf{a}$ is not strongly ι-linked to $\mathbf{b}$" means $\mathbf{a}$ is connected to $\mathbf{b}$ by a geodesic other than $\mathbf{ab}$;

(2) the negation of the inequality in the formulation is (since G_ι is a metric) the equality

$$G_\iota\mathbf{amb} = G_i\mathbf{ab}.$$

If $\mathbf{a}$ can be connected to $\mathbf{b}$ by a geodesic $\mathbf{aXb}$ with a nonempty $\mathbf{X}$, then choosing an element $\mathbf{m}$ of this chain, we get the equality above. Conversely, if this equality is satisfied for some point $\mathbf{m}$, then the concatenation of the geodesics from $\mathbf{a}$ to $\mathbf{m}$ and from $\mathbf{m}$ to $\mathbf{b}$ is a geodesic other than $\mathbf{ab}$. □

We demonstrate the use of these results by the following example.

Example 11. Consider Figure 3 again. Inspecting, say, the link from $\mathbf{a}$ to $\mathbf{d}$ in the left panel we see that

$$G_1\mathbf{ad} = 0.4 < \begin{cases} G_1\mathbf{abd} = 0.2 + 0.6 \\ G_1\mathbf{acd} = 0.2 + 0.3. \end{cases}$$

Since this exhausts all triads of the form $\mathbf{amd}$ in this space, we conclude that $\mathbf{a}$ is strongly 1-linked to $\mathbf{d}$, and

$$\Psi^{(1)}\mathbf{ad} = G_1\mathbf{ad} = 0.4.$$

We come to the same conclusion and restore the values of $\Psi^{(1)}$ and $\Psi^{(2)}$ for all links in Figure 3 with unframed values of distance. For the links with framed values, say, from $\mathbf{c}$ to $\mathbf{d}$ in the right panel, we have

$$G_2\mathbf{cd} = 0.6 = G_1\mathbf{cad} = 0.1 + 0.5,$$

whence we conclude that $\mathbf{a}$ is not strongly 2-linked to $\mathbf{d}$. The value of $\Psi^{(2)}\mathbf{cd}$ therefore cannot be reconstructed uniquely: it can be any value ≥ 0.6.

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