JOURNAL OF MATHEMATICAL PSYCHOLOGY **40**, 48–63 (1996)
ARTICLE NO. 0003

Empirical Discriminability of Two Models for Stochastic Relationship Between Additive Components of Response Time

Ehtibar N. Dzhafarov

University of Illinois at Urbana–Champaign

and

Jeffrey N. Rouder

University of California, Irvine

Dzhafarov (1992, *J. Math. Psych.* **36**, 235–268) analyzed additive decompositions of simple response time (RT) into two random variables: a signal-independent component and a component stochastically decreasing and vanishing as signal magnitude increases. The asymptotic behavior of RT (the dependence of RT of a given quantile rank on signal magnitude in the region of sufficiently large signals) was shown to be different under different models of stochastic relationship between the two RT components. As a simple alternative to the more traditional stochastic independence model, according to which the two RT components have stochastically independent sources of random variability, Dzhafarov proposed a single-variate RT decomposition model (SVRT) according to which the two components are increasing functions of a single common source of random variability. The two models predict distinctly different patterns of the asymptotic RT behavior on a population level. Our computer simulations show, however, that if Dzhafarov's test based on this difference is applied to RT samples generated according to the stochastic independence model, the results can sometimes mimic the asymptotic predictions of the SVRT model. This happens because of the uncertainty in determining the range of signals that are ''sufficiently large'' to warrant asymptotic approximations. This difficulty can be overcome if instead of choosing a fixed range of large signals one repeatedly applies the test to a sequence of nested regions of large signals. Our computer simulations show that with this approach the two models can be reliably discriminated on realistically sized RT samples.

1. INTRODUCTION: AN ASYMPTOTIC THEORY OF ADDITIVE RESPONSE TIME DECOMPOSITIONS

Dzhafarov (1992) proposed an asymptotic theory for additive decompositions of simple response time (RT) into two random variables: a *signal-independent* component and a *signal-dependent* component stochastically decreasing and vanishing as signal magnitude increases. The theory imposes no a priori constraints on possible stochastic relationships between the two RT components, and it predicts that different forms of stochastic relationship are generally reflected in different patterns of the asymptotic behavior of RT (the dependence of RT distributions on signal magnitude in the region of sufficiently large signals). In particular, the asymptotic behavior of RT is distinctly different for the two simplest (and in a sense, opposite) forms of the stochastic relationship: the stochastic independence model (Luce & Green, 1972; Kohfeld, Santee, & Wallace, 1981a, b), according to which the two RT components have stochastically independent sources of random variability, and Dzhafarov's single-variate RT decomposition model, according to which the two components have a single, common source of random variability (so that for any given signal, the two components are deterministic increasing functions of each other).

Based on the difference in the two models' population-level predictions, Dzhafarov proposed a linear regression test designed to discriminate between these models when RT distributions are represented by samples corresponding to several distinct signal magnitudes. Our computer simulations show that because of the uncertainty in choosing the range of signals that are "sufficiently large" to warrant asymptotic approximations (as required by the test), RT distributions generated according to the stochastic independence model can sometimes mimic the asymptotic predictions of the single-variate RT model. To account for this result, we analyze the operational meaning of the mathematical concepts related to the asymptotic behavior of RT. This leads to a modification of the test in which, instead of choosing a fixed region of "sufficiently large" signals, one repeatedly applies the original test to a series of nested regions of large signals and observes the pattern of changes in the test results. Using simulated RT samples of realistic sizes, we show that with this sequential approach one can reliably discriminate between the two models of stochastic relationship.

Correspondence should be sent to Ehtibar N. Dzahafarov, Department of Psychology, University of Illinois at Urbana–Champaign, 603 East Daniel Street, Champaign, Illinois 61820.

0022-2496/96 $18.00

The theory presented in Dzhafarov (1992) deals with simple RT to step-function signals whose amplitude, A, forms a "unidimensional strength continuum." The latter means that as A increases, RT stochastically (i.e., for any given quantile rank) decreases, while the subjective magnitude or detectability of the signal increases. As an example, if subjects respond to the onset of a light flash, then its physical intensity forms a "unidimensional strength continuum." Being a stochastically decreasing and nonnegative random variable, RT converges (in distribution) to some nonnegative random variable $\mathbf{R}$, as A increases.[1] It is assumed that RT can be additively decomposed as

$$\mathbf{RT}(A) = T(A, \tilde{\mathbf{Z}}) + \mathbf{R}, \tag{1}$$

where $\tilde{\mathbf{Z}}$ is some set of random variables ("internal sources of variability") such that

(a) the joint distribution of $(\mathbf{R}, \tilde{\mathbf{Z}})$ does not depend on A,

(b) for (almost)[2] any value $\tilde{z}$ of $\tilde{\mathbf{Z}}$, the nonnegative function $T(A, \tilde{z})$ decreases and vanishes as A increases, and

(c) for (almost) any two values $\tilde{z}_1$ and $\tilde{z}_2$ of $\tilde{\mathbf{Z}}$, the functions $T(A, \tilde{z}_1)$ and $T(A, \tilde{z}_2)$ vanish with asymptotically proportional rates,

$$0 < \lim_{A \to \infty} \frac{T(A, \tilde{z}_1)}{T(A, \tilde{z}_2)} < \infty \tag{2}$$

(the range of A can always be set to extend from 0 to ∞).

In this decomposition, $\mathbf{R}$ is the "maximal" signal-independent component of RT, whereas $\mathbf{T}(A) = T(A, \tilde{\mathbf{Z}})$ is the "minimal" signal-dependent component of RT. The components are called "maximal" and "minimal" in the sense that $\mathbf{T}(A)$ cannot be further additively decomposed into nonnegative components $\mathbf{R}_1$ and $\mathbf{T}_1(A)$. It is clear that the stochastic relationship between the two RT components is determined by the joint distribution of $(\mathbf{R}, \tilde{\mathbf{Z}})$. In particular, if $\mathbf{R}$ and $\tilde{\mathbf{Z}}$ are stochastically independent, then $\mathbf{R}$ and $\mathbf{T}(A)$ are stochastically independent for any given A.

The asymptotic proportionality of $T(A, \tilde{z})$ at different values of $\tilde{z}$ is stated in Dzhafarov (1992) as a different though mathematically equivalent proposition ("Asymptotic Differentiability Assumption" and Lemma 1.2.2). This proposition is that $T(A, \tilde{z})$ is asymptotically factorizable into a product of a positive function $C(\tilde{z})$ and a strictly decreasing positive function $s(A)$ vanishing at $A \to \infty$:

$$T(A, \tilde{z}) = C(\tilde{z})\, s(A) + o\{s(A)\}, \tag{3}$$

where $C(\tilde{z})$ is determined uniquely and $s(A)$ asymptotically uniquely, up to positive scaling coefficients having reciprocal values for the two functions.[3] Equation (3) means that in the region of sufficiently large A, the signal-dependent RT component $\mathbf{T}(A)$ is essentially proportional to a random variable $\mathbf{C} = C(\tilde{\mathbf{z}})$, the proportionality coefficient being some decreasing transformation s of A. It follows that the stochastic relationship between $\mathbf{R}$ and $\mathbf{T}(A)$ in the region of sufficiently large A is essentially determined by the joint distribution of $(\mathbf{R}, \mathbf{C})$. The variable $\mathbf{C}$ is termed the "criterion" in reference to the Grice-representability of time variables, as discussed in Dzhafarov (1993).

The following result is central for Dzhafarov's (1992) analysis (Theorem 3.1.1):

$$\mathrm{RT}_p(A) = R_p + \mathbb{E}[\mathbf{C} \mid \mathbf{R} = R_p]\, s(A) + o\{s(A)\}, \tag{4}$$

where $\mathrm{RT}_p(A)$ and R_p denote the rank-p quantiles of $\mathbf{RT}(A)$ and $\mathbf{R}$, respectively $(0 < p < 1)$, whereas $\mathbb{E}[\cdots \mid \cdots]$ denotes conditional expectation. This proposition serves as a basis for empirical analysis of stochastic relationship between $\mathbf{R}$ and $\mathbf{C}$, which, as we have seen, asymptotically characterizes the stochastic relationship between $\mathbf{R}$ and $\mathbf{T}(A)$. For any quantile rank p, the value of R_p in (4) can be estimated by the value of $\mathrm{RT}_p(A)$ at a very large A. The term "very large" means that no further increase in A produces a noticeable change in the RT distribution (an empirical version of Cauchy's convergence criterion). If the "asymptotically linearizing" transformation $s(A)$ is known, then one can find $\mathbb{E}[\mathbf{C} \mid \mathbf{R} = R_p]$ for different values of p by estimating slopes of the tangent lines drawn to the RT quantile curves, $\mathrm{RT}_p(A)$ versus $s(A)$, at their intercepts, R_p. We will refer to these tangent lines as *intercept tangents* (Fig. 1).

The problem of finding the asymptotically linearizing transformation $s(A)$ is simpler than it might seem. In the best case it is given by a psychophysical model of sensory processing. In Dzhafarov's (1992) analysis of RT to an abrupt displacement of a light source, the transformation used, $s(A) = A^{-2}$, is derived from a certain model of visual motion detection (Dzhafarov, Sekuler, & Allik, 1993, Equations A1 and A4). In the absence of such a model, however, one can always empirically estimate $s(A)$ (more precisely, a function asymptotically proportional to $s(A)$; see Footnote 3) by choosing an arbitrary quantile rank p and computing $\mathrm{RT}_p(A) - R_p$ (the latter term being estimated from RT to a very large signal): indeed, according to (4) this difference is asymptotically proportional to $s(A)$,

$$\mathrm{RT}_p(A) - R_p = \mathrm{const} \times s(A) + o\{s(A)\}.$$

[1] Hereafter boldface letters stand for random variables; all unidimensional random variables are assumed to be absolutely continuous on some intervals of reals.

[2] "Almost" means "except for a subset of measure zero."

[3] Recall that $o\{s\}$ is any variable such that $o\{s\}/s \to 0$ as $s \to 0$. In practical terms, $o\{s\}$ can be ignored when s is sufficiently small. The asymptotic uniqueness of s (up to positive scaling) means that s can be replaced with a function s^* if, and only if, s^*/s tends to a positive constant (as $A \to \infty$).

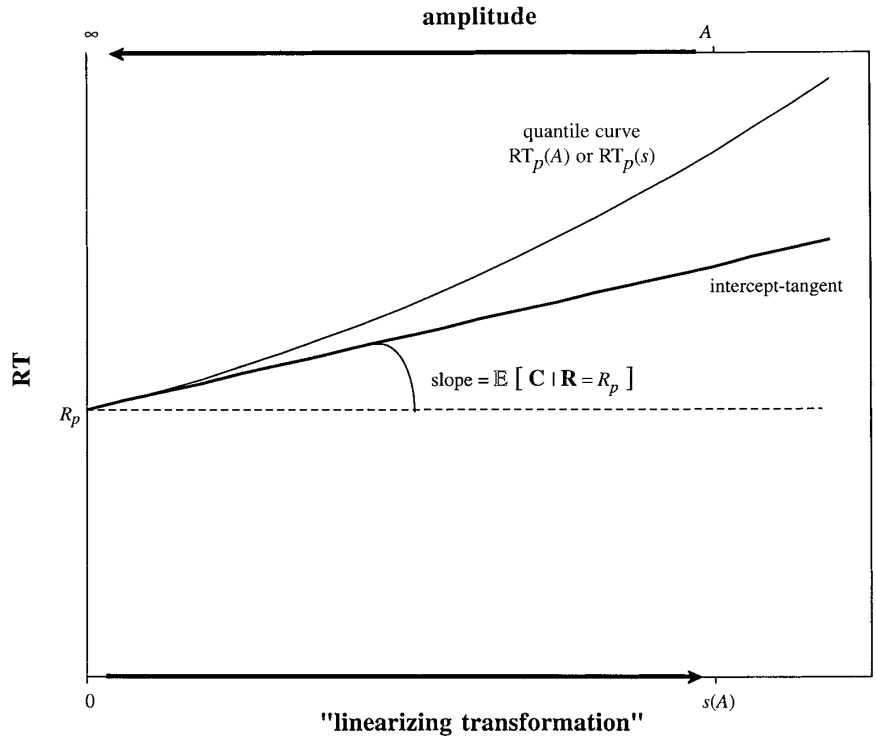

FIG. 1. Geometric meaning of Eq. (4) for an arbitrary quantile rank p.

In Dzhafarov (1992, Section 3.4) this technique is used as well, although instead of choosing a single arbitrary quantile rank, the estimator of $s(A)$ is computed there by averaging across a range of quantile ranks. In fact, our purposes would be served by plotting $\mathrm{RT}_p(A)$ against any positive linear transformation of (a function asymptotically proportional to) $s(A)$. This means that one can even avoid the necessity of estimating R_p and replace $s(A)$ with $\mathrm{RT}_p(A)$ itself, for a given quantile rank p or averaged across a range of quantile ranks. At the end of this paper we show that this technique works quite well. Until then, however, we assume that the asymptotically linearizing transformation $s(A)$ is known theoretically.

It is much more difficult a problem to estimate slopes of the intercept tangents. In a sense, this difficulty is the main issue of this paper. Dzhafarov's (1992) approach to this problem is to choose A-values so large that $o\{s(A)\}$ in (4) can be ignored. The quantile curves then are approximately linear with respect to $s(A)$, and the slopes in question can be obtained by a standard linear regression procedure. We will see that this approach may fail to tell apart two very dissimilar models of stochastic relationship between $\mathbf{R}$ and $\mathbf{T}(A)$, focal to our analysis. We consider these models in the next section.

2. TWO MODELS OF STOCHASTIC RELATIONSHIP

The single-variate RT decomposition model (SVRT model) is perhaps the simplest possible scheme of stochastic relationship between the RT components. Our account of this model is equivalent to that in Dzhafarov (1992), but follows more closely Dzhafarov (1991). Recall that for any quantile rank p, $\mathrm{RT}_p(A)$ decreases in A and converges to the same-rank quantile R_p of the signal-independent RT component. In the SVRT model it is assumed that the difference $\mathrm{RT}_p(A)-R_p$ is increasing in p (in addition to being decreasing in A). Because of this, the following special version of (1) holds:

$$\mathbf{RT}(A)=\mathrm{RT}(A,\mathbf{P})=T(A,\mathbf{P})+R(\mathbf{P}), \tag{5}$$

where $\mathbf{P}$ is uniformly distributed between 0 and 1, and $\mathrm{RT}(A,\mathbf{P})$, $T(A,\mathbf{P})$, $R(\mathbf{P})$ are quantile functions (inverse distribution functions, strictly increasing) for random variables $\mathbf{RT}(A)$, $\mathbf{T}(A)$, and $\mathbf{R}$, respectively. This implies (and is implied by) that $\mathbf{T}(A)$ and $\mathbf{R}$ are increasing deterministic functions of each other. It is assumed further that for (almost) any two quantile ranks, $0<p_2<p_1<1$,

$$1<\lim_{A\to\infty}\frac{T(A,p_1)}{T(A,p_2)}<\infty, \tag{6}$$

which is a special version of (2). It immediately follows from (5) and (6) that

$$T_p(A)=C_p s(A)+o\{s(A)\}, \tag{7}$$

where C_p is the rank-p quantile of some positive random variable $\mathbf{C}$. Since $\mathbb{E}[\mathbf{C} \mid \mathbf{R} = R_p] = C_p$, one comes to the following version of (4):

$$\mathrm{RT}_p(A) = R_p + C_p s(A) + o\{s(A)\}. \tag{8}$$

This means that slopes of the intercept tangents increase with the intercept values (both C_p and R_p increase with p). That is, the intercept tangents form a diverging fan pattern (Fig. 2).

The second simple model of stochastic relationship between the two RT components assumes that $\tilde{\mathbf{Z}}$ and $\mathbf{R}$ in (1) are stochastically independent (Luce & Green, 1972; Kohfeld *et al.*, 1981a, b), from which it follows that $\mathbf{C}$ and $\mathbf{R}$ in (4) are stochastically independent too, and $\mathbb{E}[\mathbf{C} \mid \mathbf{R} = R_p] = \mathbb{E}[\mathbf{C}] = C$, a positive constant. Equation (4) therefore acquires the form

$$\mathrm{RT}_p(A) = R_p + Cs(A) + o\{s(A)\}. \tag{9}$$

This pattern is distinctly different from that described by (8): all intercept-tangents now have one and the same slope (Fig. 3).

The patterns shown in Figs. 2 and 3 seem to be the only two plausible patterns for the asymptotic behavior of RT quantiles. A range of quantile curves with decreasing slopes of their intercept tangents would mean that as amplitude increases beyond a sufficiently high value, the RT quantiles increase. This seems highly unlikely, if not directly refuted by available evidence (and this certainly contradicts the assumption made in the Introduction that amplitude A forms a "unidimensional strength continuum"). It seems also unlikely that RT quantile curves might exhibit the parallel pattern of Fig. 3 in some quantile rank regions while being divergent in other regions.

The two models of stochastic relationship represented by Figs. 2 and 3 lead to very different computational procedures aimed at extracting the signal-dependent component $\mathbf{T}(A)$ from the overall RT—which is the primary goal of analysis if $\mathbf{T}(A)$ is interpreted, justifiably or not, as the duration of sensory processing. In the stochastic independence model, $\mathbf{T}(A)$ is extracted by deconvolving the distribution of $\mathbf{R}$ (estimated by RT to a very large signal) from the distribution of $\mathbf{RT}(A)$. In the SVRT model, the dependence of $\mathrm{RT}_p(A)$ on A is a deterministic function for any given rank p, and all one has to do to obtain a rank-p quantile of $\mathbf{T}(A)$ is to arithmetically subtract the quantile R_p from the same-rank quantile $\mathrm{RT}_p(A)$. This procedure is both simpler and more reliable. In a broader RT decomposition context, Dzhafarov and Schweickert (1995) showed that the assumption of a perfect positive interdependence between RT components leads to a much simpler analysis of RT distributions, both conceptually and technically, than the more traditional assumption of stochastic independence.

Note that competing models of stochastic relationship between RT components are generally only testable in

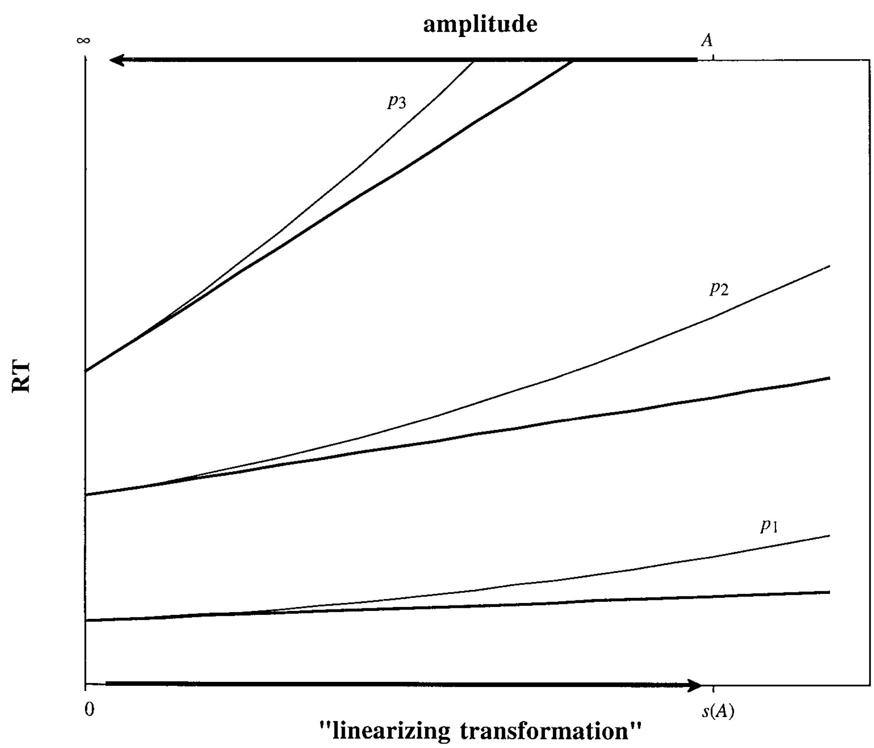

FIG. 2. The asymptotically diverging pattern of quantile curves predicted by the SVRT model ($p_1 < p_2 < p_3$ are arbitrary quantile ranks).

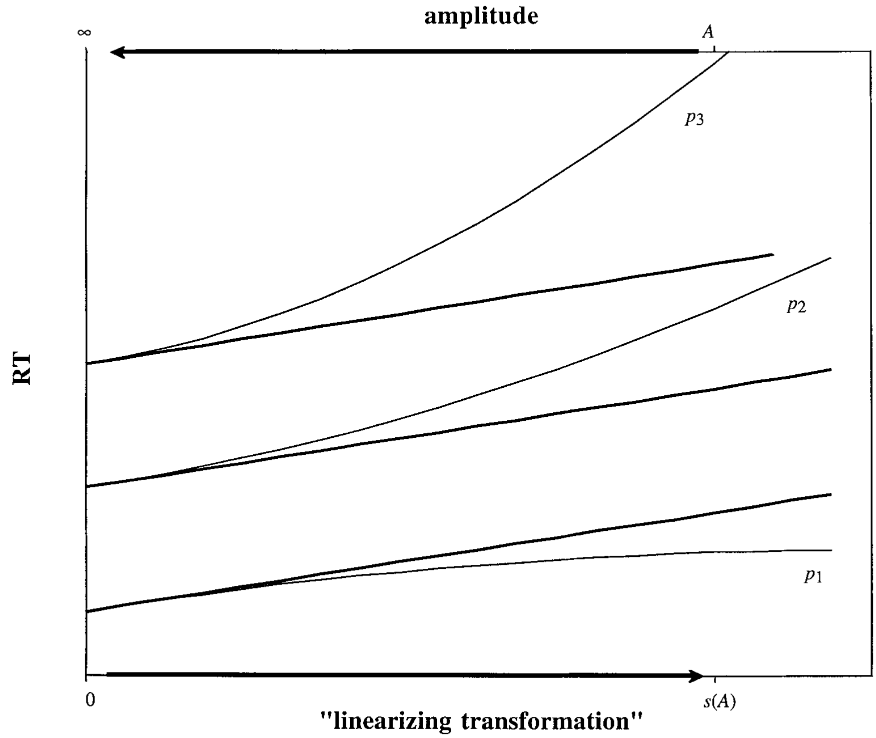

FIG. 3. The asymptotically parallel pattern of quantile curves predicted by the stochastic independence model ($p_1 < p_2 < p_3$ are arbitrary quantile ranks).

conjunction with other specific assumptions about the RT components. Thus it is critical to observe that in the present context the stochastic independence and SVRT models only yield different predictions (Figs. 2 and 3) because $\mathbf{T}(A)$ is defined as the "minimal" signal-dependent component satisfying a certain asymptotic proportionality assumption, (2). To see that these constraints cannot be weakened, assume that $\mathbf{RT}(A) = \mathbf{T}^*(A) + \mathbf{R}^*$, where $\mathbf{T}^*(A)$ stochastically decreases but does not converge to a constant as A increases.[4] Instead, assume that it converges to a non-degenerate random variable $\mathbf{L}^*$. In such a case, presenting $\mathbf{T}^*(A) = T^*(A, \mathbf{P})$ as $T(A, \mathbf{P}) + L^*(\mathbf{P})$, and assuming that properties (b) and (c) stated in the Introduction hold for $T(A, \mathbf{P})$, it is easy to show that the RT quantile curves form the diverging pattern of Fig. 2 even if $\mathbf{T}^*(A)$ and $\mathbf{R}^*$ are stochastically independent.

Having properly defined the RT components $\mathbf{T}(A)$ and $\mathbf{R}$ in the SVRT and stochastic independence models, observe that the patterns shown in Figs. 2 and 3 are only necessary consequences but not sufficient conditions for the respective models. Indeed, it is quite possible, for instance, that $\mathbb{E}[\mathbf{C} \mid \mathbf{R} = R_p] = \mathbb{E}[\mathbf{C}] = \text{const}$ (i.e., the parallel pattern of Fig. 3 holds) even though $\mathbf{C}$ and $\mathbf{R}$ are not stochastically independent. It is also quite possible that $\mathbb{E}[\mathbf{C} \mid \mathbf{R} = R_p]$ increases with quantile rank p (i.e., the diverging pattern of Fig. 2 holds) even though the two RT components are not perfectly positively interrelated. Therefore, by trying to match RT data with the patterns shown in Figs. 2 and 3, one does not directly test the independence model against the SVRT model. Rather, one tests a variety of models asymptotically indistinguishable from the independence model against a variety of models asymptotically indistinguishable from the SVRT model (see Theorem 3.2.1 in Dzhafarov, 1992). In the rest of this paper, however, we allow ourselves to simply identify Figs. 2 and 3 with the SVRT model and the stochastic independence model, respectively.

[4] Convergence to a constant is immediately reducible, by renaming the components, to convergence to zero.

3. EMPIRICAL ANALYSIS OF RT QUANTILE CURVES

If RT distributions were known precisely (on a population level) and for all signal amplitudes beyond a certain value, then no difficulty would be involved in determining which of the two patterns of the asymptotic behavior of RT quantiles holds. In reality, however, RT distributions are represented by finite samples corresponding to a few distinct amplitudes. As a result, the task of estimating slopes of the intercept tangents becomes formidable. Dzafarov's (1992) approach to this problem is to choose amplitudes so large that the quantile curves themselves appear linear with respect to $s(A)$, which means that $o\{s(A)\}$ in (4) is negligibly small. The slopes in question then can be determined by a standard linear regression procedure.

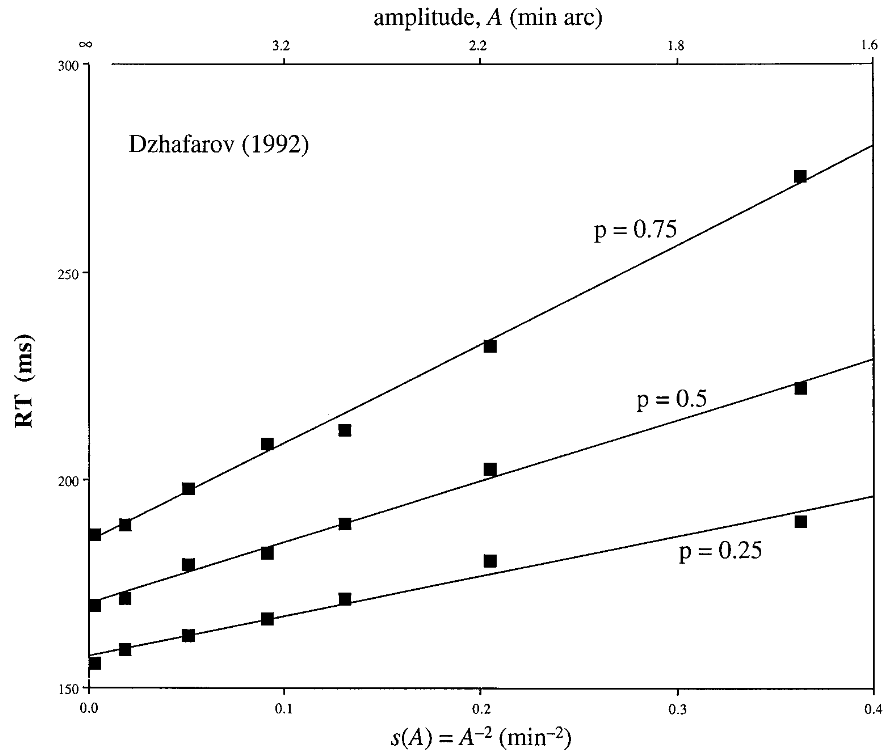

FIG. 4. RT quartile curves from Dzhafarov (1992), arithmetically average over two subjects (about 200 RTs per subject per amplitude; the averaging is justifiable to the extent the individual quantile curves are close to linear functions).

Figure 4 provides an example of this approach. It presents results of an experiment in which subjects respond to instantaneous changes in spatial position of a small light source. The amplitude A of the displacement forms a unidimensional strength continuum because as A increases, the detectability and perceptual salience of the displacement increases whereas RT stochastically decreases. In the region of A shown, the RT quantile curves (of which only three quartile curves are presented) are approximately linear with respect to $s(A) = A^{-2}$, in accordance with a motion detection model presented in Dzhafarov *et al.* (1993). The regression slopes obviously increase with quantile rank, in fact, about 5 times faster than the regression intercepts, and Dzhafarov (1992) correctly observed that this pattern was consistent with the SVRT model (Fig. 2). He also concluded, however, that the data definitely rejected the stochastic independence model (Fig. 3), and this conclusion turns out to be open to criticism.

The weakness of this conclusion lies in the following possibility. If the linearity of the RT quantile curves when plotted against $s(A)$ is only approximate, then the linear regression lines approximating the RT quantiles in a finite range of amplitudes may very well be systematically biased with respect to the true intercept tangents. The regression lines, in particular, may form a diverging fan even when the true intercept tangents are parallel. To appreciate this possibility, consider Fig. 5 that shows RT quantile curves generated according to the SVRT model and the stochastic independence model. The two upper graphs are identical to Figs. 2 and 3, with the diverging (SVRT) and parallel (independence) patterns of the intercept tangents. The middle graphs show that one can choose a region of sufficiently small values of $s(A)$ so that the linear regression lines approximating RT quantiles in this region virtually coincide with the intercept tangents, exhibiting therefore the same patterns of divergence and parallelness as in Figs. 2 and 3. This illustrates the fact that the regression technique used in Fig. 4 may successfully discriminate between the competing models in a well-chosen region of small $s(A)$-values.[5] The lower graphs, show that if one chooses a wider region, then the overall regression lines may systematically deviate from the true intercept tangents; in the case shown, the regression lines form a diverging fan pattern even when the intercept tangents are parallel. As a result, if one is satisfied with the quality of linear approximations in the right-bottom graph, one might erroneously conclude that the data reject the independence model.

To formalize this graphical illustration, let the RT quantiles be presented as

$$\mathrm{RT}_p(s) = R_p + \mu s + \varphi(s, p). \tag{10}$$

[5] Hereafter the term "region (or range) of small s-values" refers to a positive neighborhood of $s = 0$.

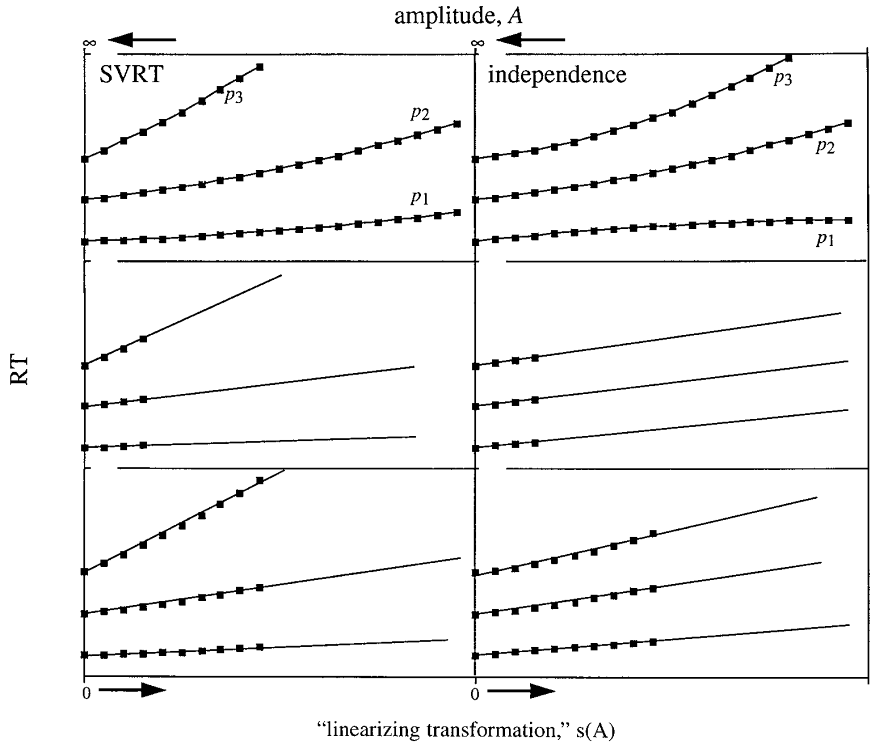

FIG. 5. The logic of the data analysis designed to distinguish between the SVRT and stochastic independence models. In the middle and lower panels the symbols represent the regions of RT quantile curves (shown "in entirety" in the corresponding upper panel) that are used to compute the linear regression lines.

If $s = s(A)$ is a decreasing and vanishing function, and if $\varphi(s, p)$ is $o\{s\}$, then (10) is equivalent to (9), that is, it describes the prediction of the stochastic independence model, with $R_p + \mu s$ describing constant-slope intercept tangents. Suppose that

$$\frac{\partial \varphi}{\partial p} > 0 \quad \text{and} \quad \frac{\partial^2 \varphi}{\partial p\, \partial s} > 0, \tag{11}$$

that is, $\varphi(s, p)$, the deviation of the quantile curve from its intercept tangent, increases with p and it increases the faster the greater the values of s. Under these assumptions, it can be shown that for any set of distinct values $\{s_1, s_2, ..., s_n\}$, the least-squares linear regression lines approximating the corresponding rank-q quantiles $\{\mathrm{RT}_p(s_1), \mathrm{RT}_p(s_2), ..., \mathrm{RT}_p(s_n)\}$ form a diverging pattern (i.e., their slopes increase with p). A generalized and sample-oriented version of this statement is proved in Appendix as Theorem 1. The proof can be easily generalized to a broad class of loss functions other than the sum of squared deviations.

It is not difficult to find sufficient conditions under which RT distributions generated according to the stochastic independence model would yield quantile curves whose deviations from their intercept-tangents satisfy (11). Let

$$\mathbf{T}(A) = \mathbf{C}s(A), \tag{12}$$

which is a special case of (3), with $o\{s(A)\} = 0$. The equation means that the signal-dependent RT component $\mathbf{T}(A)$ is precisely, rather than just asymptotically, factorizable into a product of a criterion $\mathbf{C}$ and some decreasing and vanishing function $s(A)$. Assume further that $\mathrm{RT}_p(s)$ have a continuous bounded second derivative with respect to s on some positive neighborhood of $s = 0$, for any quantile rank p. Then it can be proved (Theorem 2 in Appendix) that under the assumption of stochastic independence

$$\mathrm{RT}_p(s) = R_p + E[\mathbf{C}]\, s - \frac{1}{2}\frac{g'(R_p)}{g(R_p)} \mathrm{Var}[\mathbf{C}]\, s^2 + o\{s^2\}, \tag{13}$$

where g is the density function for $\mathbf{R}$. Comparing this with (10) we have

$$\varphi(s, p) = -\frac{1}{2}\frac{g'(R_p)}{g(R_p)} \mathrm{Var}[\mathbf{C}]\, s^2 + o\{s^2\}.$$

By choosing a region of sufficiently small s-values (i.e., sufficiently large amplitudes A) the term $o\{s^2\}$ can be dropped, and it follows that $\phi(s, p)$ satisfies (11) if, and only if,

$$\left[\frac{g'(R_p)}{g(R_p)}\right]' = [\log(g(R_p)]'' < 0. \tag{14}$$

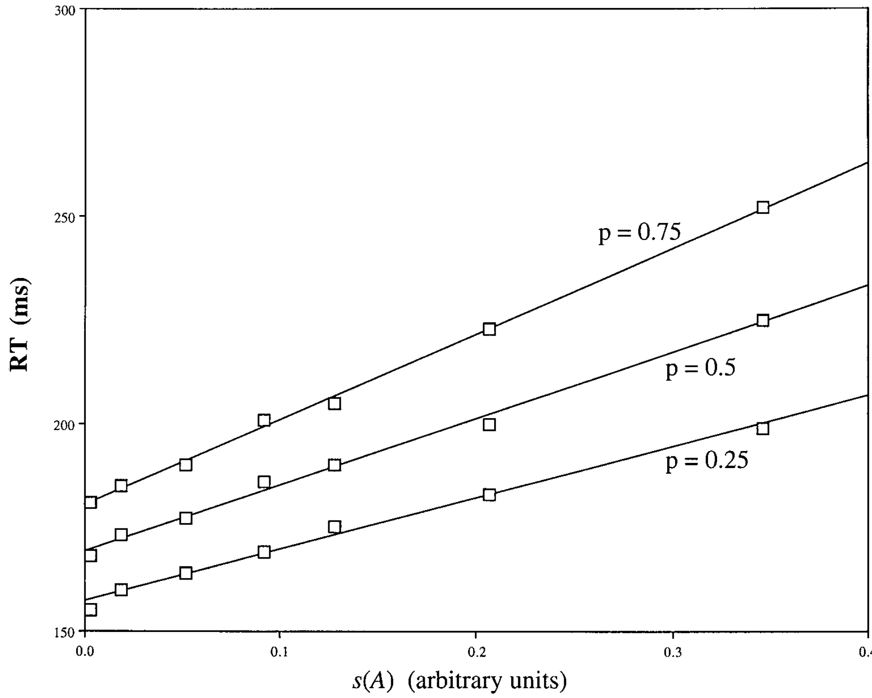

FIG. 6. Simulated RT quantiles generated according to a special case of the stochastic independence model: $\mathbf{RT}(A) = \mathbf{C}s(A) + \mathbf{R}$, where **C** and **R** are independent and gamma-distributed with parameters $\{\text{scale} = 0.0115\ \text{ms}^{-1}, \text{shape} = 2\}$ and $\{\text{scale} = 0.425\ \text{ms}^{-1}, \text{shape} = 72\}$, respectively. The RT quantiles are computed from 375 RTs per amplitude on the average.

Thus whether or not (11) is satisfied depends essentially on the signal-independent component **R** only. If (14) holds at least within a certain range of quantile ranks (as it does for many distributions, such as gamma or normal), then the regression lines within this range will form a diverging fan pattern in a certain area of small s-values, an area in which one can ignore all $o\{s^2\}$-terms but not the linear and quadratic terms.

Figure 6 presents computer simulation results that illustrate this possibility. The RT distributions whose quartiles are plotted in the figure have been generated according to the stochastic independence model with the signal-dependent component satisfying (12). That is, the model is

$$\mathbf{RT}(A) = \mathbf{C}s(A) + \mathbf{R}, \tag{15}$$

with stochastically independent **C** and **R**. Seven values of $s(A)$ have been chosen to match those in Fig. 4, with the RT sample sizes being also approximately the same, 375 RTs per amplitude on the average.[6] Both **C** and **R** are gamma-distributed with the parameters indicated in the legend; they have been chosen so that the means and standard deviations of the simulated RTs are approximately equal to those in Dzhafarov's (1992) data at the two marginal amplitudes, the highest and the lowest. The regression lines approximating the RT quantiles clearly form a diverging fan pattern, not dissimilar to that in Fig. 4 (even though less pronounced).

It should be clear from the previous discussion (see Fig. 5) that this result is due to the fact that the region of s-values shown in Fig. 6 is "too broad," and the curvilinearity of the quantile curves is sufficient to create a systematic deviation of the regression lines from the true intercept tangents. One should expect then that the diverging pattern in Fig. 6 should diminish if one restricts the regression analysis to a shorter range of small s-values: the shorter the range the closer the expected slopes of the intercept tangents should be to each other. Open symbols in Fig. 7 show this to be the case with Fig. 6: the increase of regression slopes with regression intercepts essentially disappears as one progressively restricts the abscissa range. By contrast, the filled symbols in Fig. 7 show that no such effect exists for the data presented in Fig. 4.

One should be cautious, of course, not to assign an undue weight to such an analysis, because statistical reliability of regression estimates diminishes when computed from progressively fewer data points spaced progressively denser. To rule out the possibility that the flattening of the slope-intercept curves in Fig. 7 (open symbols) is spurious, we have replicated the simulation scheme described in its legend three times, and to get a more detailed look, from each replication we have computed RT quantiles of the ranks 0.20, 0.33, 0.5, 0.67, and 0.75. Figure 8 (open symbols) shows the flattening effect quite clearly. For a comparison,

[6] Equal probabilities rather than equal numbers of presentations have been fixed for different amplitudes in all our computer simulations.

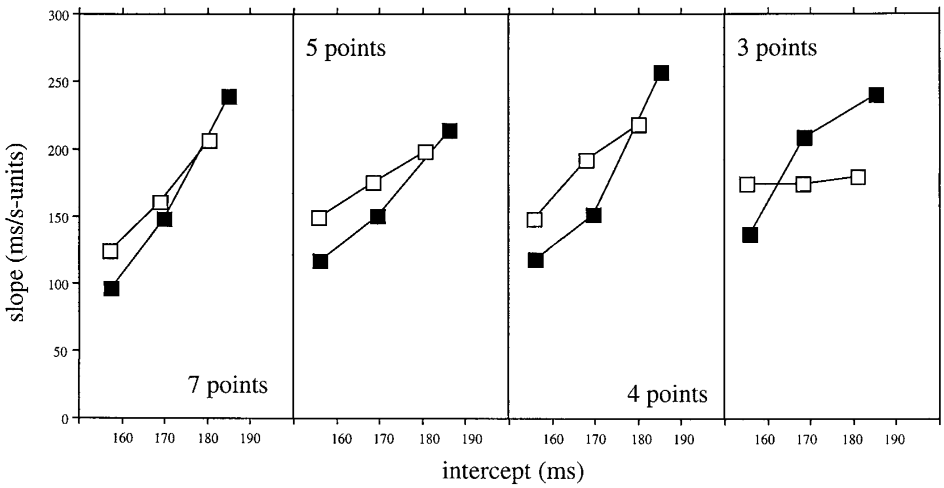

FIG. 7. Relationship between regression slopes and intercepts computed from Fig. 4 (filled symbols) and Fig. 6 (open symbols) for different ranges of small s-values (involving all 7, or only 5, 4, or 3 smalles s-values). In each curve the symbols read from left to right correspond to the quantile ranks 0.25, 0.50, and 0.75, respectively.

the figure also presents the same-rank quantiles computed from Dzhafarov's (1992) data (filled symbols) that clearly do not show this effect. All this corroborates the hypothesis that the diverging fan pattern in these data (Fig. 4) indeed reflects their conformity with the SVRT model,[7] rather than being a statistical artefact, as it is with the simulated data in Fig. 6.

It is not the aim of this paper, however, to determine which of the two competing models of stochastic relationship fits better in a particular experimental situation. Rather, we are interested in a methodological question: whether the pattern of changes in the slope-intercept curves computed from realistically sized RT samples for progressively shrinking regions of small s-values is *robustly* diagnostic in discriminating between the two models (when one of them is known to be true). Robustness is a key requirement here, because no theory of sampling distributions of the regression parameters for RT quantile curves can be constructed without strong additional assumptions about RT distributions.

To answer the question, the computer simulation scheme described earlier has been extended to incorporate different distributions of the RT components, both models of their stochastic relationship, and different sample sizes. Response times have been generated as

$$\mathbf{RT}(A) = \mathbf{C}s(A) + \mathbf{R},$$

with $\mathbf{C}$ and $\mathbf{R}$ being either stochastically independent or linearly interdependent (a special case of the SVRT model). As in Fig. 6, the seven values of $s(A)$ used in the simulations have been identical to those in Fig. 4, and the distributions of $\mathbf{C}$ and $\mathbf{R}$ have been chosen so that at the two marginal amplitudes the means and standard deviations of the simulated RTs are approximately equal to those in Dzhafarov's (1992) data. There have been four joint distributions of $(\mathbf{C}, \mathbf{R})$, two with stochastically independent and two with linearly interdependent $\mathbf{C}$ and $\mathbf{R}$, each of which has been used to generate RT samples of three sizes: 10000, 1000, and 250 RT values per amplitude on the average (see Footnote 6). Hereafter they are referred to as the large, medium, and small sample sizes, respectively. Quantile curves computed from the medium-size samples are presented in Figs. 9 and 10, with the distribution parameters being described in the figure legends. The pairing of the independence-model-generated and SVRT-model-generated data in these and subsequent figures is by the form of the distribution of $\mathbf{C}$, which could be either gamma (Fig. 9) or uniform (Fig. 10).[8]

The slope-intercept curves shown in Figs. 11 and 12 leave little doubt that at least for the medium and large RT samples the pattern of changes in the curves allows one to reliably tell the two models apart. Observe that decreasing parts of the curves can only be spurious and should be interpreted as indicating equal slopes. The difference between the two models becomes even more apparent if one focuses one's attention on the near-median quantiles only, ignoring the (less reliable) leftmost and rightmost points in each curve. The flattening of the slope-intercept curves for independence-model-generated data is faster for the uniform-$\mathbf{C}$ data (Fig. 12, open symbols) than for the gamma-$\mathbf{C}$ data (Fig. 11, open symbols). It is easy to account for this effect

[7] This argument is mentioned but not elaborated in Dzhafarov (1992) in relation to his Fig. 8, inset.

[8] Observe that the parameters of these distributions are different for the two models, in order to ensure the same mean and variance values at the marginal amplitudes in both cases.

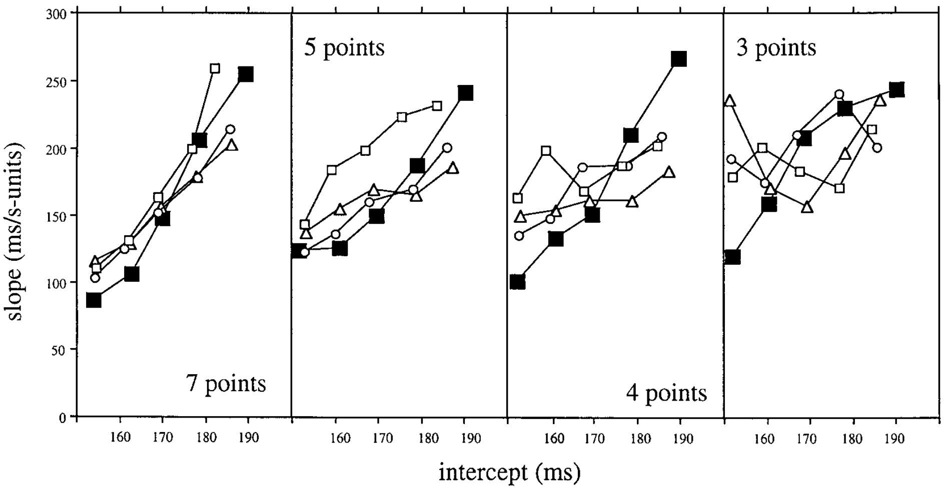

FIG. 8. Relationship between regression slopes and intercepts computed for RT quantiles of the ranks 0.20, 0.33, 0.50, 0.67, and 0.75 (from left to right in each curve) and for different ranges of small s-values (as in Fig. 7). Filled symbols represent Dzhafarov's (1992) data (the same used in Fig. 4). Open symbols represents three replications of the simulation scheme described in the legend of Fig. 6 (the same sample size).

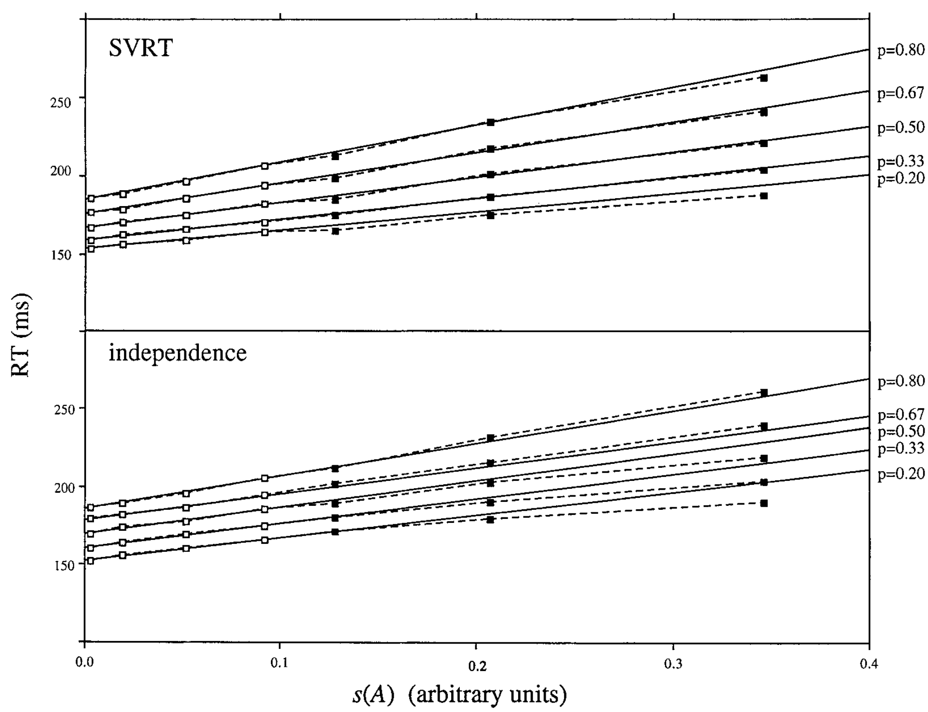

FIG. 9. Simulated RT quantile curves generated according to the model $\mathbf{RT}(A) = \mathbf{C}s(A) + \mathbf{R}$, where **C** and **R** are either independent or linearly interdependent. In the independence case **C** and **R** are distributed as described in the legend of Fig. 6. In the SVRT case **C** is gamma-distributed with parameters $\{\text{scale} = 0.0233\ \text{ms}^{-1}, \text{shape} = 4\}$, whereas **R** is a linear transformation of **C** yielding the same mean and variance as in the independence case. The RT quantiles shown are computed from 1,000 RTs per amplitude on the average. The linear approximations are computed for the regions of the quantile curves shown by open symbols.

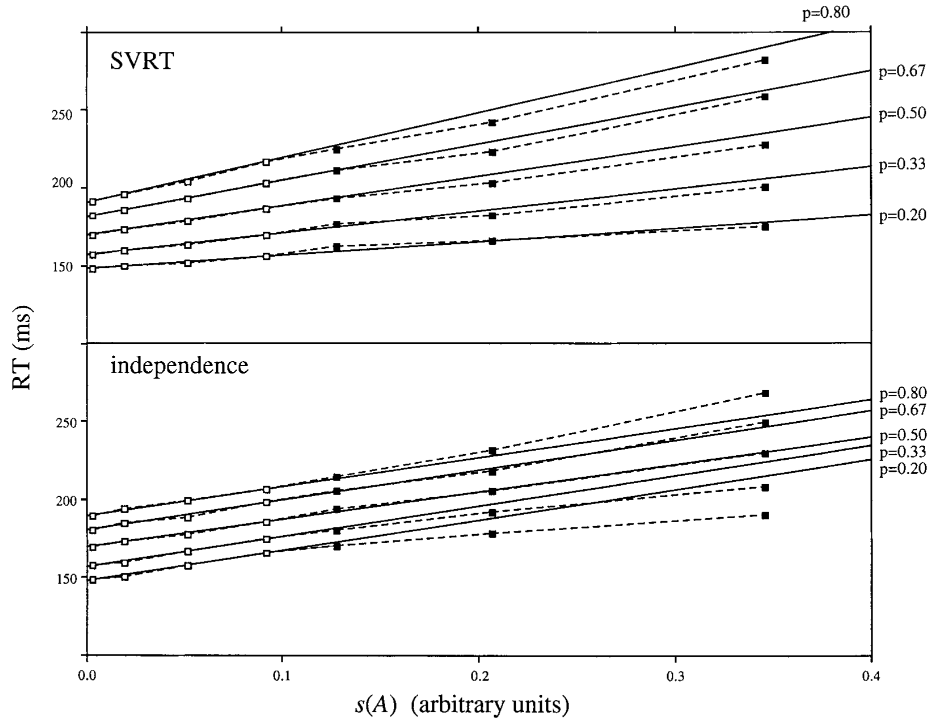

FIG. 10. The same as in Fig. 9, except that **C** and **R** are distributed uniformly. The distribution interval for **R** is (135 ms, 205 ms). The distribution interval for **C**, in ms/$s(A)$-units, is (0, 346) in the stochastic independence case, and (100, 276) in the SVRT case.

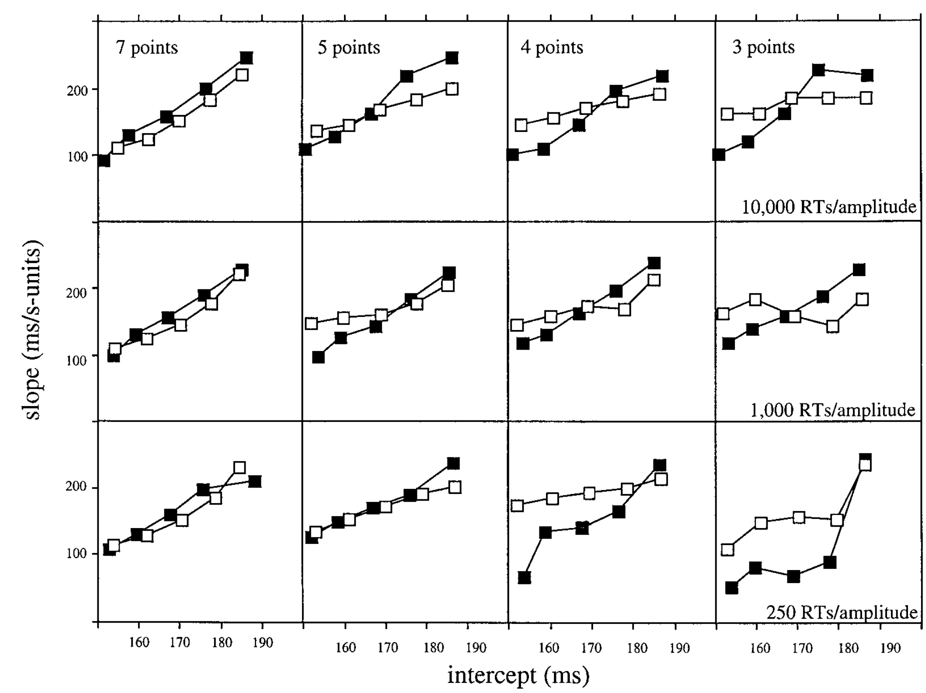

FIG. 11. Relationship between regression slopes and intercepts computed for simulated RT quantiles of the ranks 0.20, 0.33, 0.50, 0.67, and 0.75 (from left to right in each curve) and for different ranges of small s-values. The simulation scheme is as described in the legend of Fig. 9 (which corresponds to the middle panels here). SVRT-model-generated data are shown by filled symbols, open symbols represent independence-model-generated data.

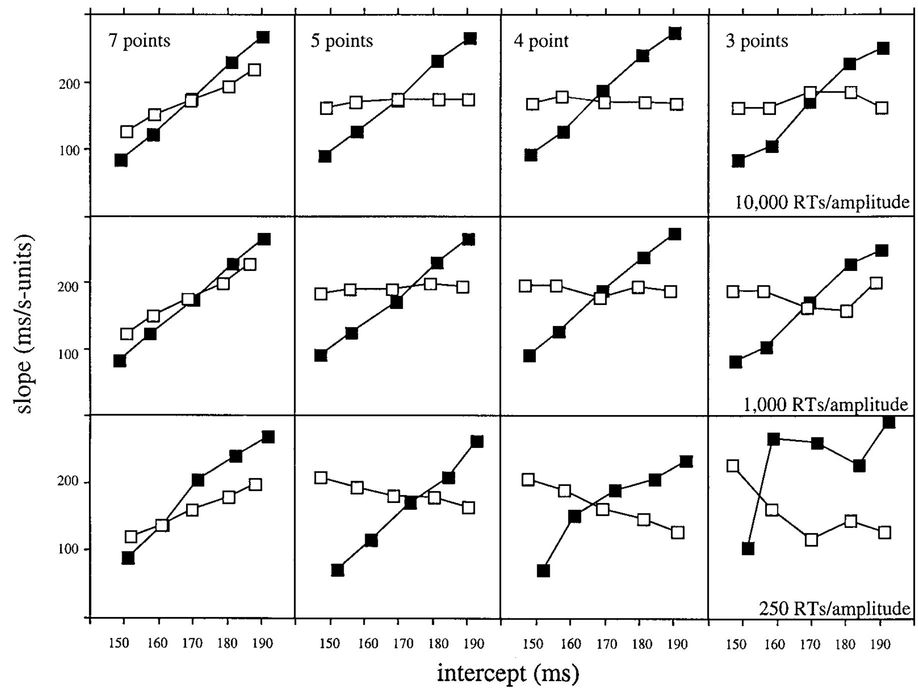

FIG. 12. The same as Fig. 11, but for the uniform-**C** simulation scheme described in Fig. 10.

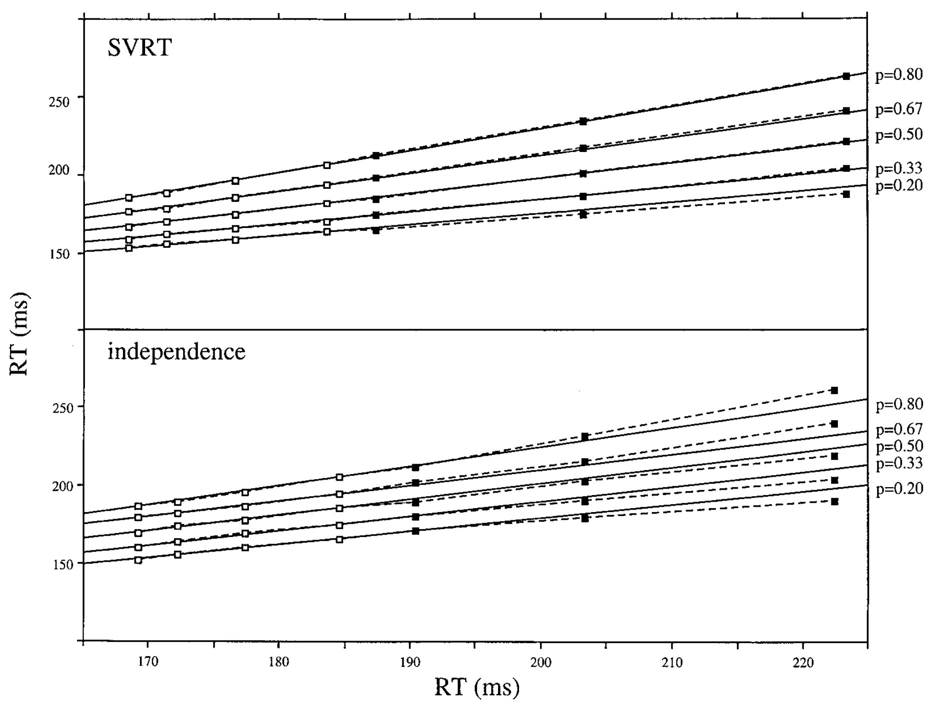

FIG. 13. The same as Fig. 9, except that the quantile curves are plotted against their arithmetic mean.

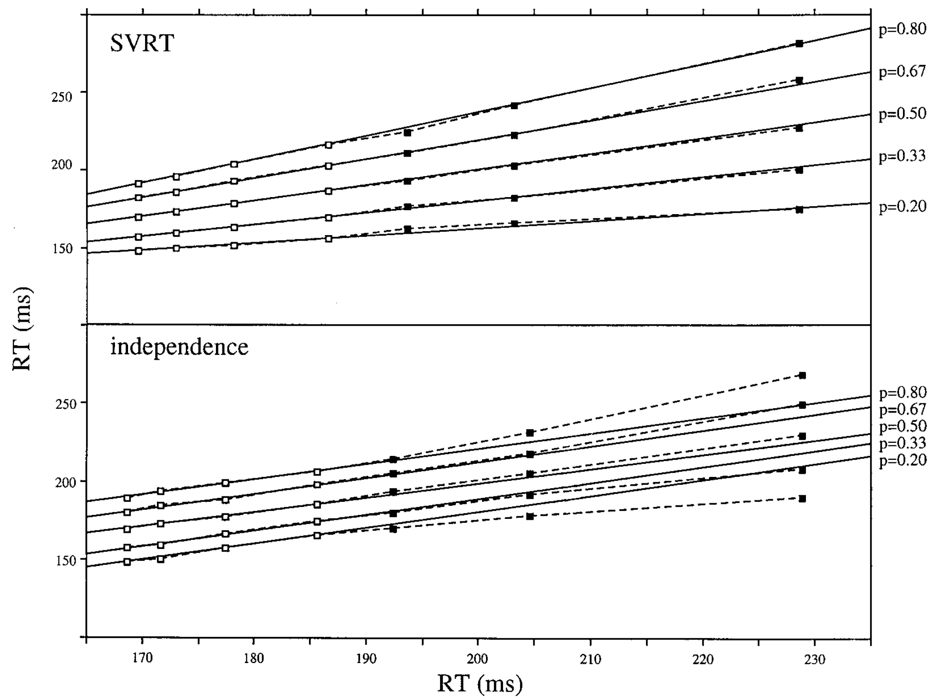

FIG. 14. The same as Fig. 10, except that the quantile curves are plotted against their arithmetic mean.

by referring to (13). Since **R** in the uniform-**C** data is also uniformly distributed (see the legend of Fig. 10), and since

$$\left[\frac{g'(R_p)}{g(R_p)}\right]' = 0$$

when g is a uniform density, the deviation of the quantile curves from their intercept-tangents in these data is $o\{s^2\}$, rather than just $o\{s\}$ as in the gamma-**C** data. This means that as the range of small s-values shrinks, the deviation in the uniform-**C** case tends to zero infinitely faster.

The analyses presented so far have been predicated upon the assumption that the asymptotically linearizing transformation $s(A)$ is known theoretically. As mentioned in the Introduction, however, this knowledge is not necessary to achieve a reliable discrimination between the two models of stochastic relationship. An alternative approach is based on the following modification of the argument presented in Dzhafarov (1992, Section 3.4). Let $\{\mathrm{RT}_p(A)\}_{p \in P}$ be a

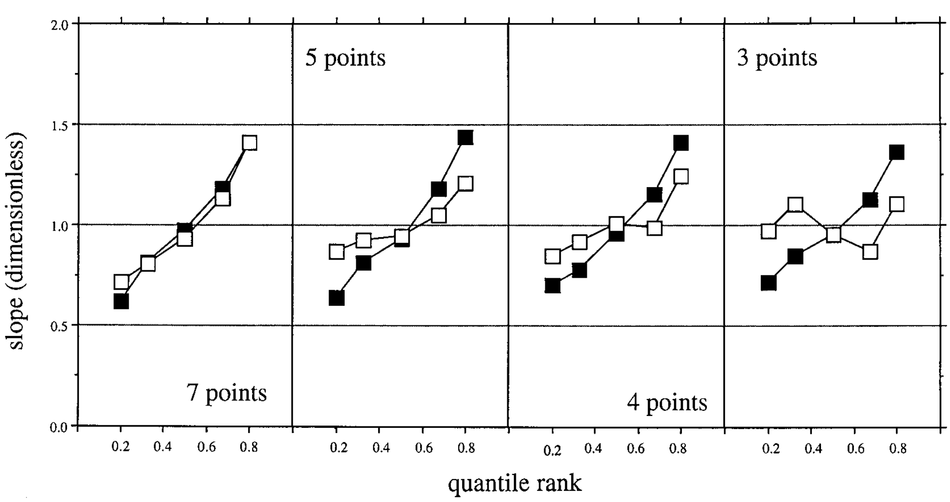

FIG. 15. Relationship between quantile ranks and regression slopes in Fig. 12. Symbols are the same as in Fig. 11.

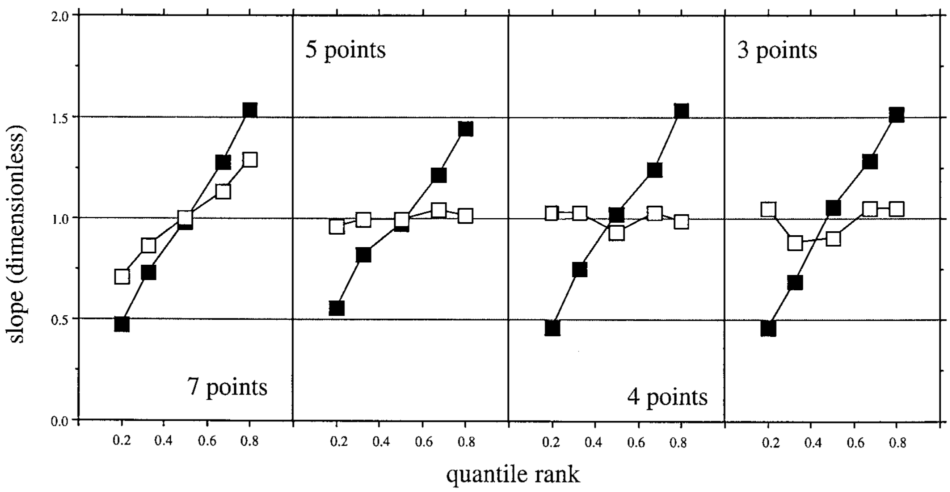

FIG. 16. Relationship between quantile ranks and regression slopes in Fig. 13. Symbols are the same as in Fig. 12

collection of RT quantile curves, and let $\mathrm{RT}_\bullet(A)$ be their mean. It is obvious from (4) that

$$\mathrm{RT}_\bullet(A) = R_\bullet + \mathbb{C}_\bullet s(A) + o\{s(A)\},$$

where $\mathbb{C}_\bullet$ is the mean of $\mathbb{E}[\mathbf{C} \mid \mathbf{R} = R_p]$-values across $p \in P$. It follows by simple algebra that

$$\mathrm{RT}_p(A) = \left[R_p - \frac{\mathbb{E}[\mathbf{C} \mid \mathbf{R} = R_p]}{\mathbb{C}_\bullet} R_\bullet \right] + \frac{\mathbb{E}[\mathbf{C} \mid \mathbf{R} = R_p]}{\mathbb{C}_\bullet} \mathrm{RT}_\bullet(A) + o\{s(A)\}. \quad (16)$$

This means that for any quantile rank $p \in P$, if $\mathrm{RT}_p(A)$ is plotted against $\mathrm{RT}_\bullet(A)$, then this line is asymptotically linear, and the slope of the tangent line drawn to it at $s(A) = 0$ is $\mathbb{E}[\mathbf{C} \mid \mathbf{R} = R_p]/\mathbb{C}_\bullet$. We will refer to the $\mathrm{RT}_p(A)$-versus-$\mathrm{RT}_\bullet(A)$ curves as the RT *quantile-versus-mean* curves; for terminological simplicity, its tangent line at $s = 0$ can be referred to as the intercept tangent, even though at $s(A) = 0$, $\mathrm{RT}_\bullet = R_\bullet$.

In the case of the SVRT model, (16) becomes

$$\mathrm{RT}_p(A) = \left[R_p - \frac{C_p}{\mathbb{C}_\bullet} R_\bullet \right] + \frac{C_p}{\mathbb{C}_\bullet} \mathrm{RT}_\bullet(A) + o\{s(A)\}, \quad (17)$$

and we see that the slopes of the intercept tangents to RT quantile-versus-mean curves in this case monotonically increase with quantile rank. Since $\mathbb{C}_\bullet$ is the mean of all C_p-values, the slopes must increase from a value below unity to a value above unity. In the case of the stochastic independence model, (16) assumes the form

$$\mathrm{RT}_p(A) = [R_p - R_\bullet] + \mathrm{RT}_\bullet(A) + o\{s(A)\}, \quad (18)$$

and we see that the slopes of the intercept tangents to RT quantile-versus-mean curves here are all equal to unity. One can now replicate virtually verbatim the previous analysis of the relationship between regression lines and intercept tangents to come to the following conclusion: as the range of small s-values progressively shrinks, the expected regression slopes for RT quantile-versus-mean curves should progressively flatten as a function of quantile rank and converge to unity in the case of the stochastic independence model; and they should converge to an increasing function crossing the unity level in the case of the SVRT model.

Figures 13 and 14 present the RT quantile-versus-mean curves for our medium sample size simulations, that is, the data previously presented in Figs. 9 and 10. Their slope versus quantile rank analysis is shown in Figs. 15 and 16.[9] Obviously, the pattern of changes in the curves here is as clear as, if not clearer than, that in the slope-intercept curves analyzed earlier. Again, the distinction between the two models becomes even more apparent if one focuses on the near-median quantiles only, and again the flattening and convergence is faster for the uniform-**C** simulations. To save space we do not show the results for the large and small sample size simulations. The conclusion is that beginning with the medium size RT samples, the pattern of changes in the slope versus quantile rank curves for progressively shrinking regions of small s-values is robustly diagnostic in discriminating between the two models.

Summarizing, whether or not the asymptotically linearizing transformation $s(A)$ is known theoretically, there exist reliable ways to empirically discriminate between the two models of stochastic relationship considered in this paper. Thinking about possible modifications of the analysis, its

[9] For simplicity, all regression lines have been computed with the abscissa taken as an independent variable (i.e., ignoring the fact that each abscissa value is the mean of the corresponding ordinate values).

empirical power can be further increased by designing the experiments so that higher (and more informative for the analyses) amplitudes are represented by larger sample sizes.

APPENDIX

THEOREM 1. *Let within some interval I of s-values and some interval P of quantile ranks p,*

$$\mathrm{RT}_p(s) = R_p + \mu s + \varphi(s, p),$$

and let

$$\frac{\partial \varphi}{\partial p} > 0 \qquad and \qquad \frac{\partial^2 \varphi}{\partial p\, \partial s} > 0.$$

Let $\{s_i\}$, $i = 1, ..., n$, *be some distinct values in I, and* $\{\widehat{\mathrm{RT}}_p(s_i)\}$ *be unbiased estimates of* $\{\mathrm{RT}_p(s_i)\}$ *for some* $p \in P$. *Finally, let* $\hat{b}_p$ *be the slope of the least-squares linear regression of* $\{\widehat{\mathrm{RT}}_p(s_i)\}$ *on* $\{s_i\}$. *Then* $E[\hat{b}_p]$ *is an increasing function of p.*

Proof. Since $\mathrm{RT}_p(s)$ is an unbiased estimator of $\mathrm{RT}_p(s)$, $E[\hat{b}_p]$ is the slope of the least-squares linear regression of $\{\mathrm{RT}_p(s_i)\}$ on $\{s_i\}$:

$$E[\hat{b}_p] = k \sum_{i=1}^{n} (s_i - \bar{s})\, \mathrm{RT}_p(s_i) = \mu + k \sum_{i=1}^{n} (s_i - \bar{s})\, \varphi(s_i, p),$$

where $k = (\sum_{i=1}^{n} s_i^2 - n\bar{s}^2)^{-1} > 0$. Then

$$\frac{dE[\hat{b}_p]}{dp} = k \sum_{i=1}^{n} (s_i - \bar{s}) \frac{\partial \varphi(s_i, p)}{\partial p} = k \left[\sum_{s_i < \bar{s}} (s_i - \bar{s}) \frac{\partial \varphi(s_i, p)}{\partial p} + \sum_{s_i \geqslant \bar{s}} (s_i - \bar{s}) \frac{\partial \varphi(s_i, p)}{\partial p} \right].$$

The value of $\partial \varphi(s, p)/\partial p$ is positive and increases with s. Hence the two summands above decrease if we replace $\partial \varphi(s, p)/\partial p$ in them with $\partial \varphi(\bar{s}, p)/\partial p$, and we have

$$\frac{dE[\hat{b}_p]}{dp} = k \sum_{i=1}^{n} (s_i - \bar{s}) \frac{\partial \varphi(s_i, p)}{\partial p} > k \sum_{i=1}^{n} (s_i - \bar{s}) \frac{\partial \varphi(\bar{s}, p)}{\partial p} = k \frac{\partial \varphi(\bar{s}, p)}{\partial p} \sum_{i=1}^{n} (s_i - \bar{s}) = 0.$$

This concludes the proof.

THEOREM 2. *Let* $\mathbf{RT}(s) = \mathbf{C}s + \mathbf{R}$, *with stochastically independent* $\mathbf{C}$ *and* $\mathbf{R}$. *For any quantile rank p, let* $\mathrm{RT}''_p(s)$ *exist and be continuous and bounded on a positive neighborhood of* $s = 0$. *Then*

$$\mathrm{RT}_p(s) = R_p + E[\mathbf{C}]\, s - \frac{1}{2} \frac{g'(R_p)}{g(R_p)} \mathrm{Var}[\mathbf{C}]\, s^2 + o\{s^2\}$$

where g is the density function for $\mathbf{R}$.

Proof. Consider the Taylor expansion of $G(\mathrm{RT}_p(s) - sc)$ about $s = 0$, where G denotes the distribution function for $\mathbf{R}$ and c a possible value of $\mathbf{C}$. Obviously,

$$\frac{dG(\mathrm{RT}_p(s) - sc)}{ds} = g(\mathrm{RT}_p(s) - sc)[\mathrm{RT}'_p(s) - c],$$

$$\frac{d^2G(\mathrm{RT}_p(s) - sc)}{ds^2} = g'(\mathrm{RT}_p(s) - sc)[\mathrm{RT}'_p(s) - c]^2 + g(\mathrm{RT}_p(s) - sc)\, \mathrm{RT}''_p(s)$$

(obviously, $\mathrm{RT}'_p(s)$ exists and is continuous and bounded in a neighborhood of $s = 0$). At $s = 0$, $\mathrm{RT}_p(0) = R_p$, and

$$G(\mathrm{RT}_p(s) - sc)|_{s=0} = G(R_p),$$

$$\left.\frac{dG(\mathrm{RT}_p(s) - sc)}{ds}\right|_{s=0} = g(R_p)[\mathrm{RT}'_p(0) - c],$$

$$\left.\frac{d^2G(\mathrm{RT}_p(s) - sc)}{ds^2}\right|_{s=0} = g'(R_p)[\mathrm{RT}'_p(0) - c]^2 + g(R_p)\, \mathrm{RT}''_p(0),$$

where $\mathrm{RT}'_p(0) = \lim \mathrm{RT}'_p(s)$ and $\mathrm{RT}''_p(0) = \lim \mathrm{RT}''_p(s)$ as $s \to 0$. Hence the Taylor-expansion has the form

$$G(\mathrm{RT}_p(s) - sc) = G(R_p) + g(R_p)[\mathrm{RT}'_p(0) - c]\, s + \{g'(R_p)[\mathrm{RT}'_p(0) - c]^2 + g(R_p)\, \mathrm{RT}''_p(0)\} \frac{s^2}{2} + o\{s^2\}.$$

Denoting by F the distribution function of $\mathbf{C}$, and integrating over the spectrum of $\mathbf{C}$, we have

$$\int G(\mathrm{RT}_p(s) - sc)\, dF(c) = G(R_p) + g(R_p)\{\mathrm{RT}'_p(0) - E[\mathbf{C}]\}s + g'(R_p)\{\mathrm{RT}'_p(0)^2 - 2\mathrm{RT}'_p(0)\, E[\mathbf{C}] + E[\mathbf{C}^2]\} \frac{s^2}{2} + g(R_p)\, \mathrm{RT}_p(0) \frac{s^2}{2} + o\{s^2\}.$$

Since

$$\int G(\mathrm{RT}_p(s) - sc)\, dF(c) = \mathrm{Prob}\{\mathbf{R} + \mathbf{C}s < \mathrm{RT}_p(s)\}$$
$$= \mathrm{Prob}\{\mathbf{RT}(s) < \mathrm{RT}_p(s)\} = p,$$

and

$$G(R_p) = \mathrm{Prob}\{\mathbf{R} < R_p\} = p,$$

we conclude that

$$g(R_p)\{\mathrm{RT}'_p(0) - E[\mathbf{C}]\}\, s + g'(R_p)\{\mathrm{RT}'_p(0)^2 - 2\mathrm{RT}'_p(0)\, E[\mathbf{C}] + E[\mathbf{C}^2]\}\, \frac{s^2}{2} + g(R_p)\, \mathrm{RT}''_p(0)\, \frac{s^2}{2} = o\{s^2\}.$$

Since the coefficients at s and s^2 are constants with respect to s, this equation can only hold if they equal zero:

$$g(R_p)\{\mathrm{RT}'_p(0) - E[\mathbf{C}]\} = 0,$$

that is,

$$\mathrm{RT}'_p(0) = E[\mathbf{C}],$$

and

$$g'(R_p)\{\mathrm{RT}'_p(0)^2 - 2\mathrm{RT}'_p(0)\, E[\mathbf{C}] + E[\mathbf{C}^2]\} + g(R_p)\, \mathrm{RT}''_p(0)$$
$$= g'(R_p)\{E[\mathbf{C}]^2 - 2E[\mathbf{C}]\, E[\mathbf{C}] + E[\mathbf{C}^2]\} + g(R_p)\, \mathrm{RT}''_p(0)$$
$$= g'(R_p)\, \mathrm{Var}[\mathbf{C}] + g(R_p)\, \mathrm{RT}''_p(0) = 0.$$

As a result,

$$\mathrm{RT}'_p(0) = E[\mathbf{C}] \qquad \text{and} \qquad \mathrm{RT}''_p(0) = -\frac{g'(R_p)}{g(R_p)}\, \mathrm{Var}[\mathbf{C}],$$

from which the statement of the theorem follows immediately.

ACKNOWLEDGMENTS

This research was supported in part by NSF Grant DBS-9614278 to the University of California. We are grateful to Xiagen Hu, Geoffrey Iverson, and especially Duncan Luce for many helpful discussions.

REFERENCES

Dzhafarov, E. N. (1991). Additive decomposition of simple reaction time. Paper presented at the 24th Meeting of the Society for Mathematical Psychology, Aug. 1991.

Dzhafarov, E. N. (1992). The structure of simple reaction time to step-function signals. *Journal of Mathematical Psychology*, **36**, 235–268.

Dzhafarov, E. N. (1993). Grice-representability of response time distribution families. *Psychometrika*, **58**, 281–314.

Dzhafarov, E. N., & Schweickert, R. (1995). Decompositions of response times: An almost general theory. *Journal of Mathematical Psychology*, **39**, 285–314.

Dzhafarov, E. N., Sekuler, R., & Allik, J. (1993). Detection of changes in speed and direction of motion: Reaction time analysis. *Perception and Psychophysics*, **54**, 733–750.

Kohfeld, D. L., Santee, J. L., & Wallace, N. D. (1981a). Loudness and reaction time, I. *Perception and Psychophysics*, **29**, 535–549.

Kohfeld, D. L., Santee, J. L., & Wallace, N. D. (1981b). Loudness and reaction time, II. Identification of detection components at different intensities and frequencies. *Perception and Psychophysics*, **29**, 550–562.

Luce, R. D. (1986). *Response Times*. New York: Oxford University Press.

Luce, R. D., & Green, D. M. (1972). A neural timing theory for response times and the psychophysics of intensity. *Psychological Review*, **79**, 14–57.

Received: February 9, 1995