The Equivalence of Two Ways of Computing Distances from Dissimilarities for Arbitrary Sets of Stimuli

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Abstract

Given a set endowed with pairwise dissimilarities, the Dissimilarity Cumulation procedure computes the (quasi)distance between any two elements of the set as the infimum of the sums of dissimilarities across all finite chains of elements connecting the two elements. For finite sets this procedure is known to be equivalent to recursive corrections for violations of the triangle inequality in any sequence of ordered triads of points which contains every triad a sufficient number of times. This paper extends this equivalence to infinite sets.

For a finite stimulus set $\mathfrak{S}$, a *dissimilarity* is a function $D : \mathfrak{S} \times \mathfrak{S} \to \mathbb{R}$ which is nonnegative and equal to zero if and only if its two arguments coincide. The dissimilarity $D(a, b)$, which we will write as Dab, is usually an empirically observable quantity, such as the mean numerical estimate of dissimilarity by an observer, or, in Fechnerian scaling, either of the following two "psychometric increments" (let x Diff y abbreviate "x is judged to be different from y"):

$$\Pr[a \text{ Diff } b] - \Pr[a \text{ Diff } a]$$

and

$$\Pr[b \text{ Diff } a] - \Pr[a \text{ Diff } a].$$

Here the probabilities are defined on $\mathfrak{S}\times\mathfrak{S}$ following certain "canonical" transformation of the stimuli (Dzhafarov, 2002; Dzhafarov and Colonius, 2006). While by itself D imposes a rather weak structure on the set $\mathfrak{S}$, it allows one to compute, for each pair of stimuli a, b, a quantity interpretable as a "subjective distance from a to b." This is done as follows. Denoting by $X = x_1 \cdots x_n$ $(n \geq 1)$ a finite sequence of stimuli (referred to as a *chain* and written as a string, without commas), denoting by aXb the chain $ax_1 \cdots x_n b$, and putting

$$DaXb = Dax_1 + \sum_{1\leq i\leq n-2} Dx_i x_{i+1} + Dx_{n-1}b, \qquad (1)$$

we define

$$Gab = \min_X DaXb, \qquad (2)$$

where we write Gab in place of $G(a, b)$, and the minimum is taken over all chains X in $\mathfrak{S}$. The function G is easily seen to be a *(quasi)metric*, that is, a dissimilarity function that satisfies the triangle inequality (but which is not necessarily symmetric). It is referred to as the *(quasi)metric induced by the dissimilarity D*.[1] A symmetrization, if needed, can be obtained by taking $Dab + Dba$, and will not concern us in this paper.

It is shown in Dzhafarov (2010a) that Gab can be also be computed from Dab by means of another procedure. We present it in a modified form to better link it to the construction we introduce in the next section. Let $Tri(\mathfrak{S})$ be the set of all ordered triples $(a, b, c) \in \mathfrak{S}^3$ with $a \neq b$, $a \neq c$, and $b \neq c$. Call each such triple a *triad*. Consider any sequence T of triads in which every triad occurs an infinite number of times. Suppose that we move along T from one triad to another and every time when we find that $Dab > Dac + Dcb$ (i.e., the triangle inequality is violated), we replace Dab with $Dac + Dcb$ and consider this sum a new, redefined value of Dab. Then, after a finite number of such steps the redefined D will coincide with the quasimetric G induced by the original D (whence subsequent steps will no longer change D). The number of these steps can be arbitrarily large, but, as it is known from the Floyd-Warshall algorithm (Floyd, 1962), it can be made no larger than n^3, where n is the cardinality of $\mathfrak{S}$.[2]

It is not immediately obvious how this recursive procedure of correcting for violations of the triangle inequality should be defined in the case of an infinite stimulus set $\mathfrak{S}$; and if appropriately defined, whether for infinite sets too the "eventual" result of such corrections is guaranteed to be achieved and coincide with the metric induced by the original dissimilarity in accordance with Dissimilarity Cumulation (DC) theory (Dzhafarov & Colonius, 2007; Dzhafarov, 2008a-b, 2009, 2010b). We show in this paper that

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[1] The reason for writing the qualifier "quasi" in parentheses is that G is a quasimetric in traditional terminology but also, as explained below, a metric in the nomenclature of Dissimilarity Cumulation theory (see Dzhafarov, 2010b).

[2] In the Floyd-Warshall algorithm, having enumerated the elements of $\mathfrak{S}$ 1 to n, the replacement of Dab with $\min\{Dab, Dac + Dcb\}$ occurs within three nested cycles (each combination whereof we call a step): $c = 1$ to n, nesting $a = 1$ to n, nesting $b = 1$ to n. At the end of this triple-cycle all violations of the triangle inequality are guaranteed to be corrected. If one excludes degenerate triads, the number of steps in the triple-cycle is $n(n-1)(n-2)$.

the answer to the latter question is affirmative after the correction procedure has been extended to arbitrary sets by means of transfinite recursion. We provide a general account of ordinals and transfinite recursion, sufficient for our purposes, in Appendix 3.

1. DISSIMILARITY AND RELATED NOTIONS FOR ARBITRARY SETS

For an arbitrary stimulus set $\mathfrak{S}$, we follow notation conventions already used in the introduction. Chains X are finite sequences of elements of $\mathfrak{S}$, written as strings: ab, abc, $x_1 \cdots x_n$, etc. XY is the concatenation of X and Y, so we can write aXb, $aXbYZc$, etc. For any function $F : \mathfrak{S} \times \mathfrak{S} \to \mathbb{R}$ the notation FX denotes 0 if X is the empty chain or a chain of length 1, and denotes $Fx_1x_2 + \cdots + Fx_{n-1}x_n$ if $X = x_1 \cdots x_n$, $n \geq 2$. Let $\mathcal{S}$ denote the set of all chains in $\mathfrak{S}$, including the empty chain.

We call any function $D : \mathfrak{S} \times \mathfrak{S} \to \mathbb{R}$ which is nonnegative and equal to zero if and only if its two arguments coincide a *pre-dissimilarity function.* The minimum in (2) need not generally exist and has to be replaced with

$$Gab = \inf_{X \in \mathcal{S}} DaXb. \tag{3}$$

The function Gab is nonnegative, equal to zero at $a = b$, and easily seen to satisfy the triangle inequality: by definition, for any $X, Y \in \mathcal{S}$,

$$Gab \leq DaXcYb = DaXc + DcYb,$$

so $Gab \leq Gac + Gcb$. The function G is not itself, however, a pre-dissimilarity function, as $Gab = 0$ does not imply $a = b$. Consider, for instance, $Dab = (b-a)^2$ with $\mathfrak{S} = \mathbb{R}$: in this case

$$\begin{aligned} Gab &= \inf_{X=x_1\cdots x_n}[(a-x_0)^2 \\ &\quad + \textstyle\sum_{1\leq i\leq n-1}(x_{i+1}-x_i)^2 + (b-x_n)^2] \\ &\leq \inf_{k\geq 1}\textstyle\sum_{1\leq i\leq k}((a+(i-1)\frac{b-a}{k}) - (a+i\frac{b-a}{k}))^2 \\ &= \inf_{k\geq 1}\textstyle\sum_{1\leq i\leq k}(\frac{b-a}{k})^2 = 0, \end{aligned}$$

for any reals a and b. The function G defined by (3) therefore is a *pseudo-quasi-metric* (*p.q.-metric*, for short) *induced by the pre-dissimilarity* D. For any $a, b \in \mathfrak{S}$, the value Gab will be referred to as the *p.q.-distance from a to b*.

A *metric*, in the nomenclature of DC theory, is a p.q.-metric M such that $Mab > 0$ for any $a \neq b$, and $Ma_nb_n \to 0$ implies $Mb_na_n \to 0$ for any sequences $\{a_n\}$ and $\{b_n\}$ in $\mathfrak{S}$.[3] To ensure that the p.q.-metric G induced by a pre-dissimilarity D is a metric, the pre-dissimilarity should be strengthened by additional properties, making it a *dissimilarity.* In DC theory these properties are: (1) $Da_nb_n - Da'_nb'_n \to 0$ for any sequences $\{a_n\}, \{a'_n\}, \{b_n\}, \{b'_n\}$ in $\mathfrak{S}$ such that $Da_na'_n \to 0$ and $Db_nb'_n \to 0$ (uniform continuity), and (2) $Da_nb_n \to 0$ for any sequences $\{a_n\}, \{b_n\}$ in $\mathfrak{S}$ such that $Da_nX_nb_n \to 0$ for some sequence $\{X_n\}$ in $\mathcal{S}$ (the "chain property"). For finite sets these properties are satisfied trivially, whence any pre-dissimilarity on a finite set is a dissimilarity, and the p.q.-metric induced by it a metric.

We shall show below that a certain procedure of recursively redefining a pre-dissimilarity function D will "eventually" (in a transfinite sense) always result in the p.q.-metric G induced by D, whether this p.q.-metric is a metric or not. In other words, we will assume that an infinite stimulus set $\mathfrak{S}$ is endowed with a pre-dissimilarity D which induces the p.q.-metric G according to (3). Then we will define a recursive procedure which consists in inspecting the ordered triads abc of the elements of $\mathfrak{S}$ one by one, and replacing Dab with $Dac + Dcb$ if the former exceeds the latter, precisely as it was done for finite $\mathfrak{S}$. The triads, however, will now have to be enumerated by transfinite numbers. We will prove that however this enumeration is performed, provided each triad is enumerated by a sufficiently large set of transfinite numbers, all violations of the triangle inequality in $\mathfrak{S}$ will have been corrected at some transfinite step, at which step the corrected pre-dissimilarity D will have been transformed into the p.q.-metric G. Moreover, we will estimate the cardinality of the set of the transfinite steps it will take to achieve G. We will show that if the inspections of the triads for violations of the triangle inequality are organized "economically," the cardinality in question coincides with that of the stimulus set $\mathfrak{S}$.

2. RECURSIVE CORRECTIONS IN INFINITE SETS

We assume in this section that the stimulus set $\mathfrak{S}$ is infinite. Denoting by Ord the class of all ordinals, let $T :$ Ord $\to Tri(\mathfrak{S})$ be any class function such that for every $abc \in Tri(\mathfrak{S})$ and every ordinal α there is an ordinal $\beta \geq \alpha$ with $T(\beta) = abc$. Fix a pre-dissimilarity $D : \mathfrak{S} \times \mathfrak{S} \to \mathbb{R}$, and let G be the induced p.q.-metric.

Definition 1. Define for each ordinal α a function $M^{(\alpha)} : \mathfrak{S} \times \mathfrak{S} \to \mathbb{R}$ by transfinite recursion:

1. $M^{(0)} = D$;

2. if $\alpha = \beta + 1$, then for all $a, b \in \mathfrak{S}$,

$$M^{(\alpha)}ab = \begin{cases} \min\{M^{(\beta)}ab, M^{(\beta)}ac + M^{(\beta)}cb\} & \text{if } T(\beta) = abc, \\ M^{(\beta)}ab & \text{otherwise;} \end{cases}$$

3. if α is a limit ordinal, then $M^{(\alpha)} = \inf_{\beta<\alpha} M^{(\beta)}$.

It is obvious from this definition that $M^{(\alpha)}ab$ is nonincreasing in α for any $a, b \in \mathfrak{S}$.

[3] This "symmetry in the small" property replaces the global symmetry, $Gab = Gba$, as the latter plays no useful role in DC theory. This notion of a metric is common in Finsler geometry (from which DC theory has evolved), where the global symmetry is often dropped and the symmetry in the small is always satisfied.

Lemma 2. *Let α be any ordinal.*

1. *For all $a, b \in \mathfrak{S}$, $M^{(\alpha)}ab \geq Gab$.*

2. *If $M^{(\alpha)}$ satisfies the triangle inequality, then for all $a, b \in \mathfrak{S}$, $M^{(\alpha)}ab = Gab$.*

Proof. We prove part 1 by transfinite induction. It clearly holds for $\alpha = 0$, since $D \geq G$. Let it hold for all $\beta < \alpha$. It clearly follows from Definition 1(3) that it holds for α if it is a limit ordinal. If $\alpha = \beta + 1$ for some β, then, by part 2 of Definition 1, for all $a, b \in \mathfrak{S}$, either

$$M^{(\alpha)}ab = M^{(\beta)}ab \geq Gab,$$

or

$$M^{(\alpha)}ab = M^{(\beta)}ac + M^{(\beta)}cb \geq Gac + Gcb \geq Gab,$$

where the last inequality holds because G satisfies the triangle inequality. This completes the proof.

To prove part 2, suppose $M^{(\alpha)}$ satisfies the triangle inequality. Then, for all $a, b \in \mathfrak{S}$ and every $X \in \mathcal{S}$, we have

$$M^{(\alpha)}ab \leq M^{(\alpha)}aXb \leq DaXb,$$

since $M^{(\alpha)} \leq M^{(0)} = D$. We conclude that

$$M^{(\alpha)}ab \leq \inf_{X \in \mathcal{S}} DaXb = Gab,$$

and the equality $M^{(\alpha)}ab = Gab$ follows then from part 1. □

Lemma 3.

1. *The class*

$$C_{ab} = \{\mu \in \mathrm{Ord} : M^{(\mu+1)}ab \neq M^{(\mu)}ab\}$$

is a countable set for every $a, b \in \mathfrak{S}$.

2. *The class*

$$C = \{\mu \in \mathrm{Ord} : M^{(\mu)} \neq M^{(\mu+1)}\}$$

is a set of cardinality $\leq |\mathfrak{S}|$.

Proof. To prove part 1, assume it is false. Then there exists a function $f : \omega_1 \to C_{ab}$, where ω_1 is the first uncountable ordinal, such that for all $\alpha < \beta < \omega_1$ we have $f(\alpha) < f(\beta)$. In particular, $f(\alpha) + 1 \leq f(\alpha + 1)$, so since $M^{(\gamma)}ab$ is nonincreasing in γ, we have,

$$M^{(f(\alpha))}ab > M^{(f(\alpha)+1)}ab \geq M^{(f(\alpha+1))}ab.$$

Thus we can pick a rational q_α such that $M^{(f(\alpha))}ab > q_\alpha > M^{(f(\alpha+1))}ab$. But then the map $\alpha \mapsto q_\alpha$ is an injection from ω_1 into $\mathbb{Q}$, which is impossible.

To prove part 2, first notice that the class C is a set since $\alpha \in C$ if and only if $\alpha \in C_{ab}$ for some $a, b \in \mathfrak{S}$. Thus

$$C = \bigcup_{a,b \in \mathfrak{S}} C_{ab},$$

whence $|C| \leq |\mathfrak{S}^2| \cdot |\omega| = |\mathfrak{S}|$. This completes the proof. □

Now we can prove our first main theorem.

Theorem 4. *There exists an ordinal α_M such that*

1. *$M^{(\alpha_M)} = M^{(\alpha)}$ for all $\alpha \geq \alpha_M$, and*

2. *$M^{(\alpha_M)}$ satisfies the triangle inequality.*

Thus, $M^{(\alpha_M)} = G$.

Proof. Let the set C_{ab} be as in Lemma 3, and let α_{ab} be the least ordinal larger than each $\alpha \in C_{ab}$. For any $\alpha \geq \alpha_{ab}$, we have $M^{(\alpha+1)}ab = M^{(\alpha)}ab$, whence if $M^{(\alpha)}ab = M^{(\alpha_{ab})}ab$ then $M^{(\alpha+1)}ab = M^{(\alpha_{ab})}ab$. If α is a limit ordinal and $M^{(\beta)}ab = M^{(\alpha_{ab})}ab$ for all $\alpha_{ab} \leq \beta < \alpha$, then

$$M^{(\alpha)}ab = \inf_{\beta < \alpha} M^{(\beta)}ab = \inf_{\alpha_{ab} \leq \beta < \alpha} M^{(\beta)}ab = M^{(\alpha_{ab})}ab,$$

where the second equality holds because $M^{(\gamma)}ab$ is nonincreasing in γ. By transfinite induction, it follows that for all $\alpha \geq \alpha_{ab}$, $M^{(\alpha)}ab = M^{(\alpha_{ab})}ab$. Now consider the class function $f : \mathfrak{S}^2 \to \mathrm{Ord}$ given by $f(a, b) = \alpha_{ab}$. As $\mathfrak{S}^2$ is a set and Ord is a proper class, f must be bounded by some ordinal, and we let α_M be the least such. It is readily seen that α_M satisfies part 1 of the statement of the theorem.

To see that α_M also satisfies part 2 of the statement, we argue by contradiction. Suppose there is a triad abc in $Tri(\mathfrak{S})$ such that

$$M^{(\alpha_M)}ab > M^{(\alpha_M)}ac + M^{(\alpha_M)}cb.$$

By definition of T, there exists some $\alpha \geq \alpha_M$ such that $T(\alpha) = abc$. By definition of α_M, $M^{(\alpha)} = M^{(\alpha_M)}$, so that

$$M^{(\alpha)}ab > M^{(\alpha)}ac + M^{(\alpha)}cb.$$

But then by Definition 1, $M^{(\alpha+1)}ab$ is $M^{(\alpha)}ac + M^{(\alpha)}cb$, whence $M^{(\alpha+1)}ab \neq M^{(\alpha)}ab = M^{(\alpha_M)}ab$. This is a contradiction, which completes the proof. That $M^{(\alpha_M)} = G$ now follows by Lemma 2. □

The cardinality of α_M in the previous theorem can be arbitrary, but only because nothing prevents one from defining the class function T in a "wasteful" way, e.g., by mapping all ordinals between two uncountable limit ordinals into one and the same triad abc. We can still estimate the cardinality of α_M under a reasonable, "economic" organization of T. For this, however, we need an auxiliary construction.

Definition 5. Let $T : \mathrm{Ord} \to Tri(\mathfrak{S})$ be any class function as described at the beginning of this section. Define, for each ordinal α, an ordinal ι_α by transfinite recursion as follows:

1. $\iota_0 = 0$;

2. if $\alpha = \beta + 1$, then ι_α is the least ordinal $> \iota_\beta$ such that for each $abc \in Tri(\mathfrak{S})$ there exists $\iota_\beta \leq \gamma < \iota_\alpha$ with $T(\gamma) = abc$;

3. if α is a limit ordinal, then $\iota_\alpha = \sup_{\beta < \alpha} \iota_\beta$.

Lemma 6. *If $M^{(\iota_\alpha)} = M^{(\beta)}$ for all $\iota_\alpha \le \beta \le \iota_{\alpha+1}$, then $M^{(\iota_\alpha)} = G$.*

Proof. If $M^{(\iota_\alpha)} = M^{(\beta)}$ for all $\iota_\alpha \le \beta \le \iota_{\alpha+1}$ then it must be that $M^{(\iota_\alpha)}$ satisfies the triangle inequality. Otherwise, fix the least $\beta \ge \iota_\alpha$ such that $T(\beta) = abc$ for some $a, b, c \in \mathfrak{S}$ with $M^{(\beta)}ab > M^{(\beta)}ac + M^{(\beta)}cb$. By definition, we must have $\beta < \iota_{\alpha+1}$, so also $\beta + 1 \le \iota_{\alpha+1}$. But then $M^{(\beta+1)}ab$ is defined to be $M^{(\beta)}ac + M^{(\beta)}cb$, meaning that $M^{(\beta+1)}ab \ne M^{(\beta)}ab = M^{(\iota_\alpha)}ab$, a contradiction. By Lemma 2, we must thus have $M^{(\iota_\alpha)} = G$. □

We now use this result to estimate the cardinality of α_M in Theorem 4 under an "economic" organization of the class function T. Denoting $[\iota_\alpha, \iota_{\alpha+1}) = \{\beta : \iota_\alpha \le \beta < \iota_{\alpha+1}\}$, by Definition 5, $T([\iota_\alpha, \iota_{\alpha+1})) = Tri(\mathfrak{S})$ for any α. The class function T is "economic" if, for any α, T maps $[\iota_\alpha, \iota_{\alpha+1})$ onto $Tri(\mathfrak{S})$ injectively. Such a class function can be constructed as follows. Fix a well-ordering $\preccurlyeq$ of $Tri(\mathfrak{S})$. Define T by transfinite recursion as follows. Fix an ordinal α, and assume $T(\beta)$ is defined for each $\beta < \alpha$, with $T(\beta) \in Tri(\mathfrak{S})$. We define $T(\alpha) \in Tri(\mathfrak{S})$. Let S_α be the set of all $abc \in Tri(\mathfrak{S})$ for which there is a $\beta < \alpha$ such that $T(\beta) \preccurlyeq abc$ and $T(\beta') \ne abc$ for all $\beta \le \beta' < \alpha$. If S_α is nonempty, set $T(\alpha)$ equal to the $\preccurlyeq$-least element of S_α. Otherwise, let $T(\alpha)$ be the $\preccurlyeq$-least element of $Tri(\mathfrak{S})$.

Theorem 7. *If the class function T is such that for any ordinal α, T maps the interval $[\iota_\alpha, \iota_{\alpha+1})$ onto $Tri(\mathfrak{S})$ injectively, then $|\alpha_M| \le |\mathfrak{S}|$.*

Proof. Let α_0 be the least ordinal such that $\iota_{\alpha_0} \ge \alpha_M$. From Lemma 6 it follows that for each $\alpha < \alpha_0$, there is some β with $i_\alpha \le \beta \le \iota_{\alpha+1}$ such that $M^{\iota_\alpha} \ne M^{\beta}$. The least such β cannot be a limit, since otherwise, for all $a, b \in \mathfrak{S}$, we would have $M^{(\beta)}ab = \inf_{\beta'<\beta} M^{(\beta')}ab = \inf_{\iota_\alpha \le \beta' < \beta} M^{(\beta')}ab = M^{(\iota_\alpha)}ab$. The least such β is therefore a successor, and we denote its predecessor by β_α. It follows that $M^{(\beta_\alpha)} \ne M^{(\beta_\alpha+1)}$, so that β_α belongs to the set C of Lemma 3. This defines an injective function $\{\iota_\alpha : \alpha < \alpha_0\} \to C$, so by Lemma 3, $\{\iota_\alpha : \alpha < \alpha_0\}$ has cardinality $\le |\mathfrak{S}|$. Since $|[\iota_\alpha, \iota_{\alpha+1})| \le |Tri(\mathfrak{S})|$ by assumption, we have

$$|\alpha_M| \le |\iota_{\alpha_0}| = \left| \bigcup_{\alpha<\alpha_0} [\iota_\alpha, \iota_{\alpha+1}) \right| = |\{\iota_\alpha : \alpha < \alpha_0\}| \cdot |Tri(\mathfrak{S})|$$
$$\le |\mathfrak{S}| \cdot |Tri(\mathfrak{S})| = |\mathfrak{S}| \cdot |\mathfrak{S}| = |\mathfrak{S}|.$$

This completes the proof. □

3. CONCLUSION

For both finite and infinite sets $\mathfrak{S}$, finding the induced p.q.-metric G from a pre-dissimilarity D is equivalent to correcting violations of the triangle inequality in all triads abc with elements in $\mathfrak{S}$. The corrections consist in replacing Dab with $Dac + Dcb$ whenever the former exceeds the sum. These corrections can be done sequentially, having enumerated the triads by ordinals (in the finite case, by natural numbers) in an arbitrary way, provided only that each triad is enumerated by unboundedly many ordinals (natural numbers). The latter means that every triad remains accessible after any position in the enumeration. In Dzhafarov (2010a) and the present paper it is shown that the corrections will stop at some step (at some natural number in the finite case, and at some infinite ordinal otherwise) because all violations of the triangle inequality at this step will have been corrected; the redefined pre-dissimilarity D will then coincide with G. The well-known Floyd-Warshall algorithm for finding $\min aXb$ for all elements a, b and chains X in a finite $\mathfrak{S}$ shows that the sequence of triads can be organized so that the cardinality of the eventual step does not exceed the cardinality of $Tri(\mathfrak{S})$. The same statement is shown in this paper to hold for infinite sets, where $|Tri(\mathfrak{S})| = |\mathfrak{S}|$.

APPENDIX: A Brief Account of Ordinals and Cardinals

We recall here some basic facts about ordinals and cardinals. We refer the reader to any basic text on set theory for complete details, e.g., Halmos (1974). We work within Zermelo-Fraenkel set theory with the axiom of choice (ZFC). The basic objects we deal with are *sets*, in terms of which all our other terms (functions, relations, etc.) may be defined. A *class* is formally a formula $\varphi(x)$ in the language of ZFC (with paremeters), and informally it is the "collection" C of all x that satisfy this formula. We abuse notation, and write $x \in C$ in place of $\varphi(x)$, and we say C *contains* x. Not all classes are sets, and these are called *proper* classes. (The class of all sets, for example, is a proper class.) We can also define *class functions* to be formulas $\varphi(x, y)$ such that for every x there is at most one y satisfying $\varphi(x, y)$. Intuitively, a class function is a map f from one class into another, and as a result we use the suggestive notation $f : C_1 \to C_2$, where C_1 is the class of all x for which there exists a y such that $\varphi(x, y)$ holds, and C_2 is any class containing all y such that $\varphi(x, y)$ holds for some x in the class C_1. In this case, for $x \in C_1$ we write $f(x)$ to indicate the unique $y \in C_2$ such that $\varphi(x, y)$ holds.

An *ordinal* is a set α such that each $\beta \in \alpha$ is a set and $\beta \subseteq \alpha$. Thus, for example,

$$\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \ldots$$

are all ordinals. It can be shown that for any two ordinals α and β, either $\alpha = \beta$, $\alpha \in \beta$, or $\beta \in \alpha$. We write $\alpha \le \beta$ to denote $\alpha = \beta$ or $\alpha \in \beta$; we write $\alpha < \beta$ to denote $\alpha \in \beta$. In the above example, we thus have

$$\emptyset < \{\emptyset\} < \{\emptyset, \{\emptyset\}\} < \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} < \cdots.$$

The class of all ordinals is not a set, and thus $\le$ is not a set relation. But $\le$ can be thought of as well-ordering the class of ordinals in the following sense: if S is any set of ordinals, then $\le$ restricted to the elements of S is a well-ordering of S. Thus, in particular, each ordinal is well-ordered by $\le$. Conversely, every well-ordered set can be mapped by an order-preserving bijection onto some ordinal. The ordinals are therefore canonical representatives of all possible well-order *types*, i.e., isomorphism classes of well-ordered sets.

For each ordinal α, $\alpha \cup \{\alpha\}$ is also an ordinal, and we call it the *successor* of α and denote it by $\alpha + 1$. If α is the successor of some ordinal, we call α a *successor* ordinal; otherwise, we call α a *limit* ordinal. If we identify $\emptyset$ with 0, and, having identified the natural number $n \geq 0$ with an ordinal, identify $n + 1$ with $n \cup \{n\}$, then the above displayed sequence becomes

$$0, 1, 2, 3, \ldots$$

and $\leq$ coincides with the standard ordering of the natural numbers. Here, 0 is not a successor ordinal, but each of $1, 2, 3, \ldots$ are. The ordinal

$$\{0, 1, 2, 3, \ldots\}$$

is denoted ω, and it is the ($\leq$-)least infinite ordinal, and the least limit ordinal after 0.

An ordinal α is a *cardinal* if it cannot be put into one-to-one correspondence with any ordinal $\beta < \alpha$. Thus, for example, each of $0, 1, 2, \ldots$ are cardinals (called the *finite* cardinals), as is ω. The ordinal $\omega + 1$, on the other hand, is not a cardinal, since it can be put into one-to-one correspondence with $\omega < \omega + 1$. The *cardinality* of a set S, denoted $|S|$, is the least ordinal S can be bijected with; that such an ordinal always exists follows by the axiom of choice, since it implies that S can be well-ordered. Thus, for example, $|\omega + 1| = |\omega|$ and $|\mathbb{Q}| = \omega$. It is clear that the cardinality of a set is a cardinal.

Systems of arithmetic can be developed on each of the class of ordinals and class of cardinals. These satisfy many familiar properties from arithmetic on the natural numbers, but also many properties which the naturals do not have. For the purposes of our work here, the only relevant properties are that if κ and λ are cardinals at least one of which is infinte then $\kappa \cdot \lambda = \max\{\kappa, \lambda\}$ (in particular, if S is any infinite set, then $|S| \cdot \omega = |S|$), and that if β is an ordinal and $\{A_\alpha : \alpha < \beta\}$ is a collection of disjoint sets each of cardinality $\leq \kappa$, then

$$\left| \bigcup \{A_\alpha : \alpha < \beta\} \right| \leq |\beta| \cdot \kappa.$$

Theorems involving ordinals are often proved by *transfinite induction*, which asserts that if some class contains an ordinal α whenever it contains all ordinals $\beta < \alpha$, then the class contains all ordinals. Similarly, we can define a class function on the ordinals by means of *transfinite recursion*: if, having defined it on all $\beta < \alpha$, we can use our definition to define it on α, then we define it on all ordinals. Within the induction or recursion procedure, it is sometimes more convenient to handle separately the case $\alpha = 0$, α a successor ordinal, and α a limit ordinal.

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