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Double Skew-Dual Scaling: A Conjoint Scaling of Two Sets of Objects Related by a Dominance Matrix

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Consider two sets of objects, $\{\alpha_1, ..., \alpha_n\}$ and $\{\beta_1, ..., \beta_m\}$, such as n subjects solving m tasks, or n stimuli presented first and m stimuli presented second in a pairwise comparison experiment. Let any pair (α_i, β_j) be associated with a real number a_{ij}, interpreted as the degree of dominance of α_i over β_j (e.g., the probability of α_i relating in a certain way to β_j). Intuitively, the problem addressed in this paper is how to conjointly, in a "naturally" coordinated fashion, characterize the α-objects and β-objects in terms of their overall tendency to dominate or be dominated. The gist of the solution is as follows. Let A denote the $n \times m$ matrix of a_{ij} values, and let there be a class of monotonic transformations ϕ with nonnegative codomains. For a given ϕ, a complementary matrix B is defined so that $\phi(a_{ij}) + \phi(b_{ij}) = const$, and one computes vectors D^α and D^β (the dominance values for α-objects and β-objects) by solving the equations $\phi(A)\,\phi(D^\beta)/\Sigma\phi(D^\beta) = \phi(D^\alpha)$ and $\phi(B^T)\,\phi(D^\alpha)/\Sigma\phi(D^\alpha) = \phi(D^\beta)$, where ^{T} is transposition, Σ is the sum of elements, and ϕ applies elementwise. One also computes vectors S^α and S^β (the subdominance values for α-objects and β-objects) by solving the equations $\phi(B)\,\phi(S^\beta)/\Sigma\phi(S^\beta) = \phi(S^\alpha)$ and $\phi(A^T)\,\phi(S^\alpha)/\Sigma\phi(S^\alpha) = \phi(S^\beta)$. The relationship between S-vectors and D-vectors is complex: intuitively, D^α characterizes the tendency of an α-object to dominate β-objects with large dominance values, whereas S^α characterizes the tendency of an α-objects to fail to dominate β-objects with large subdominance values. For classes containing more than one ϕ-transformation, one can choose an optimal ϕ as the one maximizing some measure of discrimination between individual elements of vectors $\phi(D^\alpha)$, $\phi(D^\beta)$, $\phi(S^\alpha)$, and $\phi(S^\beta)$, such as the product or minimum of these vectors' variances. The proposed analysis of dominance matrices has only superficial similarities with the classical dual scaling (Nishisato, 1980). © 1999 Academic Press

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1. INFORMAL INTRODUCTION

The purpose of this paper is to introduce a new scaling (or measurement) procedure that I call the double *skew-dual scaling*. The procedure provides numerical values for any two sets of objets $\{\alpha_1, ..., \alpha_n\}$ and $\{\beta_1, ..., \beta_m\}$ that are related to each other by a matrix interpretable (at least loosely) as a dominance matrix. The numerical values assigned to the objects reflect their overall tendency to dominate or be dominated. The double skew-dual scaling is not based on and does not constitute an empirical model of dominance matrices: it does not impose falsifiable constraints on the relationship between the matrices' elements. Rather the logical status of the procedure is similar to that of computing the mean values for probability distributions: it is a quantitative descriptive language that provides a "reasonable" summary of the relationship between $\{\alpha_1, ..., \alpha_n\}$ and $\{\beta_1, ..., \beta_m\}$ and can be utilized in formulating falsifiable models for the structure of dominance matrices. The double skew-dual scaling has certain similarities with the classical dual scaling (Guttman, 1941; Nishisato, 1980, 1996), but as shown in the concluding section of this paper, the similarities are more superficial than deep.

The three matrices below can be viewed as dominance matrices:

$A_{4\times 7}$	β_1	β_2	β_3	β_4	β_5	β_6	β_7
α_1	1	1	1	1	0	0	0
α_2	1	0	0	1	0	1	1
α_3	1	1	0	0	1	1	1
α_4	0	0	0	0	1	0	1

,

$A_{4\times 4}$	β_1	β_2	β_3	β_4
α_1	0.50	0.57	0.67	0.80
α_2	0.43	0.50	0.60	0.75
α_3	0.33	0.40	0.50	0.67
α_4	0.20	0.25	0.33	0.50

,

$A_{3\times 3}$	β_1	β_2	β_3
α_1	0	2	3
α_2	-1	1	1
α_3	-2	0	2

.

A typical situation depicted by the left matrix could be one involving examinees, $\{\alpha_1, ..., \alpha_4\}$, attempting to solve a set of problems, $\{\beta_1, ..., \beta_7\}$, the entries of the matrix being Boolean outcomes of these attempts. The middle matrix could represent a classical pairwise comparison experiment, with $\{\alpha_1, ..., \alpha_4\}$ and $\{\beta_1, ..., \beta_4\}$ being the same four stimuli when presented first and when presented second, respectively, and the entries being the probabilities with which row stimuli are preferred to column stimuli. The right matrix could represent a similar pairwise comparison experiment, in which the judgments consist in indicating the degree or confidence with which one prefers row stimuli (presented first within pairs) to column stimuli (presented second), or vice versa (on a seven point scale, ranging from -3, a strong preference for a stimulus presented first, to $+3$, a strong preference for a stimulus presented second). The three matrices, however, might also represent situations less readily interpretable in dominance terms. For instance, $\{\alpha_1, ..., \alpha_4\}$ in the left matrix could be four observation conditions (e.g., sensory adaptation states), while $\{\beta_1, ..., \beta_7\}$ could be seven stimuli (say, colors), the Boolean entries indicating whether a given color is detectable in a given adaptation state.

From a substantive point of view, *any matrix indicating the truth/falsity, probability, or degree of a row object standing in a certain relationship to a column object can be considered a dominance matrix*. From a mathematical point of view, the applicability of the double skew-dual scaling to a class of matrices only requires that all possible entries of these matrices are confined to a finite interval of reals (but see Section 4.2 for a possible relaxation of this constraint).

Consider now the first of the three matrices above, $A_{4\times 7}$,[1] and let the problem be how to ascribe to α-objects values reflecting their tendency to dominate β-objects, and vice versa. It might seem that the simplest and natural way of assigning a dominance value to a given α-object is by counting the number (or better still, the proportion, to keep everything in the same range) of β-objects that this α-object dominates:

$A_{4\times 7}$	β_1	β_2	β_3	β_4	β_5	β_6	β_7	α-*dominance*
α_1	1	1	1	1	0	0	0	4/7
α_2	1	0	0	1	0	1	1	4/7
α_3	1	1	0	0	1	1	1	5/7
α_4	0	0	0	0	1	0	1	2/7

Analogously, it might seem natural to assign a dominance value to a β-object by computing the proportion of α-objects that fail to dominate it. For our purposes, it is convenient to present this computation by introducing a matrix complementary to $A_{4\times 7}$:

$$B_{4\times 7} = 1_{4\times 7} - A_{4\times 7};$$

β-dominance is now computed from the columns of $B_{4\times 7}$ in the same way α-dominance is computed from the rows of $A_{4\times 7}$:

$B_{4\times 7}$	β_1	β_2	β_3	β_4	β_5	β_6	β_7
α_1	0	0	0	0	1	1	1
α_2	0	1	1	0	1	0	0
α_3	0	0	1	1	0	0	0
α_4	1	1	1	1	0	1	0
β-*dominance*	1/4	2/4	3/4	2/4	2/4	2/4	1/4

Denoting the α-dominance and β-dominance vectors by $D^{\alpha}_{4\times 1}$ and $D^{\beta}_{7\times 1}$, respectively, the two computations can be presented as

$$\left[\begin{array}{l} A_{4\times 7}\dfrac{1_{7\times 1}}{7} = D^{\alpha}_{4\times 1} \\ B^{T}_{7\times 4}\dfrac{1_{4\times 1}}{4} = D^{\beta}_{7\times 1} \end{array}\right. .$$

[1] For the reader's convenience, all matrices in this paper are presented with their dimensions as subscripts, $M_{n\times m}$, and all vectors are presented as one-column matrices, $V_{n\times 1}$. This notation makes the customary boldfacing of matrices and vectors unnecessary. The transpose of $M_{n\times m}$ is denoted as $M^{T}_{m\times n} = (M_{n\times m})^{T}$. $1_{n\times m}$ and $0_{n\times m}$ are matrices with all entries equal to 1 and 0, respectively. $\Sigma V_{n\times 1}$ denotes the sum of the elements of $V_{n\times 1}$.

A closer look reveals, however, that when computed in this way, the vectors $D^{\alpha}_{4\times 1}$ and $D^{\beta}_{7\times 1}$ are not *mutually consistent*, in the following sense. Intuitively, the dominance value of a β-object should depend on how high the dominance values of the α-objects it dominates are, and vice versa. Compare, for example, β_2 and β_5. Both have been assigned dominance values equal to 2/4, because they dominate two α-objects each. But β_2 dominates α-objects whose dominance values are 4/7 and 2/7, whereas β_5 dominates α-objects whose dominance values are higher, 4/7 and 4/7. It seems more reasonable then to have ascribed a lower dominance value to β_2 than to β_5. This can be achieved by means of weighting the dominated α-objects by their own dominance values;

$$\frac{4/7\times 0+4/7\times 1+5/7\times 0+2/7\times 1}{4/7+4/7+5/7+2/7}=\frac{6}{15}$$

for β_2, and

$$\frac{4/7\times 1+4/7\times 1+5/7\times 0+2/7\times 0}{4/7+4/7+5/7+2/7}=\frac{8}{15}$$

for β_5. Analogously, it seems unfair to have ascribed equal dominance values to α_1 and α_2. Even though they dominate four β-objects each, α_1 dominates β-objects with greater dominance values (1/4, 2/4, 2/4, 3/4) than does α_2(1/4, 1/4, 2/4, 2/4). A more reasonable way of computing the α-dominance values would be

$$\frac{1/4\times 1+2/4\times 1+3/4\times 1+2/4\times 1+2/4\times 0+2/4\times 0+1/4\times 0}{1/4+2/4+3/4+2/4+2/4+2/4+1/4}=\frac{8}{13}$$

for α_1, and

$$\frac{1/4\times 1+2/4\times 0+3/4\times 0+2/4\times 1+2/4\times 0+2/4\times 1+1/4\times 1}{1/4+2/4+3/4+2/4+2/4+2/4+1/4}=\frac{6}{13}$$

for α_2.

Using this logic, one could compute a new pair of dominance vectors $D^{\alpha}_{4\times 1}$ and $D^{\beta}_{7\times 1}$, but this would not eliminate the problem: although the new α-dominance values would now depend on the old β-dominance values, they would remain, as one can easily verify, inconsistent with the new β-dominance values (that, in turn, depend on the old α-dominance values but are not consistent with the new α-dominance values). These considerations lead one to a more sophisticated formulation of the problem of how to assign dominance values to the two sets of objects. The desideratum now is to make these assignments so that

(i) α-dominance values characterize the rows of matrix $A_{4\times 7}$ in such a way that the dominance value for α_i ($i=1, ..., 4$) be the mean of the ith row of $A_{4\times 7}$ weighted by the β-dominance values;

(ii) β-dominance values characterize the columns of matrix $B_{4\times7}$ in such a way that the dominance values for β_j ($j = 1, ..., 7$) be the mean of the jth column of $B_{4\times7}$ weighted by the α-dominance values.

These requirements translate into the formula representing what I call a *skew-dual relationship* between α-dominance and β-dominance:

$$\left[\begin{array}{l} A_{4\times7}\dfrac{D^{\beta}_{7\times1}}{\sum D^{\beta}_{7\times1}} = D^{\alpha}_{4\times1} \\ B^{T}_{7\times4}\dfrac{D^{\alpha}_{4\times1}}{\sum D^{\alpha}_{4\times1}} = D^{\beta}_{7\times1} \end{array}\right. . \qquad (1)$$

If the numerical iterations described earlier continued indefinitely, the vectors $(D^{\alpha}_{4\times1}, D^{\beta}_{7\times1})$ would have eventually converged to the solution of (1). As explained in Section 2, this system of equations has one and only one pair of nonnegative vectors $(D^{\alpha}_{4\times1}, D^{\beta}_{7\times1})$ as its solution:

$A_{4\times7}$	β_1	β_2	β_3	β_4	β_5	β_6	β_7	α-*dominance*
α_1	1	1	1	1	0	0	0	.57
α_2	1	0	0	1	0	1	1	.46
α_3	1	1	0	0	1	1	1	.61
α_4	0	0	0	0	1	0	1	.28

$B_{4\times7}$	β_1	β_2	β_3	β_4	β_5	β_6	β_7
α_1	0	0	0	0	1	1	1
α_2	0	1	1	0	1	0	0
α_3	0	0	1	1	0	0	0 .
α_4	1	1	1	1	0	1	0
β-*dominance*	.15	.38	.70	.46	.54	.45	.30

I call these vectors the *dominance vectors*, and together they form a *skew-dual dominance scale* for objects $\{\alpha_1, ..., \alpha_4\}$ and $\{\beta_1, ..., \beta_7\}$.

The dominance vector $D^{\alpha}_{4\times1}$ measures the objects $\{\alpha_1, ..., \alpha_4\}$ through the corresponding rows of the matrix $A_{4\times7}$, while the dominance vector $D^{\beta}_{7\times1}$ measures the objects $\{\beta_1, ..., \beta_7\}$ through the corresponding columns of the matrix $B_{4\times7}$. The two matrices are mutually complementary,

$$B_{4\times7} = 1_{4\times7} - A_{4\times7},$$

and the complete symmetry of this relationship,

$$A_{4\times7} = 1_{4\times7} - B_{4\times7},$$

suggests that one could also approach the situation in a symmetrically opposite way. Namely, one could assign numerical values to $\{\alpha_1, ..., \alpha_4\}$ by characterizing the corresponding rows of the matrix $B_{4\times 7}$, while the numerical values for $\{\beta_1, ..., \beta_7\}$ could be computed from the corresponding columns of the matrix $A_{4\times 7}$. The rows of $B_{4\times 7}$, when related to the α-objects, can be interpreted as these objects' *subdominance* patterns (as opposed to the rows of $A_{4\times 7}$, interpreted as the α-objects' dominance patterns): a larger entry here indicates a larger degree (or, in the Boolean case, the fact) of failure of the corresponding α-object to dominate the corresponding β-object. Analogously, the β-objects' subdominance patterns are represented by the columns of $A_{4\times 7}$. The measurement problem, therefore, becomes one of assigning to α-objects their subdominance values, reflecting their tendency to be dominated by β-objects, and vice versa. By a complete analogy with the dominance-related procedure, the desideratum here is to assign subdominance values so that

(iii) α-subdominance values characterize the rows of matrix $B_{4\times 7}$ in such a way that the subdominance value for α_i $(i=1, ..., 4)$ be the mean of the ith row of $B_{4\times 7}$ weighted by the β-subdominance values;

(iv) β-subdominance values characterize the columns of matrix $A_{4\times 7}$ in such a way that the subdominance value for β_j $(i=1, ..., 7)$ be the mean of the jth column of $A_{4\times 7}$, weighted by the α-subdominance values.

This translates into the formula representing the second *skew-dual relationship*, this time between α-subdominance and β-subdominance. Denoting the α-subdominance and β-subdominance vectors by $S^{\alpha}_{4\times 1}$ and $S^{\beta}_{7\times 1}$, respectively,

$$\left[\begin{array}{l} B_{4\times 7}\dfrac{S^{\beta}_{7\times 1}}{\sum S^{\beta}_{7\times 1}}=S^{\alpha}_{4\times 1} \\ A^{T}_{7\times 4}\dfrac{S^{\alpha}_{4\times 1}}{\sum S^{\alpha}_{4\times 1}}=S^{\beta}_{7\times 1} \end{array}\right. . \tag{2}$$

Like (1), this system of equations has one and only one pair of nonnegative vectors $(S^{\alpha}_{4\times 1}, S^{\beta}_{7\times 1})$ as its solution:

$B_{4\times 7}$	β_1	β_2	β_3	β_4	β_5	β_6	β_7	α-*subdominance*
α_1	0	0	0	0	1	1	1	.47
α_2	0	1	1	0	1	0	0	.36
α_3	0	0	1	1	0	0	0	.23
α_4	1	1	1	1	0	1	0	.64

,

$A_{4\times 7}$	β_1	β_2	β_3	β_4	β_5	β_6	β_7
α_1	1	1	1	1	0	0	0
α_2	1	0	0	1	0	1	1
α_3	1	1	0	0	1	1	1
α_4	0	0	0	0	1	0	1
β-*subdominance*	.62	.41	.28	.49	.51	.35	.72

.

These are the *subdominance vectors* forming together a *skew-dual subdominance scale*, $(S^{\alpha}_{4\times 1}, S^{\beta}_{7\times 1})$. One could also combine $D^{\alpha}_{4\times 1}$ with $S^{\alpha}_{4\times 1}$ and consider the pair $(D^{\alpha}_{4\times 1}, S^{\alpha}_{4\times 1})$ as a *double* (dominance–subdominance) scale for the α-objects; analogously, $(D^{\beta}_{7\times 1}, S^{\beta}_{7\times 1})$ is a double scale for the β-objects. Figure 1 shows all these relationships schematically.

Since the dominance and subdominance patterns of one and the same object (i.e., the corresponding rows, or columns, of the matrices $A_{7\times 4}$ and $B_{4\times 7}$) are mutually complementary, one might have expected *a priori* that the dominance and subdominance values assigned to the same objects would be mutually complementary too,

$$\begin{array}{l} D^{\alpha}_{4\times 1} + S^{\alpha}_{4\times 1} = 1_{4\times 1} \\ D^{\beta}_{7\times 1} + S^{\beta}_{7\times 1} = 1_{7\times 1} \end{array}.$$

This is obviously not the case. As one can see in the double scales depicted in the top two panels of Fig. 2, the relationship between dominance and subdominance values for the same objects is not even monotonically decreasing. In accordance

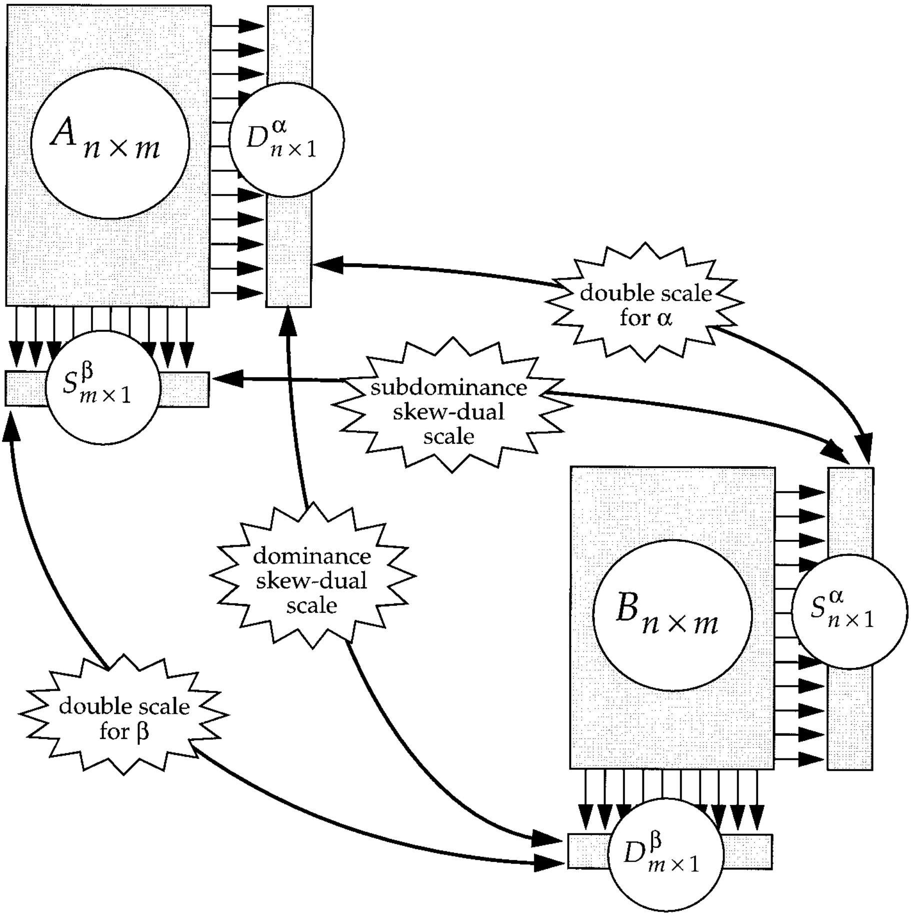

FIG. 1. Relationships between two matrices and four vectors of the double skew-dual scaling.

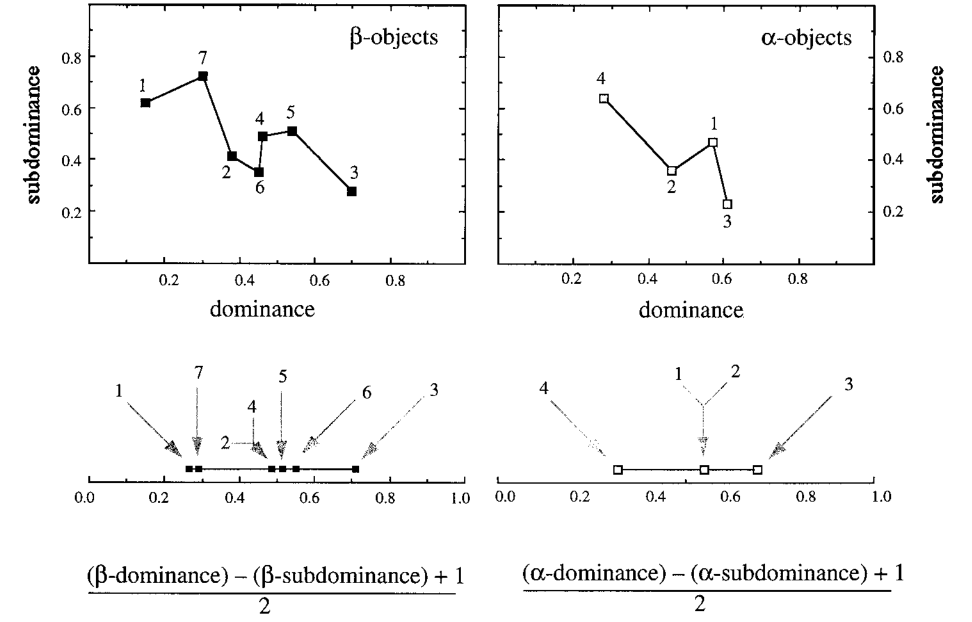

FIG. 2. Analysis of matrix $A_{4\times 7}$ of Section 1.

with Section 2, the only mathematically predictable relationship between the dominance and subdominance vectors in our example is

$$\sum D^{\alpha}_{4\times 1}\sum D^{\beta}_{7\times 1}=\sum S^{\alpha}_{4\times 1}\sum S^{\beta}_{7\times 1}.$$

This is unexpected and might even appear disappointing. I suggest, however, that the lack of an apparent relationship between the two skew-dual scales, $(D^{\alpha}_{4\times 1}, D^{\beta}_{7\times 1})$ and $(S^{\alpha}_{4\times 1}, S^{\beta}_{7\times 1})$, has a plausible interpretation. Intuitively, a high (low) dominance value for, say, a β-object indicates its tendency to dominate (fail to dominate) α-objects with high (low) dominance values of their own; analogously, a low (high) subdominance value for the β-object indicates its tendency to dominate (fail to dominate) α-objects with low (high) subdominance values of their own. There is no logical reason why a β-object cannot, say, both dominate very dominant α-objects and be dominated by very subdominant α-objects. Think about β-objects as keys with which one tries to open a series of locks, α-objects. Obviously, one can have a key that unlocks sufficient number of very difficult locks (that few other keys can unlock) while failing to unlock several very simple locks (that many other keys can unlock): this key would have simultaneously a high dominance value and a high subdominance value. Figure 3 shows that if "high" means exceeding 0.5 (in the range from 0 to 1), then this is what characterizes objects β_5 in our example.

The lack of an apparent relationship between the two skew-dual scales, $(D^{\alpha}_{4\times 1}, D^{\beta}_{7\times 1})$ and $(S^{\alpha}_{4\times 1}, S^{\beta}_{7\times 1})$, is also a desirable property of these scales, for two reasons. First, in purely descriptive terms, two numbers provide a better summary for entire rows or entire columns. If one so wishes, one can always compute a single measure for an object from its dominance and subdominance values, as shown in

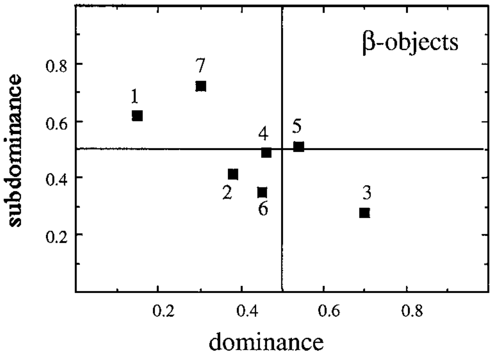

FIG. 3. Demonstration of the four combinations, (low–high) × (low–high), for dominance and subdominance values.

the bottom panels of Fig. 2: the algorithm used there is chosen to keep the composite value in the range and has no deeper justification. A second and more important reason stems from the fact that the double skew-dual scaling, not being a model itself, is designed to allow one to formulate falsifiable models using it as a quantitative language. That the measurement (computational) procedure itself does not relate the two skew-dual scales in any restrictive way is significant, for it allows one to impose such restrictions as falsifiable models. For instance, if α-objects are examinees and β-objects problems, one could hypothesize that both α-dominance and α-subdominance are determined by a unidimensional property called "ability" and that as this "ability" increases, α-dominance increases while α-subdominance decreases. This would restrict the class of empirically possible matrices, and $A_{4\times 7}$ then would have provided a falsifying counterexample if obtained experimentally.[2]

One can now generalize (1) and (2) to apply to any matrix $A_{n\times m}$ whose elements are confined to the interval $[0, 1]$:

$$\left[\begin{array}{l} A_{n\times m}\dfrac{D^{\beta}_{m\times 1}}{\sum D^{\beta}_{m\times 1}} = D^{\alpha}_{n\times 1} \\ B^{T}_{m\times n}\dfrac{D^{\alpha}_{n\times 1}}{\sum D^{\alpha}_{n\times 1}} = D^{\beta}_{m\times 1} \end{array}\right., \qquad \left[\begin{array}{l} B_{n\times m}\dfrac{S^{\beta}_{m\times 1}}{\sum S^{\beta}_{m\times 1}} = S^{\alpha}_{n\times 1} \\ A^{T}_{m\times n}\dfrac{S^{\alpha}_{n\times 1}}{\sum S^{\alpha}_{n\times 1}} = S^{\beta}_{m\times 1} \end{array}\right.. \tag{3}$$

Applying these formulas to our pairwise comparison example,

$A_{4\times 4}$	β_1	β_2	β_3	β_4
α_1	0.50	0.57	0.67	0.80
α_2	0.43	0.50	0.60	0.75
α_3	0.33	0.40	0.50	0.67
α_4	0.20	0.25	0.33	0.50

[2] A monotonically decreasing dependence (or some specific form thereof) between dominance and subdominance, say, for α, can even be taken to be a criterion of the unidimensionality of the dominance–subdominance characterization of α. It would be interesting to investigate the constraints on the dominance matrix that imply and are implied by thus understood unidimensionality. No theory of such constraints is known to me presently.

one can prove (see the next section) that they have one and only one quadruple of nonnegative vectors $[(D^{\alpha}_{4\times 1}, D^{\beta}_{4\times 1}), (S^{\alpha}_{4\times 1}, S^{\beta}_{4\times 1})]$ as their solution:

$A_{4\times 4}$	β_1	β_2	β_3	β_4	α-*dominance*
α_1	0.50	0.57	0.67	0.80	0.61
α_2	0.43	0.50	0.60	0.75	0.54
α_3	0.33	0.40	0.50	0.67	0.44
α_4	0.20	0.25	0.33	0.50	0.29
β-*subdominance*	0.34	0.40	0.49	0.65	

$B_{4\times 4}$	β_1	β_2	β_3	β_4	α-*subdominance*
α_1	0.50	0.43	0.33	0.20	0.34
α_2	0.57	0.50	0.40	0.25	0.40
α_3	0.67	0.60	0.50	0.33	0.49
α_4	0.80	0.75	0.67	0.50	0.65
β-*dominance*	0.61	0.54	0.44	0.29	

The results are graphically shown in Fig. 4. Since the matrix $A_{4\times 4}$ is skew-symmetrical with respect to its main diagonal, that is,

$$a_{ij} + a_{ji} = 1, \qquad i, j = 1, 2, 3, 4$$

(which indicates the absence of a time error), the double scales for the α-objects (stimuli presented first) and β-objects (stimuli presented second) coincide:

$$(D^{\alpha}_{4\times 1}, S^{\alpha}_{4\times 1}) = (D^{\beta}_{4\times 1}, S^{\beta}_{4\times 1}).$$

Figure 4 also shows that in this case the relationship between the dominance and subdominance values happens to be monotonically decreasing (see footnote 2).

The generalization of (3) to matrices $A_{n\times m}$ whose elements belong to finite intervals $[u, v]$ other than $[0, 1]$ is easy: all that is required is an appropriate redefinition of the complementary matrix $B_{n\times m}$ (whose elements will also belong to $[u, v]$). Observe that not any four vectors satisfying (3) are admissible as comprising the two skew-dual scales: in addition, all elements of these vectors must be interpretable as weighted means of the corresponding rows and columns of $A_{n\times m}$ and $B_{n\times m}$. In particular, all these elements should belong to $[u, v]$. According to the theory presented in Section 2, the two skew-dual scales satisfying (3), $(D^{\alpha}_{n\times 1}, D^{\beta}_{m\times 1})$ and $(S^{\alpha}_{n\times 1}, S^{\beta}_{m\times 1})$, always exist and are determined uniquely if $[u, v]$ contains only positive values. It turns out that for intervals $[u, v] = [0, v]$ (which includes $[0, 1]$ dealt with previously) two skew-dual scales satisfying (3) always exist, but in some special cases they may not be determined uniquely (this may happen only if the matrix products $A_{n\times m}B^{T}_{m\times n}$, $B^{T}_{n\times m}A_{m\times n}$, $B_{n\times m}A^{T}_{m\times n}$, $A^{T}_{n\times m}B_{m\times n}$ all belong to the class of so-called *reducible* matrices, discussed later). The situation is the worst when (3) is applied to matrices with both positive and negative elements (i.e., with $u < 0$, $v > 0$): neither uniqueness nor existence of skew-dual scales is guaranteed in

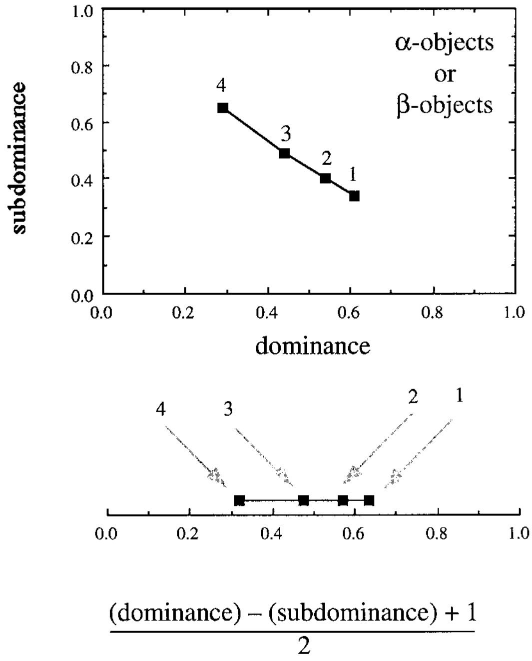

FIG. 4. Analysis of matrix $A_{4\times 4}$ of Section 1.

this case. Moreover, the concept of a weighted mean is not well defined in this situation to begin with: if weights can be both positive and negative, the "weighted means" can attain values beyond the range of the values being averaged and sometimes even be unrelated to these values completely (e.g., the "weighted mean" of 0 and 1 taken with respective weights -0.4 and 0.5 is 5, while the "weighted mean" of any set of numbers whose weights sum to zero is $\pm\infty$).

A way of dealing with these difficulties lies in observing that the main principles of (and the basic intuition behind) the skew-dual scaling do not depend on the special form of weighted averaging involved in (3). It turns out that one can guarantee the existence and uniqueness of the skew-dual scales for any given class of matrices by replacing, if necessary, the arithmetic *weighted means* computed in (3) by appropriately chosen alternative forms of *weighted averages.* The principle of skew-dual scaling (i–iv) can now be generalized to read:

(i*–ii*) α-dominance values are averages of the corresponding rows $A_{n\times m}$ weighted by β-dominance values, while β-dominance values are averages of the corresponding columns of $B_{n\times m}$ weighted by α-dominance values;

(iii*–iv*) α-subdominance values are averages of the corresponding rows of $B_{n\times m}$ weighted by β-subdominance values, while β-subdominance values are averages of the corresponding columns of $A_{n\times m}$ weighted by α-subdominance values.

From an algebraic point of view (systematically developed in Section 2), the construction of alternative weighted averages consists in replacing the operations of addition and multiplication involved in the formula for weighted means by two other operations, $\oplus$ and $\otimes$, with appropriately defined "addition-like" and "multiplication-like" properties, respectively. That is, the generalized weighted average $\bar{x}$ of x_1 and x_2 taken with respective weights c_1 and c_2 is computed as the value satisfying

$$\bar{x} \otimes (c_1 \oplus c_2) = (c_1 \otimes x_1) \oplus (c_2 \otimes x_2),$$

instead of the conventional

$$\bar{x} \times (c_1 + c_2) = (c_1 \times x_1) + (c_2 \times x_2).$$

As shown in Section 2, any suitable pair of operations $\oplus$ and $\otimes$ is associated with a continuous monotonic transformations ϕ in such a way that

$$\bar{x} \otimes (c_1 \oplus c_2) = (c \otimes x_1) \oplus (c \otimes x_2) \Leftrightarrow \phi(\bar{x}) = \frac{\phi(c_1)\, \phi(x_1) + \phi(c_2)\, \phi(x_2)}{\phi(c_1) + \phi(c_2)}.$$

The transformation ϕ translates the interval $[u, v]$ (to which all possible elements of $A_{n \times m}$ belong) onto an interval of nonnegative reals. Given this transformation, (3) can be generalized to the following formulas for the two skew-dual scales,

$$\left[\begin{array}{l} \phi(A_{n\times m}) \dfrac{\phi(D^{\beta}_{m\times 1})}{\sum \phi(D^{\beta}_{m\times 1})} = \phi(D^{\alpha}_{n\times 1}) \\ \phi(B^{T}_{m\times n}) \dfrac{\phi(D^{\alpha}_{n\times 1})}{\sum \phi(D^{\alpha}_{n\times 1})} = \phi(D^{\beta}_{m\times 1}) \end{array}\right., \qquad \left[\begin{array}{l} \phi(B_{n\times m}) \dfrac{\phi(S^{\beta}_{m\times 1})}{\sum \phi(S^{\beta}_{m\times 1})} = \phi(S^{\alpha}_{n\times 1}) \\ \phi(A^{T}_{m\times n}) \dfrac{\phi(S^{\alpha}_{n\times 1})}{\sum \phi(S^{\alpha}_{n\times 1})} = \phi(S^{\beta}_{m\times 1}) \end{array}\right., \qquad (4)$$

where all matrices and vectors are transformed elementwise, and $B_{n \times m}$ is defined from

$$\frac{\phi(A_{n\times m}) + \phi(B_{n\times m})}{\phi(u) + \phi(v)} = 1_{n\times m}.$$

As an example, consider our second pairwise comparison matrix,

$A_{3\times 3}$	β_1	β_2	β_3
α_1	0	2	3
α_2	-1	1	1
α_3	-2	0	2

for which $[u, v] = [-3, 3]$. The transformation

$$\phi(x) = \frac{x+3}{6}$$

translates this interval onto $[0, 1]$. Using this transformation and applying (4) to $A_{3\times 3}$, one can prove that the skew-dual scales $(D^{\alpha}_{n\times 1}, D^{\beta}_{m\times 1})$ and $(S^{\alpha}_{n\times 1}, S^{\beta}_{m\times 1})$ exist and are determined uniquely:

$A_{3\times 3}$	β_1	β_2	β_3	α-*dominance*
α_1	0	2	3	0.96
α_2	−1	1	1	−0.17
α_3	−2	0	2	−0.90
β-*subdominance*	−1.27	0.73	1.74	

,

$B_{3\times 3}$	β_1	β_2	β_3	α-*subdominance*
α_1	0	−2	−3	−2.12
α_2	1	−1	−1	−0.66
α_3	2	0	−2	−0.59
β-*dominance*	0.79	−1.21	−2.13	

.

An inspection of the skew-dual scales obtained would convince one that the values do appear to be reasonable averages of the corresponding rows and columns, and that they satisfy the principles (i*–iv*). One could, of course, use other transformations translating $[u, v] = [-3, 3]$ onto an interval of nonnegative reals: the resulting skew-dual scales would be different from the ones above, but they would be equally "reasonable." There is no *a priori*, universally applicable criterion for preferring one particular transformation to another. This observation leads one to the idea of using (4) in conjunction with appropriately chosen *classes of transformations* ϕ, rather than a single fixed transformation: then for any given matrix $A_{n\times m}$, one could select from this class a transformation that provides skew-dual scales that are *optimal* in some well defined sense. This idea is taken on in Section 3.

2. A SYSTEMATIC CONSTRUCTION OF DOUBLE SKEW-DUAL SCALING

Double skew-dual scaling involves computing generalized weighted averages, with respect to two operations, $\oplus$ and $\otimes$, that replace the conventional addition and multiplication in the formula for weighted means. The generalized weighted average is defined as a function ρ that translates vector $\{x_1, ..., x_k\}$, of an arbitrary dimension, into its average by means of a vector of weights $\{c_1, ..., c_k\}$ (and the two operations $\oplus$, $\otimes$).

Definition 1. Let $\oplus$ and $\otimes$ be two binary operations defined and closed on some interval I of reals, such that

(i) $\oplus$ is associative, continuous, and increasing in both arguments;

(ii) $\otimes$ is continuous, increasing in the second argument, and left- and right-distributed over $\oplus$, the distributivity meaning that

$$c \otimes (x_1 \oplus x_2) = (c \otimes x_1) \oplus (c \otimes x_2), \qquad (c_1 \oplus c_2) \otimes x = (c_1 \otimes x) \oplus (c_2 \otimes x).$$

(Somewhat anticipating, the interval I is supposed to contain the interval $[u, v]$ within which the entries of the analyzed dominance matrices attain their values. The purpose of introducing I is to extend $[u, v]$ so that the operations $\oplus$ and $\otimes$ performed on points of $[u, v]$ are algebraically closed.)

Let ρ be a function that, for any $k=1, 2, ...,$ and any pair of vectors $(\{x_1, ..., x_k\}, \{c_1, ..., c_k\})$ whose elements all belong to the interval I, maps this pair of vectors into a value of I:

$$\rho\colon \bigcup_{k=1}^{\infty} I^k \times I^k \to I.$$

This function is said to be *characterizing* its first argument, $\{x_1, ..., x_k\}$, by *means of* its second argument, $\{c_1, ..., c_k\}$ (and the operations $\oplus$ and $\otimes$) if there is a function

$$\lambda\colon \bigcup_{k=1}^{\infty} I^k \to I,$$

such that

$$\lambda(c_1, ..., c_k) \otimes \rho(\{x_1, ..., x_k\}, \{c_1, ..., c_k\}) = (c_1 \otimes x_1) \oplus \cdots \oplus (c_k \otimes x_k), \tag{5}$$

and if, in addition, ρ satisfies the "unanimity" property

$$\rho(\{x_1 = x, ..., x_k = x\}, \{c_1, ..., c_k\}) = x. \tag{6}$$

A conventional weighted mean, obviously, satisfies this definition:

$$\underbrace{(c_1 + \cdots + c_k)}_{\lambda(c_1, ..., c_k)} \times \underbrace{\frac{c_1 \times x_1 + \cdots + c_k \times x_k}{c_1 + \cdots + c_k}}_{\rho(\{x_1, ..., x_k\}, \{c_1, ..., c_k\})} = (c_1 \underset{\otimes}{\underset{\uparrow}{\times}} x_1) \underset{\oplus}{\underset{\uparrow}{+}} \cdots + (c_k \times x_k).$$

THEOREM 1. *A function $\rho(\{x_1, ..., x_k\}, \{c_1, ..., c_k\})$ characterizes $\{x_1, ..., x_k\}$ by means of $\{c_1, ..., c_k\}$ (and operations $\oplus$ and $\otimes$) if and only if there is a monotonic continuous transformation $\phi\colon I \to [q, \infty)$ (onto, $q \geqslant 0$), such that*

$$\phi(\rho) = \frac{\sum_{i=1}^{k} \phi(c_i)\, \phi(x_i)}{\sum_{i=1}^{k} \phi(c_i)}. \tag{7}$$

The transformation ϕ and the operation $\oplus$ and $\otimes$ are interrelated as

$$\left[\begin{array}{l} x \oplus y = \phi^{-1}[\phi(x) + \phi(y)] \\ x \otimes y = \phi^{-1}[p\phi(x)\, \phi(y)] \end{array}\right., \tag{8}$$

where p is a positive constant, and if $q \neq 0$, $p \geqslant q^{-1}$.

See the Appendix for the proof.

The characterizing functions ρ in the double skew-dual scaling are applied to rows and columns of dominance matrices, and this makes necessary the following.

DEFINITION 2. Let $X_{n\times m}$, $U_{m\times 1}$, $V_{n\times 1}$ be a matrix and two vectors whose elements all belong to I. Let ρ be a characterizing function on I satisfying Definition 1, and let the operations $\oplus$ and $\otimes$ in (5) be determined by a transformation ϕ, according to Theorem 1. The vector $V_{n\times 1}$ is said to ϕ-characterize $X_{n\times m}$ (row-wise) by means of $U_{m\times 1}$ if

$$v_i = \rho(\{x_{i1}, ..., x_{im}\}, \{u_1, ..., u_m\}), \qquad i = 1, ..., n.$$

I present this relationship symbolically as $V_{n\times 1}[[X_{n\times m}]]_\phi\, U_{m\times 1}$. The vector $U_{m\times 1}$ is said to ϕ-characterize $X_{n\times m}$ (column-wise) by means of $V_{n\times 1}$ if

$$u_j = \rho(\{x_{1j}, ..., x_{nj}\}, \{v_1, ..., v_n\}), \qquad j = 1, ..., m.$$

Symbolically, $U_{m\times 1}[[X_{n\times m}]]_\phi\, V_{n\times 1}$.

Theorem 1 allows one to reformulate this definition in a more explicit way:

$$\begin{aligned} V_{n\times 1}[[X_{n\times m}]]_\phi\, U_{m\times 1} &\Leftrightarrow \phi(X_{n\times m})\,\frac{\phi(U_{m\times 1})}{\sum \phi(U_{m\times 1})} = \phi(V_{n\times 1}) \\ U_{m\times 1}[[X_{n\times m}]]_\phi\, V_{n\times 1} &\Leftrightarrow \phi(X^T_{m\times n})\,\frac{\phi(V_{n\times 1})}{\sum \phi(V_{n\times 1})} = \phi(U_{m\times 1}). \end{aligned} \tag{9}$$

A matrix in the double skew-dual scaling is viewed as a *multidimensional variable* whose entries can assume all possible values selected from a specified set of reals X. The only restriction imposed on X is that it is bounded from above and from below, i.e., it lies within a finite interval

$$[u, v] = [\inf X, \sup X].$$

This includes cases ranging from the one where X is an open set, and the matrices cannot contain the values u, v precisely, to the one where (as in Boolean matrices) u, v are the only attainable values. The interval I considered previously is supposed to contain $[u, v]$, and its only purpose is to extend $[u, v]$ so that the operations $\oplus$ and $\otimes$ performed on elements of $X \subseteq [u, v] \subset I$ are algebraically closed. Since the monotonic transformation ϕ associated with $\oplus$ and $\otimes$ by Theorem 1 translates I onto some intervals $[q, \infty)$, $q \geqslant 0$, it translates $[u, v]$ onto some subinterval of $[q, \infty)$. One can now view $[q, \infty)$ as an extension of $\phi([u, v])$ that is necessary for algebraically closing the operations $x + y$ and pxy. It immediately follows from the properties of these two operations that, with an appropriate choice of p, the minimal $[q, \infty)$ that contains $\phi([u, v])$ is obtained by putting $q = \min \phi([u, v])$. The lemma below follows from these considerations immediately.

LEMMA. *Let $[u, v]$ be as above and given. Then the minimal domain for a transformation ϕ specified in Theorem 1 is either of the form $[u, v^*)$, $v < v^*$, if ϕ is* *increasing* (*in which case* $\phi(v^*) = \infty$), *or of the form* $[u^*, v]$, $u > u^*$, *if* ϕ *is decreasing* (*in which case* $\phi(u^*) = \infty$).

Hereafter, I assume that I is always given in its minimal version. The reason for this convention is that the class of ϕ-transformations satisfying Theorem 1 is the broadest when they are defined on the minimal versions of I. Thus, the transformation $\phi(x) = (x+3)/6$ used in Section 1 for the matrix $A_{3\times 3}$ satisfies Theorem 1 if I is defined as $[-3, \infty)$, but it does not satisfy Theorem 1 for any broader interval (and narrower intervals will not have $x + y$ closed on them).

DEFINITION 3. For a matrix $A_{n\times m}$, let interval $[u, v]$, transformation ϕ, and interval I be as above and given. Matrix $B_{n\times m}$ is said to be ϕ-*complementary* to $A_{n\times m}$ (or, simply, complementary) if

$$\phi(A_{n\times m}) + \phi(B_{n\times m}) = \omega 1_{n\times m}, \tag{10}$$

where

$$\omega = \phi(u) + \phi(v) \tag{11}$$

(a finite value) is referred to as the ϕ-*size* (or simply, size) of $A_{n\times m}$.

Obviously, the complementarity is a symmetrical relationship, and $B_{n\times m}$ is associated with the same interval $[u, v]$ and has the same size as $A_{n\times m}$.

Now all preparations are completed for defining the notion of a skew-dual scale.

DEFINITION 4. Let ϕ be fixed, and let $A_{n\times m}$ and $B_{n\times m}$ be mutually complementary. Vectors $U_{m\times 1}$ and $V_{n\times 1}$ are said to form a *skew-dual scale* if simultaneously $V_{n\times 1}[[A_{n\times m}]]_\phi\, U_{m\times 1}$ and $U_{m\times 1}[[B_{n\times m}]]_\phi\, V_{n\times 1}$, that is, if

(i) $V_{n\times 1}$ characterizes $A_{n\times m}$ (row-wise) by means of $U_{m\times 1}$,

(ii) $U_{m\times 1}$ characterizes $B_{n\times m}$ (column-wise) by means of $V_{n\times 1}$.

Since the complementarity is a symmetrical relationship, it immediately follows from Definition 4 that one can define two skew-dual scales for one and the same pair of complementary matrices: one for which simultaneously $V_{n\times 1}[[A_{n\times m}]]_\phi$ $U_{m\times 1}$ and $U_{m\times 1}[[B_{n\times m}]]_\phi\, V_{n\times 1}$, and another for which simultaneously $V_{n\times 1}[[B_{n\times m}]]_\phi\, U_{m\times 1}$ and $U_{m\times 1}[[A_{n\times m}]]_\phi\, V_{n\times 1}$.

To distinguish between these two skew-dual scales, the following convention is adopted. If $A_{n\times m}$ is interpreted as depicting the degree of dominance of row objects, $\{\alpha_1, ..., \alpha_n\}$, over column objects, $\{\beta_1, ..., \beta_m\}$, then the vectors forming the first skew-dual scale are called *dominance vectors* (for α-objects and β-objects) and are denoted as $D^{\alpha}_{n\times 1}$ and $D^{\beta}_{m\times 1}$, respectively (with implicit reference to ϕ). These vectors are defined by the simultaneous relationships

$$\left[\begin{array}{l} D^{\alpha}_{n\times 1}[[A_{n\times m}]]_\phi\, D^{\beta}_{m\times 1} \\ D^{\beta}_{m\times 1}[[B_{n\times m}]]_\phi\, D^{\alpha}_{n\times 1} \end{array}\right.$$

The vectors forming the second skew-dual scale are called *subdominance vectors* (for α-objects and β-objects), and are denoted as $S^{\alpha}_{n\times 1}$ and $S^{\beta}_{m\times 1}$, respectively (with implicit reference to ϕ). They are defined by the simultaneous relationships

$$\left[\begin{array}{l} S^{\alpha}_{n\times 1}[[B_{n\times m}]]_{\phi}\, S^{\beta}_{m\times 1} \\ S^{\beta}_{m\times 1}[[A_{n\times m}]]_{\phi}\, S^{\alpha}_{n\times 1} \end{array}\right.$$

Theorem 1 and formula (9) allow one to present the two skew-dual relationships *in extenso*:

$$\left[\begin{array}{l} D^{\alpha}_{n\times 1}[[A_{n\times m}]]_{\phi}\, D^{\beta}_{m\times 1} \\ D^{\beta}_{m\times 1}[[B_{n\times m}]]_{\phi}\, D^{\alpha}_{n\times 1} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \phi(A_{n\times m})\dfrac{\phi(D^{\beta}_{m\times 1})}{\sum \phi(D^{\beta}_{m\times 1})} = \phi(D^{\alpha}_{n\times 1}) \\ \phi(B^{T}_{m\times n})\dfrac{\phi(D^{\alpha}_{n\times 1})}{\sum \phi(D^{\alpha}_{n\times 1})} = \phi(D^{\beta}_{m\times 1}) \end{array}\right. \tag{12}$$

for the dominance vectors, and

$$\left[\begin{array}{l} S^{\alpha}_{n\times 1}[[B_{n\times m}]]_{\phi}\, S^{\beta}_{m\times 1} \\ S^{\beta}_{m\times 1}[[A_{n\times m}]]_{\phi}\, S^{\alpha}_{n\times 1} \end{array}\right. \Leftrightarrow \left[\begin{array}{l} \phi(B_{n\times m})\dfrac{\phi(S^{\beta}_{m\times 1})}{\sum \phi(S^{\beta}_{m\times 1})} = \phi(S^{\alpha}_{n\times 1}) \\ \phi(A^{T}_{m\times n})\dfrac{\phi(S^{\alpha}_{n\times 1})}{\sum \phi(S^{\alpha}_{n\times 1})} = \phi(S^{\beta}_{m\times 1}) \end{array}\right. \tag{13}$$

for the subdominance vectors.

These definitions, of course, do not tell us anything about the existence, uniqueness, or relationships between the two skew-dual scales, and these are the issues elucidated by the two theorems presented next.

The reader is reminded of the meaning of two terms from matrix analysis (see Horn & Johnson, 1985) used in the first theorem. An eigenvalue of a square matrix is said to be *simple* if all associated eigenvectors are multiples of each other. A matrix, or a vector, is *positive* (*nonnegative*) if all of its elements are positive (nonnegative). It is always assumed below that a nonnegative matrix (vector) has some non-zero elements.

THEOREM 2. *Let $\phi: I \to \mathrm{Re}^{+}$ be fixed, and let $A_{n\times m}$ and $B_{n\times m}$ be mutually complementary, with elements in some $[u, v] \subset I$.*

(a) *If $D^{\alpha}_{n\times 1}$, $D^{\beta}_{m\times 1}$, $S^{\alpha}_{n\times 1}$, $S^{\beta}_{m\times 1}$ exist, then all elements of these vectors belong to $[u, v]$.*

(b1) *If dominance vectors $D^{\alpha}_{n\times 1}$, $D^{\beta}_{m\times 1}$ exist, then $\phi(D^{\alpha}_{n\times 1})$ and $\phi(D^{\beta}_{m\times 1})$ are eigenvectors of $\phi(A_{n\times m})\,\phi(B^{T}_{m\times n})$ and $\phi(B^{T}_{m\times n})\,\phi(A_{n\times m})$, respectively, associated with one and the same positive eigenvalue $\delta = \sum \phi(D^{\alpha}_{n\times 1}) \sum \phi(D^{\beta}_{m\times 1})$.*

(b2) *If subdominance vectors $S^{\alpha}_{n\times 1}$, $S^{\beta}_{m\times 1}$ exist, then $\phi(S^{\alpha}_{n\times 1})$ and $\phi(S^{\beta}_{m\times 1})$ are eigenvectors of $\phi(B_{n\times m})\,\phi(A^{T}_{m\times n})$ and $\phi(A^{T}_{m\times n})\,\phi(B_{n\times m})$, respectively, associated with one and the same positive eigenvalue $\sigma = \sum \phi(S^{\alpha}_{n\times 1}) \sum \phi(S^{\beta}_{m\times 1})$.*

(c1) *If the matrices $\phi(A_{n\times m})\,\phi(B^{T}_{m\times n})$ and $\phi(B^{T}_{m\times n})\,\phi(A_{n\times m})$ have nonnegative eigenvectors associated with one and the same simple eigenvalue δ, then dominance* *vectors* $D^{\alpha}_{n\times 1}$, $D^{\beta}_{m\times 1}$ *exist; if, in addition, at least one of the matrices* $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$ *and* $\phi(B^T_{m\times n})\,\phi(A_{n\times m})$ *has no nonnegative eigenvectors associated with positive eigenvalues other than* δ, *then* $D^{\alpha}_{n\times 1}$, $D^{\beta}_{m\times 1}$ *are determined uniquely.*

(c2) *If the matrices* $\phi(B_{n\times m})\,\phi(A^T_{m\times n})$ *and* $\phi(A^T_{m\times n})\,\phi(B_{n\times m})$ *have nonnegative eigenvectors corresponding to one and the same simple eigenvalue* σ, *then subdominance vectors* $S^{\alpha}_{n\times 1}$, $S^{\beta}_{m\times 1}$ *exist; if, in addition, at least one of the matrices* $\phi(B_{n\times m})\,\phi(A^T_{m\times n})$ *and* $\phi(A^T_{m\times n})\,\phi(B_{n\times m})$ *has no nonnegative eigenvectors associated with positive eigenvalues other than* σ, *then* $S^{\alpha}_{n\times 1}$, $S^{\beta}_{m\times 1}$ *are determined uniquely.*

See the Appendix for the proof.

The next theorem utilizes the following several facts and definitions from matrix analysis. Matrices $X_{n\times n}$ and $X^T_{n\times n}$ have the same eigenvalues (and, consequently, the same set of simple eigenvalues). Matrix products $X_{n\times m}Y_{m\times n}$ and $Y_{m\times n}X_{n\times m}$ have the same set of non-zero eigenvalues (and, consequently, the same set of simple eigenvalues) (Horn & Johnson, 1985; p. 53). It follows that the sets of non-zero eigenvalues (as well as the sets of simple eigenvalues) coincide for the four matrix products of the type dealt with below, $X_{n\times m}Y_{m\times n}$, $Y_{m\times n}X_{n\times m}$, $Y^T_{m\times n}X^T_{n\times m}$, $X^T_{n\times m}Y^T_{m\times n}$ ("the rule of four products").

A nonnegative matrix $X_{n\times n}$ possesses a nonnegative eigenvalue, called the *Perron root* of the matrix, whose modulus is not exceeded by that of any other eigenvalue of the matrix. The Perron root is associated with at least one nonnegative eigenvector (Horn & Johnson, 1985; p. 503) ("the rule of Perron roots").

A nonnegative matrix $X_{n\times n}$ is said to be *reducible* if, by some permutation of its rows and columns, one can form a rectangle of zeros whose dimensions add up to n:

$$X_{n\times n}=\begin{bmatrix} \cdots & \cdots \\ 0_{r\times(n-r)} & \cdots \end{bmatrix}, \qquad 1\leqslant r\leqslant n.$$

Otherwise the matrix is *irreducible*, and then it has the following properties: its Perron root is positive and simple, the associated eigenvectors are all positive (and multiples of each other), and the matrix has no other nonnegative eigenvectors (Seneta, 1973; pp. 20–21) ("the Perron–Frobenius rule").

Obviously, all positive square matrices are irreducible and the inverse of an irreducible matrix is irreducible. However, if $X_{n\times m}Y_{m\times n}$ is irreducible, $Y_{m\times n}X_{n\times m}$ may be reducible.

THEOREM 3. *Let* $\phi = I \to \mathrm{Re}^{+}$ *be fixed, and let* $A_{n\times m}$ *and* $B_{n\times m}$ *be mutually complementary. If at least one of the matrices* $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$, $\phi(B^T_{m\times n})\,\phi(A_{n\times m})$, $\phi(B_{n\times m})\,\phi(A^T_{m\times n})$, $\phi(A^T_{m\times n})\,\phi(B_{n\times m})$ *is irreducible, then*

(a) *vectors* $D^{\alpha}_{n\times 1}$, $D^{\beta}_{m\times 1}$, $S^{\alpha}_{n\times 1}$, $S^{\beta}_{m\times 1}$ *exist and are determined uniquely;*

(b) *vectors* $\phi(D^{\alpha}_{n\times 1})$, $\phi(D^{\beta}_{m\times 1})$, $\phi(S^{\alpha}_{n\times 1})$, $\phi(S^{\beta}_{m\times 1})$ *are eigenvectors of* $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$, $\phi(B^T_{m\times n})\,\phi(A_{n\times m})$, $\phi(B_{n\times m})\,\phi(A^T_{m\times n})$, $\phi(A^T_{m\times n})\,\phi(B_{n\times m})$, *respectively, associated with their common Perron root* π;

(c) $\sum \phi(D^{\alpha}_{n\times 1}) \sum \phi(D^{\beta}_{m\times 1}) = \sum \phi(S^{\alpha}_{n\times 1}) \sum \phi(S^{\beta}_{m\times 1}) = \pi.$

See the Appendix for the proof.

I summarize now the information provided by theorems 2 and 3 in the form of an *algorithm* for computing skew-dual scales, given $\phi: I \to \mathrm{Re}^+$ and a dominance matrix $A_{n\times m}$ with elements in some intervals $[u, v] \subset I$

(1) *Compute* $\phi(A_{n\times m})$ *and determine* $\omega = \phi(u) + \phi(v)$.

(2) *Compute* $\phi(B_{n\times m}) = \omega 1_{n\times m} - \phi(A_{n\times m})$.

(3) *Compute* $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$, $\phi(B^T_{m\times n})\,\phi(A_{n\times m})$, $\phi(B_{n\times m})\,\phi(A^T_{m\times n})$, *and* $\phi(A^T_{m\times n})\,\phi(B_{n\times m})$.

(4) (This step can be omitted if $\phi(A_{n\times m})$ is strictly positive.) *Verify that at least one of these matrix products, say,* $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$, *is irreducible.* (*Then* $\phi(B_{n\times m})\,\phi(A^T_{m\times n})$ *is irreducible too.*)

(5) *Compute the Perron root* π *of* $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$. (*This is the Perron root for all four matrix products.*)

(6) *Compute the Perron vectors for* $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$, $\phi(B^T_{m\times n})\,\phi(A_{n\times m})$, $\phi(B_{n\times m})\,\phi(A^T_{m\times n})$, $\phi(A^T_{m\times n})\,\phi(B_{n\times m})$, *that is, the eigenvectors* $V^\alpha_{n\times 1}$, $V^\beta_{m\times 1}$, $W^\alpha_{n\times 1}$, $W^\beta_{m\times 1}$, *of these matrices associated with the Perron root* π *and normalized as* $\sum V^\alpha_{n\times 1} = \sum V^\beta_{m\times 1} = \sum W^\alpha_{n\times 1} = \sum W^\beta_{m\times 1} = 1$.

(7) *Compute*

$$\left[\begin{array}{l} \phi(D^\alpha_{n\times 1}) = \phi(A_{n\times m})\ V^\beta_{m\times 1} \\ \phi(D^\beta_{m\times 1}) = \phi(B^T_{m\times n})\ V^\alpha_{n\times 1} \end{array}\right.$$

(8) *Compute*

$$\left[\begin{array}{l} \phi(S^\alpha_{n\times 1}) = \phi(B_{n\times m})\ W^\beta_{m\times 1} \\ \phi(S^\beta_{m\times 1}) = \phi(A^T_{m\times n})\ W^\alpha_{n\times 1} \end{array}\right.$$

(9) (Optional) *Apply the* ϕ^{-1}*-transformation to* $\phi(D^\alpha_{n\times 1})$ *and* $\phi(D^\beta_{m\times 1})$ *to obtain the dominance skew-dual scale,* $(D^\alpha_{n\times 1}, D^\beta_{m\times 1})$.

(10) (Optional) *Apply the* ϕ^{-1}*-transformation to* $\phi(S^\alpha_{n\times 1})$ *and* $\phi(S^\beta_{m\times 1})$ *to obtain the subdominance skew-dual scale,* $(S^\alpha_{n\times 1}, S^\beta_{m\times 1})$.

The last two steps of the algorithm are labeled optional, because insofar as one does not forget the identity of the ϕ-transformation used, one can describe the objects being scaled, $\{\alpha_1, ..., \alpha_n\}$ and $\{\beta_1, ..., \beta_m\}$, in terms of the ϕ-transformed dominance and subdominance vectors directly.

A way of dealing with situations where all four matrix products, $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$, $\phi(B^T_{m\times n})\,\phi(A_{n\times m})$, $\phi(B_{n\times m})\,\phi(A^T_{m\times n})$, and $\phi(A^T_{m\times n})\,\phi(B_{n\times m})$, are reducible is discussed in the next section.

3. VARIABLE ϕ-TRANSFORMATIONS

The theory presented so far may appear unnecessarily complicated in the following respect. Since the skew-dual relationships represented in (12)–(13),

$$\begin{bmatrix} D^{\alpha}_{n\times 1}[[A_{n\times m}]]_{\phi}\, D^{\beta}_{m\times 1} \\ D^{\beta}_{m\times 1}[[B_{n\times m}]]_{\phi}\, D^{\alpha}_{n\times 1} \end{bmatrix}, \qquad \begin{bmatrix} S^{\alpha}_{n\times 1}[[B_{n\times m}]]_{\phi}\, S^{\beta}_{m\times 1} \\ S^{\beta}_{m\times 1}[[A_{n\times m}]]_{\phi}\, S^{\alpha}_{n\times 1} \end{bmatrix},$$

are equivalent to the relationships

$$\begin{bmatrix} \phi(D^{\alpha}_{n\times 1})[[\phi(A_{n\times m})]]\, \phi(D^{\beta}_{m\times 1}) \\ \phi(D^{\beta}_{m\times 1})[[\phi(B_{n\times m})]]\, \phi(D^{\alpha}_{n\times 1}) \end{bmatrix}, \qquad \begin{bmatrix} \phi(S^{\alpha}_{n\times 1})[[\phi(B_{n\times m})]]\, \phi(S^{\beta}_{m\times 1}) \\ \phi(S^{\beta}_{m\times 1})[[\phi(A_{n\times m})]]\, \phi(S^{\alpha}_{n\times 1}) \end{bmatrix}$$

(where the absence of a subscript at $[[\cdots]]$ indicates no transformation), why would one not define the original dominance matrix as $\phi(A_{n\times m})$ at the outset? Having done so, and having named this dominance matrix $A_{n\times m}$, one would be able to define the corresponding $B_{n\times m}$ (numerically coinciding with the $\phi(B_{n\times m})$ above) and deal only with the simple relationships computed according to (3),

$$\begin{bmatrix} D^{\alpha}_{n\times 1}[[A_{n\times m}]]\, D^{\beta}_{m\times 1} \\ D^{\beta}_{m\times 1}[[B_{n\times m}]]\, D^{\alpha}_{n\times 1} \end{bmatrix}, \qquad \begin{bmatrix} S^{\alpha}_{n\times 1}[[B_{n\times m}]]\, S^{\beta}_{m\times 1} \\ S^{\beta}_{m\times 1}[[A_{n\times m}]]\, S^{\alpha}_{n\times 1} \end{bmatrix}.$$

The vectors $D^{\alpha}_{n\times 1}$, $D^{\beta}_{m\times 1}$, $S^{\alpha}_{n\times 1}$, $S^{\beta}_{m\times 1}$ then would be treated as dominance and subdominance vectors *per se*, rather than ϕ-transformations thereof. In other words, by appropriately defining the original dominance matrix, one could avoid, as it seems, the necessity of dealing with any ϕ-transformations and any formulas other than (3).

This argument is certainly valid in all cases when the original entries of $A_{n\times m}$ are nonnumerical categories arbitrarily translated into numerical codes. This could be the case, for example, with the pairwise comparison matrix $A_{3\times 3}$ considered in Section 1,

$A_{3\times 3}$	β_1	β_2	β_3
α_1	0	2	3
α_2	-1	1	1
α_3	-2	0	2

.

The numerical values $\{-3, -2, -1, 0, 1, 2, 3\}$ could very well have been arbitrarily assigned to seven verbal judgments {"strong preference for the second stimulus", ..., "strong preference for the first stimulus"}, in which case one could instead have used some nonnegative numerical labels, say, $\{0, 1/6, 2/6, 3/6, 4/6, 5/6, 1\}$, and subject the matrix to (3) directly, with no transformations involved.

Arbitrary numerical labeling, however, is confined to cases with a finite number of distinct categories. No continuous or countably infinite set of numerical values can be obtained without having a measurement or counting procedure producing the numerical value of an object as its outcome. In such a case one cannot obtain alternative numerical assignments other than through numerical transformations of the original measurements, which means, in our present terminology, that the ϕ-transformations have to be performed explicitly. This can also be true when the number of possible values is finite, notably, when the dependent variable is a count or probability. In all such cases $\phi(A_{n\times m})$ is distinct from the original matrix $A_{n\times m}$, which makes the general formulas (12)–(13) necessary and the opening argument of this section purely terminological.[3]

There is an even more compelling argument in favor of the generality of (12)–(13). Since the choice of a ϕ-transformation for a given dominance matrix is never unique, it is both reasonable and desirable, instead of making this choice *a priori*, to try on this matrix a broad class of distinct ϕ-transformations in order to choose one that produces *optimal* skew-dual scales, in some predefined sense. I call this approach the *double skew-dual scaling with variable ϕ-transformation*, and it is the focus of the present section. Obviously, this form of scaling precludes the possibility of identifying $\phi(A_{n\times m})$ with the original dominance matrix, because $\phi(A_{n\times m})$ is no longer fixed. Observe that if $A_{n\times m}$ is viewed as a multidimensional variable, with its entries attaining all possible combinations of values from some set X, then, under the variable transformation approach, different realizations (values) of this matrix may require different ϕ-transformations to produce optimal skew-dual scales, for any given criterion of optimality. Observe also that for a fixed $A_{n\times m}$, its complement $B_{n\times m}$ will generally vary with ϕ.

In order for the variable transformation approach to work, the class of ϕ-transformations applied to matrix $A_{n\times m}$ should be appropriately parametrized, that is, presented as a single transformation with free parameters,

$$\phi(x) = f(x; \theta_1, ..., \theta_R).$$

For a fixed $A_{n\times m}$, the ϕ-transformed dominance and subdominance vectors then can be viewed as vectorial functions of these parameters,

$$\left[\begin{array}{l} \phi(D^{\alpha}_{n\times 1}) = X^{\alpha}_{n\times 1}(\theta_1, ..., \theta_R) \\ \phi(D^{\beta}_{m\times 1}) = X^{\beta}_{m\times 1}(\theta_1, ..., \theta_R) \\ \phi(S^{\alpha}_{n\times 1}) = Y^{\alpha}_{n\times 1}(\theta_1, ..., \theta_R) \\ \phi(S^{\beta}_{m\times 1}) = Y^{\beta}_{m\times 1}(\theta_1, ..., \theta_R) \end{array}\right., \tag{14}$$

and the problem can now be presented as one of maximizing a certain functional defined on these vectorial functions,

$$V(X^{\alpha}_{n\times 1}, X^{\beta}_{m\times 1}, Y^{\alpha}_{n\times 1}, Y^{\beta}_{m\times 1}),$$

generally subject to certain constraints imposed on parameters $\{\theta_1, ..., \theta_R\}$.

There is always a degree of arbitrariness involved in choosing an optimality criterion, but this is more of a flexibility than impediment. A variety of reasonable desiderata or normative relations can be readily proposed and the optimality formulated in terms of maximizing the conformity of the vectors $X^{\alpha}_{n\times 1}$, $X^{\beta}_{m\times 1}$, $Y^{\alpha}_{n\times 1}$, $Y^{\beta}_{m\times 1}$ to these desiderata or normative relations. One such normative relation, for example, can be the unidimensionality of the dominance–subdominance characterization mentioned in Section 1 (see footnote 2). With this normative relation in

[3] The language and content of this paragraph are based on a general approach to the theory of measurement (Dzhafarov, 1995) that differs from the more familiar representational approach. A deeper discussion of measurement issues is, however, outside the scope of this paper.

mind one should seek the transformation $\phi(x) = f(x; \theta_1, ..., \theta_R)$ that minimizes (i.e., brings as close as possible to -1) an appropriately chosen correlation coefficient between $X^{\alpha}_{n\times 1}$ and $Y^{\alpha}_{n\times 1}$ and/or between $X^{\beta}_{m\times 1}$ and $Y^{\beta}_{m\times 1}$. Another reasonable approach can be to choose a certain metric on the space of columns and the space of rows (so that one can compute a distance between any two columns or any two rows) and to seek the transformation $\phi(x) = f(x; \theta_1, ..., \theta_R)$ that maximizes the conformity of the inter-element distances within the vectors $X^{\alpha}_{n\times 1}$, $X^{\beta}_{m\times 1}$, $Y^{\alpha}_{n\times 1}$, $Y^{\beta}_{m\times 1}$ to a certain monotonic function (say, linear) of the corresponding inter-column and inter-row distances within the matrices $\phi(A_{n\times m})$ and $\phi(B_{n\times m})$.

In the examples given below, however, yet another optimization procedure is used, based on the desideratum that the elements within each of the vectors $X^{\alpha}_{n\times 1}$, $X^{\beta}_{m\times 1}$, $Y^{\alpha}_{n\times 1}$, $Y^{\beta}_{m\times 1}$ be as distinct as possible. This approach can be realized by maximizing across all possible transformations $\phi(x) = f(x; \theta_1, ..., \theta_R)$ such functionals as

$$\min\{\mathrm{Var}[X^{\alpha}_{n\times 1}], \mathrm{Var}[X^{\beta}_{m\times 1}], \mathrm{Var}[Y^{\alpha}_{n\times 1}], \mathrm{Var}[Y^{\beta}_{m\times 1})]\}$$

or

$$\mathrm{Var}[X^{\alpha}_{n\times 1}]\,\mathrm{Var}[X^{\beta}_{m\times 1}]\,\mathrm{Var}[Y^{\alpha}_{n\times 1}]\,\mathrm{Var}[Y^{\beta}_{m\times 1}],$$

where Var stands for variance (but could also be another measure of dispersion). It is obvious from (12)–(13) that a multiplication of $\phi(A_{n\times m})$ by a positive constant leads to the multiplication of the dominance and subdominance vectors by the same constant, thereby increasing their variance. Therefore, to guarantee that maxima of such functionals as above exist, one should constrain the free parameters $\{\theta_1, ..., \theta_R\}$ so that $\phi(A_{n\times m})$ preserves some measure of its size. One can, for example, use the technical meaning of matrix size defined in Section 2, (10)–(11), and to seek maxima of the variance-based functionals above under the constraint $\omega = const$.

A degree of arbitrariness, although substantially smaller than when choosing a single ϕ-transformation, is also involved in the choice of a finitely parametrizable class of such transformations. If the usual mixture of intuitive reasonableness and mathematical simplicity considerations do not provide satisfactory guidance in this choice, one can make use of the following simple but remarkable fact: although the class of all $\phi: I \to \mathrm{Re}^+$ is too broad to allow for finite parametrization, the latter can always be achieved for the *specializations of all possible transformations on the entries of a matrix* $A_{n\times m}$. Indeed, the matrix contains $R \leqslant nm$ distinct entries, say, $a_1 < \cdots < a_R$, and one can use them as free parameters $\{\theta_1, ..., \theta_R\}$ in (14), or map them on these parameters by any one-to-one R-dimensional transformation

$$\theta_i = \theta_i(a_1, ..., a_R), \qquad i = 1, ..., R.$$

With reasonable constraints imposed on $\{a_1, ..., a_R\}$, an optimal solution can exist and be computable, across all possible ϕ-transformations. The constraints should include, obviously, the requirement that the ordering $a_1 < \cdots < a_R$ be preserved (for increasing ϕ) or reversed (for decreasing ϕ), combined with the size preservation requirement mentioned earlier. Once the optimal values of $\{a_1, ..., a_R\}$ are found, however, there will be an infinity of functions $\phi: I \to \mathrm{Re}^+$ whose specializations on the corresponding entries of $A_{n\times m}$ equal these optimal values.

Below I use Boolean matrices to construct a specific procedure of double skew-dual scaling with variable ϕ-transformation: this procedure provides a simple illustration for all the general considerations above, and it also provides a technique for dealing with the situations involving reducible matrices (see step 4 of the algorithm at the end of Section 2).

Consider again the matrix $A_{4\times 7}$ used as an example in Section 1:

$A_{4\times 7}$	β_1	β_2	β_3	β_4	β_5	β_6	β_7
α_1	1	1	1	1	0	0	0
α_2	1	0	0	1	0	1	1
α_3	1	1	0	0	1	1	1
α_4	0	0	0	0	1	0	1

.

Any monotonic transformation $\phi: I \to \mathrm{Re}^+$ translates this matrix into

$\phi(A_{4\times 7})$	β_1	β_2	β_3	β_4	β_5	β_6	β_7
α_1	b	b	b	b	a	a	a
α_2	b	a	a	b	a	b	b
α_3	b	b	a	a	b	b	b
α_4	a	a	a	a	b	a	b

,

with $a = \phi(0) \geqslant 0$, $b = \phi(1) \geqslant 0$, and $\omega = a + b$. Since $\omega = 1$ for the original matrix (corresponding to $\phi(x) = x$), the size-preservation constraint should be $\omega = 1$, and we get

$\phi(A_{4\times 7})$	β_1	β_2	β_3	β_4	β_5	β_6	β_7
α_1	$(1-\varepsilon)$	$(1-\varepsilon)$	$(1-\varepsilon)$	$(1-\varepsilon)$	ε	ε	ε
α_2	$(1-\varepsilon)$	ε	ε	$(1-\varepsilon)$	ε	$(1-\varepsilon)$	$(1-\varepsilon)$
α_3	$(1-\varepsilon)$	$(1-\varepsilon)$	ε	ε	$(1-\varepsilon)$	$(1-\varepsilon)$	$(1-\varepsilon)$
α_4	ε	ε	ε	ε	$(1-\varepsilon)$	ε	$(1-\varepsilon)$

,

where $0 \leqslant \varepsilon \leqslant 1$. The complementary matrix $\phi(B_{4\times 7}) = 1_{4\times 7} - \phi(A_{4\times 7})$ is obtained by exchanging the two values for each other. By symmetry considerations, ε can be considered in the interval $[0, 1/2]$ only. Using the second of the two variance-based functionals mentioned earlier, the problem now can be formulated as one of finding

$$\max_{0 \leqslant \varepsilon < 1/2} \mathrm{Var}[X^{\alpha}_{4\times 1}(\varepsilon)]\ \mathrm{Var}[X^{\beta}_{7\times 1}(\varepsilon)]\ \mathrm{Var}[Y^{\alpha}_{4\times 1}(\varepsilon)]\ \mathrm{Var}[Y^{\beta}_{7\times 1}(\varepsilon)],$$

where the vectors are $\phi(D^{\alpha}_{4\times 1})$, $\phi(D^{\beta}_{7\times 1})$, $\phi(S^{\alpha}_{4\times 1})$, $\phi(S^{\beta}_{7\times 1})$ computed from the matrix above according to (12)–(13). The maximum sought must exist because the functional being maximized is a continuous function of ε attaining a finite value at $\varepsilon = 0$ and zero at $\varepsilon = 1/2$. This problem can be solved quite easily by, for example, straightforward scanning of the interval $0 \leqslant \varepsilon < 1/2$ with sufficiently small steps. In the case of $A_{4\times 7}$ the optimal solution turns out to be at $\varepsilon = 0$, that is, the skew-dual scales obtained in Section 1 (see Fig. 2) are optimal with respect to the set of all possible transformations. (This outcome may be a consequence of the following conjectured property: for Boolean matrices, whenever all four matrix products of Theorem 3 are irreducible at $\varepsilon = 0$, the product of variances is a decreasing function between $\varepsilon = 0$ and $\varepsilon = 1/2$.)

Consider now another matrix,

$A_{4\times 5}$	β_1	β_2	β_3	β_4	β_5
α_1	1	0	1	1	0
α_2	1	1	1	1	0
α_3	1	1	0	0	0
α_4	1	0	0	0	0

,

It can be verified that all four matrix products $A_{4\times 5}B^T_{5\times 4}$, $B^T_{5\times 4}A_{4\times 5}$, $B_{4\times 5}A^T_{5\times 4}$, $A^T_{5\times 4}B_{4\times 5}$ are now reducible, but that they nevertheless satisfy the requirements of Theorem 2(c1)–(c2). Because of the latter, the two skew-dual scales for $A_{4\times 5}$ exist and are determined uniquely:

$A_{4\times 5}$	β_1	β_2	β_3	β_4	β_5	$D^{\alpha}_{4\times 1}$
α_1	1	0	1	1	0	.24
α_2	1	1	1	1	0	.41
α_3	1	1	0	0	0	.17
α_4	1	0	0	0	0	0
$S^{\beta}_{5\times 1}$	1	.29	.21	.21	.21	

,

$B_{4\times 5}$	β_1	β_2	β_3	β_4	β_5	$S^{\alpha}_{4\times 1}$
α_1	0	1	0	0	1	.17
α_2	0	0	0	0	1	0
α_3	0	0	1	1	1	.24
α_4	0	1	1	1	1	.41
$D^{\beta}_{5\times 1}$	0	.29	.21	.21	1	

.

One can notice, however, that these solutions, although formally acceptable, have the following counter-intuitive property: a dominance or subdominance value of an object may equal zero even if some of the entries in the corresponding row are non-zero (see the dominance value for α_4 and the subdominance value for α_2).

Turning now to the problem of finding the optimal solution for $A_{4\times 4}$, by using the same procedure as before, one arrives at the conclusion that the maximum of the variance product is achieved at about $\varepsilon = 0.01$, the optimal ϕ-transformed dominance and subdominance vectors being

$\phi(A_{4\times 5})$	β_1	β_2	β_3	β_4	β_5	$\phi(D^{\alpha}_{4\times 1})$
α_1	.99	.01	.99	.99	.01	.27
α_2	.99	.99	.99	.99	.01	0.44
α_3	.99	.99	.01	.01	.01	.19
α_4	.99	.01	.01	.01	.01	.02
$\phi(S^{\beta}_{5\times 1})$	.99	.31	.23	.23	.01	

,

$\phi(B_{4\times 5})$	β_1	β_2	β_3	β_4	β_5	$\phi(S^{\alpha}_{4\times 1})$
α_1	.01	.99	.01	.01	.99	0.19
α_2	.01	.01	.01	.01	.99	.02
α_3	.01	.01	.99	.99	.99	.27
α_4	.01	.99	.99	.99	.99	.44
$\phi(D^{\beta}_{5\times 1})$	.01	.31	.23	.23	.99	

.

This would conclude the analysis if one forgoes the optional steps 9 and 10 of the algorithm in Section 2 (the ϕ^{-1}-transformations). If one does wish to perform these steps however, one has to, essentially arbitrarily, choose a class of ϕ-transformations, with ε as its free parameter, such that

$$\phi(0; \varepsilon) = \varepsilon, \qquad \phi(1; \varepsilon) = 1 - \varepsilon.$$

The simplest choice is clearly the class of linear functions

$$\phi(x; \varepsilon) = (1 - 2\varepsilon)\, x + \varepsilon, \qquad 0 \leqslant \varepsilon \leqslant 0.5,$$

and applying the inverse of $\phi(x; \varepsilon)$ at $\varepsilon = 0.01$ to the optimal ϕ-transformed dominance and subdominance vectors above, one arrives at

$A_{4\times 5}$	β_1	β_2	β_3	β_4	β_5	$D^{\alpha}_{4\times 1}$
α_1	1	0	1	1	0	.26
α_2	1	1	1	1	0	.44
α_3	1	1	0	0	0	.18
α_4	1	0	0	0	0	.01
$S^{\beta}_{5\times 1}$	1	.31	.22	.22	0	

,

$B_{4\times 5}$	β_1	β_2	β_3	β_4	β_5	$S^{\alpha}_{4\times 1}$
α_1	0	1	0	0	1	.18
α_2	0	0	0	0	1	.01
α_3	0	0	1	1	1	.26
α_4	0	1	1	1	1	.44
$D^{\beta}_{5\times 1}$	0	.31	.22	.22	1	

.

Note that the optimal vectors do not possess the counter-intuitive property observed in the vectors associated with the untransformed matrix: no row or column that contains nonzero entries has a zero value of optimal dominance or subdominance. The optimal double scales are graphically presented in Fig; 5.

As a final example of the variable-transformation double skew-dual scaling, consider the matrix

${}^{G}A_{4\times 5}$	β_1	β_2	β_3	β_4	β_5
α_1	1	1	1	1	0
α_2	1	1	1	1	0
α_3	1	1	0	0	0
α_4	1	0	0	0	0

,

where the prescript G indicates that the rows and columns of this matrix form perfect scales in Guttman's sense (Guttman, 1950/1974): no 1's are preceded by 0's. The four matrix products ${}^{G}A_{4\times 5}{}^{G}B^{T}_{5\times 4}$, ${}^{G}B^{T}_{5\times 4}{}^{G}A_{4\times 5}$, ${}^{G}B_{4\times 5}{}^{G}A^{T}_{5\times 4}$, ${}^{G}A^{T}_{5\times 4}{}^{G}B_{4\times 5}$ here are all reducible, and even worse, have no nonzero eigenvalues. As a result, no skew-dual scales can be computed for this matrix—certainly a disappointing result when compared to Boolean matrices that do not form perfect Guttman's scales.

In our variable-ε procedure, however, the four matrix products above can only be reducible at $\varepsilon = 0$ (because all matrices are strictly positive for all other values of ε). One can, therefore, formally exclude $\varepsilon = 0$ from consideration and seek the

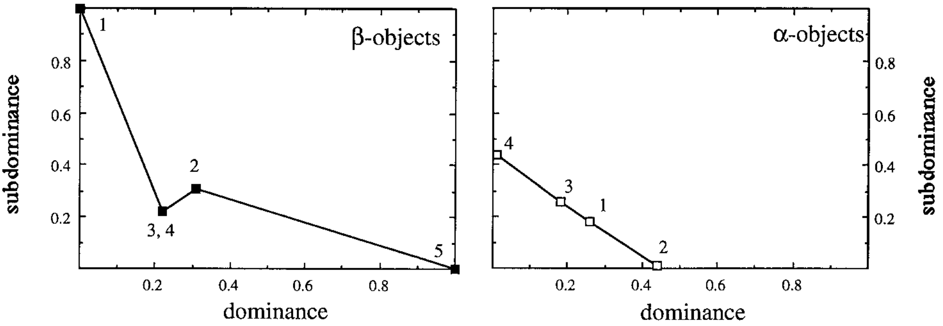

FIG. 5. Analysis of matrix $A_{4\times 5}$ of Section 3.

maximum of the variance product in the open interval $0<\varepsilon<1/2$. This maximum turns out to be achieved at about $\varepsilon=0.05$:

$\phi({}^{G}A_{4\times 5})$	β_1	β_2	β_3	β_4	β_5	$\phi(D^{\alpha}_{4\times 1})$
α_1	.95	.95	.95	.95	.05	.41
α_2	.95	.95	.95	.95	.05	.41
α_3	.95	.95	.05	.05	.05	.14
α_4	.95	.05	.05	.05	.05	.08
$\phi(S^{\beta}_{5\times 1})$	.95	.46	.20	.20	.05	

,

$\phi({}^{G}B_{4\times 5})$	β_1	β_2	β_3	β_4	β_5	$\phi(S^{\alpha}_{4\times 1})$
α_1	.05	.05	.05	.05	.95	.07
α_2	.05	.05	.05	.05	.95	.07
α_3	.05	.05	.95	.95	.95	.27
α_4	.05	.95	.95	.95	.95	.49
$\phi(D^{\beta}_{5\times 1})$	.05	.12	.24	.24	.95	

.

Using the same linear function as above, $\phi(x;\varepsilon)=(1-2\varepsilon)\,x+\varepsilon,\ 0\leqslant\varepsilon\leqslant 0.5$, one finally arrives at the following optimal dominance and subdominance vectors:

${}^{G}A_{4\times 5}$	β_1	β_2	β_3	β_4	β_5	$D^{\alpha}_{4\times 1}$
α_1	1	1	1	1	0	.41
α_2	1	1	1	1	0	.41
α_3	1	1	0	0	0	.10
α_4	1	0	0	0	0	.03
$S^{\beta}_{5\times 1}$	1	.46	.16	.16	0	

,

${}^{G}B_{4\times 5}$	β_1	β_2	β_3	β_4	β_5	$S^{\alpha}_{4\times 1}$
α_1	0	0	0	0	1	.03
α_2	0	0	0	0	1	.03
α_3	0	0	1	1	1	.24
α_4	0	1	1	1	1	.49
$D^{\beta}_{5\times 1}$	0	.07	.21	.21	1	

.

The optimal double scales are graphically presented in Fig. 6.

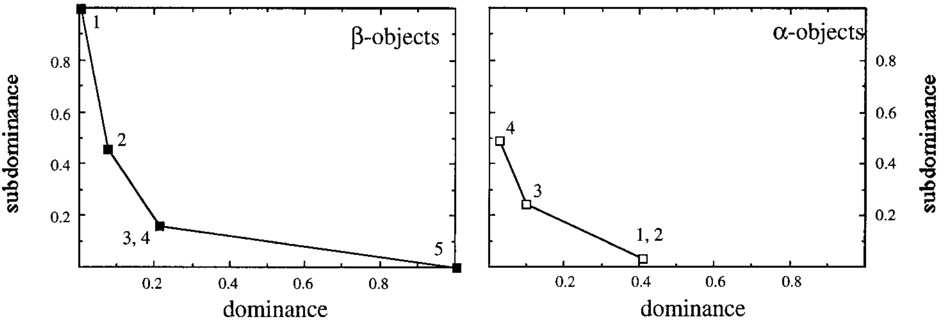

FIG. 6. Analysis of matrix ${}^{G}A_{4\times 5}$ of Section 3.

Summarizing, the three matrices used in this section to illustrate the variable-transformation double skew-dual scaling have very different algebraic properties, from the point of view of Theorems 2 and 3. These differences, however, become irrelevant and all three matrices produce interpretable dominance and subdominance vectors when subjected to one and the same variable-transformation procedure. This is a significant advantage of this procedure over a fixed-transformation one.

4. CONCLUSION

A summary of the double skew-dual scaling is provided in the paper's abstract. In this concluding section I briefly discuss several generalizations (or modifications) of this procedure, as well as its relation to the classical dual scaling.

4.1. *Classical Dual Scaling* (see Guttman, 1941; Nishisato, 1980, 1996)

Both dual scaling and the double skew-dual scaling are designed to assign numerical values to two sets of objects related by a matrix, and in both procedures the assignments received by the two sets are mutually interrelated. Here the similarities end, however: the classical dual scaling quantifies the two sets of objects in a different way, the duality of their quantification is of a different nature, and the interpretation of the quantification in terms of overall tendency to dominate or be dominated is not applicable. Also, the classical dual scaling would typically deal with different kinds of matrices. To focus on the most essential aspects of the differences between the two procedures, I assume here that they apply to one and the same dominance matrix $A_{n\times m}$, and I present the formula for the classical dual scaling in the form most resembling (3):

$$\left[\begin{array}{l} N(A_{n\times m})\, X^{\beta}_{m\times 1} = \eta X^{\alpha}_{n\times 1} \\ N(A^{T}_{m\times n})\, X^{\alpha}_{n\times 1} = \nu X^{\beta}_{m\times 1} \end{array}\right. . \tag{15}$$

Here, $X^{\alpha}_{n\times 1}$ and $X^{\beta}_{m\times 1}$ are the numerical values assigned to α-objects and β-objects, respectively, $N(\cdots)$ is a normalization operator that divides each entry of a matrix by the corresponding row sum, and η is some positive constant.

Comparing this expression with, say, the dominance part of (3),

$$\left[\begin{array}{l} A_{n\times m} \dfrac{D^{\beta}_{m\times 1}}{\sum D^{\beta}_{m\times 1}} = D^{\alpha}_{n\times 1} \\[2ex] B^{T}_{m\times n} \dfrac{D^{\alpha}_{n\times 1}}{\sum D^{\alpha}_{n\times 1}} = D^{\beta}_{m\times 1} \end{array}\right. , \tag{16}$$

one can observe the following. In (16) either of the two dominance vectors is the weighted mean of the corresponding rows or columns, the other vector serving as a set of weights. For example,

$$d^{\alpha}_{1} = \frac{a_{11} d^{\beta}_{1} + \cdots + a_{1m} d^{\beta}_{m}}{d^{\beta}_{1} + \cdots + d^{\beta}_{m}}.$$

In (15) the situation is, in a sense, opposite: either of the two vectors is (proportional to) the weighted mean of the other, the corresponding rows or columns serving as weights:

$$x_1^\alpha = \frac{a_{11}x_1^\beta + \cdots + a_{1m}x_m^\beta}{\eta(a_{11} + \cdots + a_{1m})}.$$

Since the entries of $A_{n\times m}$ are interpreted as dominance values, this form of averaging makes it impossible to interpret $X_{n\times 1}^\alpha$ and $X_{m\times 1}^\beta$ in dominance/subdominance terms.

The second and obvious difference is that the two dominance vectors in (16) are computed from two complementary matrices, whereas (15) involves only $A_{n\times m}$. This, too, indicates a considerable difference in the interpretation of the dual vectors. The involvement of both $A_{n\times m}$ and $B_{n\times m}$, due to the complete symmetry of their relationship, necessitates the introduction of the second dual pair in the double skew-dual scaling.

The difference between the two scaling procedures is best illustrated on the following example.[4] Consider the matrix of probabilities

$A_{3\times 3}$	β_1	β_2	β_3
α_1	0.7	0	0
α_2	0	0.5	0
α_3	0	0	0.1

Due to the involvement of the normalization operator $N(\cdots)$, in the classical dual scaling this matrix is indistinguishable from the identity matrix

$U_{3\times 3}$	β_1	β_2	β_3
α_1	1	0	0
α_2	0	1	0
α_3	0	0	1

Because of this, (15) yields no meaningful results: $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\beta_1, \beta_2, \beta_3\}$ can be assigned any identical triads of numbers, $X_{3\times 1}^\alpha = X_{3\times 1}^\beta$ $(\eta = 1)$.

By contrast, the double-skew-dual scaling yields numerical assignments that seem intuitively quite reasonable:

$A_{3\times 3}$	β_1	β_2	β_3	$D_{3\times 1}^\alpha$
α_1	0.7	0	0	.19
α_2	0	.5	0	.16
α_3	0	0	.1	.04
$S_{3\times 1}^\beta$	.19	.16	.04	

$B_{3\times 3}$	β_1	β_2	β_3	$S_{3\times 1}^\alpha$
α_1	0.3	1	1	.66
α_2	1	.5	1	.79
α_3	1	1	.9	.98
$D_{3\times 1}^\beta$	.66	.79	.98	

[4] I am indebted to Ulf Böcknholt for suggesting to me the idea of this comparison (personal communication, March 1997).

By symmetry, we have here $(D^{\alpha}_{3\times 1}, D^{\beta}_{3\times 1}) = (S^{\beta}_{3\times 1}, S^{\alpha}_{3\times 1})$; the α-dominance values decrease from α_1 to α_3, while the β-dominance values increase from β_1 to β_3.

None of the comparisons above should be construed as indicating the inferiority or insufficient generality of the classical dual scaling. A proper way of putting it is that when applied to dominance matrices the classical dual scaling and the double skew-dual scaling depict different aspects of these matrices. Unlike the double skew-dual scaling, the classical dual scaling is not intended to quantify such intuitive notions as "one's ability to solve difficult problems," "one's inability to solve easy problems," "the difficulty with which a problem is solved by good problem-solvers," and "the easiness with which a problem is solved with poor problem-solvers." The many intended uses of the classical dual scaling are comprehensively presented in Nishisato (1980, 1996).

4.2. *Generalizations/Modifications*

The theory presented in Sections 2 and 3 views entries of a dominance matrix as variables attaining all possible values selected from a set of reals X. The bounds of this set, $\inf X$ and $\sup X$, are to be finite and known in order to define the size of a matrix, $\omega = \phi(\inf X) + \phi(\sup X)$, and the complementarity relationship, $\phi(A_{n\times m}) + \phi(B_{n\times m}) = \omega 1_{n\times m}$. There might be situations, however, when the set of theoretically possible values for the dominance matrix entries is not known or is unbounded. The latter often happens, for example, when the matrix to be analyzed is a model-guided transformation of a data matrix, such as the standard Thurstonian translation of pairwise comparison probabilities into differences of normal distributions' means. In such a case, one of the values, $\phi(\inf X)$ or $\phi(\sup X)$, is necessarily infinite, and neither the complementarity relationship nor the size-preservation constraint can be defined as they have been in this paper.

The only way of circumventing this difficulty seems to be the replacement of the theoretical bounds $\inf X$ and $\sup X$ with the factual maximal and minimal elements found in the matrix to be analyzed: $\omega = \phi(\max A_{n\times m}) + \phi(\min A_{n\times m})$. This modification is more serious than it might appear, and I would caution against using it when the theoretical bounds $\inf X$ and $\sup X$ are finite and known. To appreciate the difference, observe, for example, that according to the "standard" scheme, the complement of the matrix of probabilities

$A_{2\times 2}$	β_1	β_2
α_1	0.1	0.1
α_2	0.1	0.1

is

$B_{2\times 2}$	β_1	β_2
α_1	0.9	0.9
α_2	0.9	0.9

,

whereas according to the modified scheme, the complement of $A_{2\times 2}$ is $A_{2\times 2}$ itself.

Numerical values of dominance or subdominance assigned to an object in the double skew-dual scaling are expressly context-dependent. A removal or addition of a single object to either of the two sets of objects corresponds to a removal or addition of a row or column to an existing matrix, and may appreciably change numerical values for objects in both sets. It seems appealing, therefore, to think of a "true" dominance or subdominance value of, say, an α-object as the one that would be assigned to it in a matrix relating all possible α-objects to all possible β-objects. Such a theoretical matrix would typically be infinitely large, and the question arises whether the theory of the present paper could be applied to matrices with countably infinite numbers of rows and/or columns. The answer to this questions seems to be a qualified yes: most of the computational operations and concepts involved in the double-skew-dual scaling (such as matrix products, eigenequations, irreducibility, etc.), as well as some of the existence and uniqueness theorems, can be directly extended or suitably modified to apply to infinite matrices (Seneta, 1973).

Finally, it should be pointed out that the basic equations of the double skew-dual scaling, (12) and (13), can be generalization in a straightforward fashion from matrices $A_{n\times m}=\{a(i, j)\}$ to arbitrary functions $a(x, y)$, provided the integrals below exist in the Lebesgue sense with respect to some suitably defined measures on x and y:

$$\left[\begin{array}{l} \dfrac{\int \phi[a(x, y)]\,\phi[d^{\beta}(y)]\,\mathrm{d}y}{\int \phi[d^{\beta}(y)]\,\mathrm{d}y}=\phi[d^{\alpha}(x)] \\ \dfrac{\int \phi[b(x, y)]\,\phi[d^{\alpha}(x)]\,\mathrm{d}x}{\int \phi[d^{\alpha}(x)]\,\mathrm{d}x}=\phi[d^{\beta}(y)] \end{array}\right.,$$

$$\left[\begin{array}{l} \dfrac{\int \phi[a(x, y)]\,\phi[s^{\alpha}(x)]\,\mathrm{d}x}{\int \phi[s^{\alpha}(x)]\,\mathrm{d}x}=\phi[s^{\beta}(y)] \\ \dfrac{\int \phi[b(x, y)]\,\phi[s^{\beta}(y)]\,\mathrm{d}y}{\int \phi[s^{\beta}(y)]\,\mathrm{d}y}=\phi[s^{\alpha}(x)] \end{array}\right..$$

The function $a(x, y)$ here is assumed to take its values in some bounded interval X, the function $b(x, y)$ is defined from

$$\phi[a(x, y)]+\phi[b(x, y)]=\phi(\inf X)+\phi(\sup X),$$

and $d^{\alpha}(x)$, $d^{\beta}(y)$, $s^{\alpha}(x)$, $s^{\beta}(y)$ are dominance and subdominance functions for certain "object-functions" $\alpha(x)$ and $\beta(y)$.

APPENDIX: PROOFS

Proof of Theorem 1. As shown in Aczél (1966, pp. 256–257), since $\oplus$ is closed, associative, continuous, and increasing in both arguments on an interval I,

$$x\oplus y=\phi^{-1}[\phi(x)+\phi(y)],$$

for some monotonic continuous function $\phi: I \to \mathrm{Re}$, determined uniquely up to a scaling coefficient, $h\phi$, $h \neq 0$.

Using this function, the right-distributivity of $\otimes$ over $\oplus$ can be written as

$$\phi^{-1}[\phi(c_1)+\phi(c_2)]\otimes x=\phi^{-1}[\phi(c_1\otimes x)+\phi(c_2\otimes x)],$$

or, equivalently,

$$\phi\{\phi^{-1}[\phi(c_1)+\phi(c_2)]\otimes x\}=\phi\{\phi^{-1}[\phi(c_1)]\otimes x\}+\phi\{\phi^{-1}[\phi(c_2)]\otimes x\}.$$

Fixing x and denoting $F_x(u)=\phi[\phi^{-1}(u)\otimes x]$, the identity above can be presented as

$$F_x[\phi(c_1)+\phi(c_2)]=F_x[\phi(c_1)]+F_x[\phi(c_2)],$$

which is the fundamental Cauchy equation with respect to an unknown function defined and continuous on $\phi(I)$. The closedness of $\oplus$ on I implies that $+$ is closed on $\phi(I)$, because of which $\phi(I)$ can only be an interval of one of the three types, $[q, \infty)$, $(-\infty, -q]$, or $(-\infty, \infty)$, where $q \geqslant 0$ (the finite end can always be closed by continuity). With another reference to Aczél (1966; pp. 31–49), the solution of the Cauchy equation above for any of these three intervals is $F_x[\phi(c)]=\gamma(x)\,\phi(c)$, which, due to the definition of F_x, means that

$$\phi(c\otimes x)=\phi(c)\,\gamma(x).$$

By analogous reasoning, we obtain from the left-distributivity of $\otimes$ over $\oplus$ that

$$\phi(c\otimes x)=\delta(c)\,\phi(x).$$

Combining these two identities,

$$\phi(c\otimes x)=\phi(c)\,\gamma(x)=\delta(c)\,\phi(x).$$

Observe that $\phi(x)$, being monotonic, cannot vanish at more than one point; $\gamma(x)$ cannot vanish at more than one point either, because $c\otimes x$ is assumed to increase with x. Fixing x at a value where both $\gamma(x)$ and $\phi(x)$ are nonzero, and denoting $\gamma(x)/\phi(x)=p$, we get

$$\delta(c)=p\phi(c),$$

from which it immediately follows that δ and γ are identical. We conclude that

$$x\otimes y=\phi^{-1}[p\phi(x)\,\phi(y)].$$

Observing that

$$\phi(x\otimes y)=p\phi(x)\,\phi(y)\Leftrightarrow -\phi(x\otimes y)=[-p][-\phi(x)][-\phi(y)],$$

any two pairs (p, ϕ) and $(-p, -\phi)$ correspond to one and the same operation $\otimes$. Since it is also true that ϕ and $-\phi$ correspond to one and the same operation $\oplus$, one can assume that $p > 0$ without any loss of generality.

It is easy to show that ϕ cannot change its sign as its argument changes. Indeed, assume otherwise, and let x and y be chosen so that $\phi(x) < 0$ and $\phi(y) > 0$. Due to the continuity of ϕ, these signs must be preserved in sufficiently small neighborhoods of x and y. Therefore, whether ϕ is increasing or decreasing, a small increment in y should lead to a decrement in $x \otimes y = \phi^{-1}[p\phi(x)\,\phi(y)]$, which cannot be the case because $x \otimes y$ is postulated to increase in y. This excludes $(-\infty, \infty)$ as a possibility for $\phi(I)$.

Of the two remaining possibilities,[5] $\phi\colon I \to \mathrm{Re}^+$ and $\phi\colon I \to \mathrm{Re}^-$, the latter is immediately excluded by the agreement that $p > 0$, because the identity $\phi(x \otimes y) = p\phi(x)\,\phi(y)$ cannot be satisfied with all three quantities being negative. We conclude that $\phi\colon I \to \mathrm{Re}^+$, and, for some $p > 0$,

$$\left[\begin{array}{l} x \oplus y = \phi^{-1}[\phi(x) + \phi(y)] \\ x \otimes y = \phi^{-1}[p\phi(x)\,\phi(y)] \end{array}\right. .$$

It is obvious that given a pair of operations $\oplus$, $\otimes$, the pair (p, ϕ) is determined uniquely, up to reciprocal similarity transformations $(p/h, h\phi)$, $h > 0$. A pair (p, ϕ) determines $(\oplus, \otimes)$ uniquely, provided that the two operations exist. Now, $\oplus$ exists for any ϕ, because, for any $x, y \in \phi(I) = [q, \infty)$ $(q \geqslant 0)$, $\phi^{-1}[\phi(x) + \phi(y)]$ is well defined and belongs to I. Since $\phi^{-1}[p\phi(x)\,\phi(y)]$ is also well defined, the only concern is that it may fail to belong to I, or equivalently, that $p\phi(x)\,\phi(y)$ may fall below q. To prevent this from happening, obviously, it is sufficient and necessary that

$$p \geqslant q^{-1} \qquad \text{if} \quad q \neq 0.$$

The definitional formula for ρ, (5), can now be written as

$$p\phi(\lambda)\,\phi(\rho) = p\phi(c_1)\,\phi(x_1) + \cdots + p\phi(c_k)\,\phi(x_k).$$

Due to the "unanimity" property, (6),

$$p\phi(\lambda)\,\phi(x) = p\phi(c_1)\,\phi(x) + \cdots + p\phi(c_k)\,\phi(x),$$

from which it follows that

$$\phi(\lambda) = \phi(c_1) + \cdots + \phi(c_k).$$

Substituting this in the definitional formula for ρ we derive (7). Observe that p plays no role in this expression, so that all pairs of operations $\oplus$, $\otimes$ corresponding to the same ϕ (up to a scaling transformation) yield the same function ρ.

[5] Symbols Re^+ and Re^- in this paper stand for nonnegative and nonpositive reals, respectively.

Proof of Theorem 2. (a) This immediately follows from (12) and (13): any element of the vectors $\phi(D^{\alpha}_{n\times 1})$, $\phi(D^{\beta}_{m\times 1})$, $\phi(S^{\alpha}_{n\times 1})$, $\phi(S^{\beta}_{m\times 1})$ is a weighted mean of a row or a column of the matrices $\phi(A_{n\times m})$ and $\phi(B_{n\times m})$, whose elements all belong to $\phi([u, v])$.

(b1) and (b2) These are proved by straightforward algebra. In (12),

$$\left[\begin{array}{l} \phi(A_{n\times m})\dfrac{\phi(D^{\beta}_{m\times 1})}{\sum\phi(D^{\beta}_{m\times 1})}=\phi(D^{\alpha}_{n\times 1}) \\ \phi(B^{T}_{m\times n})\dfrac{\phi(D^{\alpha}_{n\times 1})}{\sum\phi(D^{\alpha}_{n\times 1})}=\phi(D^{\beta}_{m\times 1}) \end{array}\right.,$$

substitute the left-hand side of the first equation for $\phi(D^{\alpha}_{n\times 1})$ in the second, and substitute the left-hand side of the second equation for $\phi(D^{\beta}_{m\times 1})$ in the first, to obtain

$$\left[\begin{array}{ll} [\phi(A_{n\times m})\,\phi(B^{T}_{m\times n})]\,\phi(D^{\alpha}_{n\times 1}) & =[\sum\phi(D^{\alpha}_{n\times 1})\sum\phi(D^{\beta}_{m\times 1})]\,\phi(D^{\alpha}_{n\times 1}) \\ & =\delta\phi(D^{\alpha}_{n\times 1}) \\ [\phi(B^{T}_{m\times n})\,\phi(A_{n\times m})]\,\phi(D^{\beta}_{m\times 1}) & =[\sum\phi(D^{\alpha}_{n\times 1})\sum\phi(D^{\beta}_{m\times 1})]\,\phi(D^{\beta}_{m\times 1}) \\ & =\delta\phi(D^{\beta}_{m\times 1}) \end{array}\right..$$

Analogously, (13) yields

$$\left[\begin{array}{ll} [\phi(B_{n\times m})\,\phi(A^{T}_{m\times n})]\,\phi(S^{\alpha}_{n\times 1}) & =[\sum\phi(S^{\alpha}_{n\times 1})\sum\phi(S^{\beta}_{m\times 1})]\,\phi(S^{\alpha}_{n\times 1}) \\ & =\sigma\phi(S^{\alpha}_{n\times 1}) \\ [\phi(A^{T}_{m\times n})\,\phi(B_{n\times m})]\,\phi(S^{\beta}_{m\times 1}) & =[\sum\phi(S^{\alpha}_{n\times 1})\sum\phi(S^{\beta}_{m\times 1})]\,\phi(S^{\beta}_{m\times 1}) \\ & =\sigma\phi(S^{\beta}_{m\times 1}) \end{array}\right..$$

(c1) By hypothesis, there exist vectors $V^{\alpha}_{n\times 1}$ and $V^{\beta}_{m\times 1}$ such that

$$\left[\begin{array}{l} [\phi(A_{n\times m})\,\phi(B^{T}_{m\times n})]\,V^{\alpha}_{n\times 1}=\delta V^{\alpha}_{n\times 1} \\ [\phi(B^{T}_{m\times n})\,\phi(A_{n\times m})]\,V^{\beta}_{m\times 1}=\delta V^{\beta}_{m\times 1} \end{array}\right..$$

Since they are nonnegative, we can assume $\sum V^{\alpha}_{n\times 1}=\sum V^{\beta}_{m\times 1}=1$. Premultiplying both sides of the first equation by $\phi(B^{T}_{m\times n})$ and denoting $X^{\beta}_{m\times 1}=\phi(B^{T}_{m\times n})\,V^{\alpha}_{n\times 1}$,

$$\left[\begin{array}{l} [\phi(B^{T}_{m\times n})\,\phi(A_{n\times m})]\,X^{\beta}_{m\times 1}=\delta X^{\beta}_{m\times 1} \\ [\phi(B^{T}_{m\times n})\,\phi(A_{n\times m})]\,V^{\beta}_{m\times 1}=\delta V^{\beta}_{m\times 1} \end{array}\right..$$

Since the eigenvalue is simple, $X^{\beta}_{m\times 1}$ must be a multiple of $V^{\beta}_{m\times 1}$, and we have

$$\phi(B^{T}_{m\times n})\,V^{\alpha}_{n\times 1}=\lambda V^{\beta}_{m\times 1}, \qquad \lambda>0.$$

Analogously, by premultiplying the second eigenvalue–eigenvector equation by $\phi(A_{n\times m})$ we get

$$\phi(A_{n\times m})\, V^{\beta}_{m\times 1} = \kappa V^{\alpha}_{n\times 1}, \qquad \kappa > 0.$$

Multiplying the first equation by κ and the second with λ,

$$\begin{bmatrix} \phi(B^T_{m\times n})[\kappa V^{\alpha}_{n\times 1}] = \kappa[\lambda V^{\beta}_{m\times 1}] \\ \phi(A_{n\times m})[\lambda V^{\beta}_{m\times 1}] = \lambda[\kappa V^{\alpha}_{n\times 1}] \end{bmatrix}.$$

Observe that $\sum[\kappa V^{\alpha}_{n\times 1}] = \kappa$ and $\sum[\lambda V^{\beta}_{m\times 1}] = \lambda$. Thus the vectors $[\kappa V^{\alpha}_{n\times 1}]$ and $[\lambda V^{\beta}_{m\times 1}]$ satisfy the requirements (12) for $\phi(D^{\alpha}_{n\times 1})$ and $\phi(D^{\beta}_{m\times 1})$, respectively. This proves the existence.

According to part (b1), if some dominance vectors $Z^{\alpha}_{n\times 1}$, $Z^{\beta}_{m\times 1}$ exist, then $\phi(Z^{\alpha}_{n\times 1})$, $\phi(Z^{\beta}_{m\times 1})$ are nonnegative eigenvectors of $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$, $\phi(B^T_{m\times n})\,\phi(A_{n\times m})$, respectively, associated with their common eigenvalue $\delta' = \sum \phi(Z^{\alpha}_{n\times 1}) \sum \phi(Z^{\beta}_{m\times 1}) > 0$. If one of the two matrices has no nonnegative eigenvectors associated with positive eigenvalues other than δ, then $\delta' = \delta$. Since δ is simple, $\phi(Z^{\alpha}_{n\times 1})$ and $\phi(Z^{\beta}_{m\times 1})$ must be multiples of $\phi(D^{\alpha}_{n\times 1})$ and $\phi(D^{\beta}_{m\times 1})$, respectively, and by simple algebra, (12) implies that $\phi(Z^{\alpha}_{n\times 1}) = \phi(D^{\alpha}_{n\times 1})$ and $\phi(Z^{\beta}_{m\times 1}) = \phi(D^{\beta}_{m\times 1})$. This proves the uniqueness.

(c1) This is proved analogously.

Proof of Theorem 3. It follows from the "rule of four products" and the "Perron–Frobenius rule" that the matrices $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$, $\phi(B^T_{m\times n})\,\phi(A_{n\times m})$, $\phi(B_{n\times m})\,\phi(A^T_{m\times n})$, $\phi(A^T_{m\times n})\,\phi(B_{n\times m})$ all share the same positive and simple Perron root π. Without loss of generality, assume that $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$ is irreducible. Then it has a positive eigenvector associated with π and no nonnegative eigenvectors corresponding to other eigenvalues (the "Perron–Frobenius rule"). By the "rule of Perron roots," $\phi(B^T_{m\times n})\,\phi(A_{n\times m})$ has a nonnegative eigenvector associated with π. The conditions of theorem 2(c1) are satisfied, and we conclude that dominance vectors $D^{\alpha}_{n\times 1}$, $D^{\beta}_{m\times 1}$ exist and are determined uniquely. The analogous statement for subdominance vectors $S^{\alpha}_{n\times 1}$, $S^{\beta}_{m\times 1}$ follows from the fact that $\phi(B_{n\times m})\,\phi(A^T_{m\times n})$, being the inverse of $\phi(A_{n\times m})\,\phi(B^T_{m\times n})$, is irreducible. This proves (i). The statements (ii) and (iii) now trivially follow from Theorem 2(b1–b2).

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