

# Contextuality is About Identity of Random Variables

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**Abstract.** Contextual situations are those in which seemingly “the same” random variable changes its identity depending on the conditions under which it is recorded. Such a change of identity is observed whenever the assumption that the variable is one and the same under different conditions leads to contradictions when one considers its joint distribution with other random variables (this is the essence of all Bell-type theorems). In our Contextuality-by-Default approach, instead of asking why or how the conditions force “one and the same” random variable to change “its” identity, any two random variables recorded under different conditions are considered different “automatically”. They are never the same, nor are they jointly distributed, but one can always impose on them a joint distribution (probabilistic coupling). The special situations when there is a coupling in which these random variables are equal with probability 1 are considered non-contextual. Contextuality means that such couplings do not exist. We argue that the determination of the identity of random variables by conditions under which they are recorded is not a causal relationship and cannot violate laws of physics.

*Keywords:* contextuality, distribution, entanglement, Kolmogorovian probability, random variables.

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## 1. Introduction

The main purpose of this paper is to explain the principle of Contextuality-by-Default (CbD) in nontechnical terms, and to demonstrate the conceptual clarity this principle brings in the analysis of random variables recorded under varying conditions. The formulation of the principle in our previous publications [1–5] involves the following component principles:

**(Indexation-by-conditions)** A random variable is *identified* (indexed, tagged) by all conditions under which its realizations are recorded.

**(Unrelatedness)** Two or more random variables recorded under mutually incompatible conditions are *stochastically unrelated*, i.e., they possess no joint distribution.

**(Coupling)** A set of pairwise stochastically unrelated random variables can be *probabilistically coupled*, i.e., imposed a joint distribution on; the choice of a coupling is generally non-unique.

CbD is complemented by the *All-Possible-Couplings* approach [1–4], according to which the constraints satisfied by a set of random variables observed under different conditions (e.g., the Tsirelson inequalities [9] satisfied by spins in the EPR/Bohm paradigm [10]) can be characterized by studying the set of all possible ways in which these random variables can be coupled. A general discussion of this approach is left out of this paper. We focus, however, on the (im)possibility of imposing on a set of random variables special, *identity couplings*, representing situations considered *non-contextual*. An identity coupling is one in which random variables recorded under different conditions are equal to each other with probability one. This approach to (non-)contextuality was first explored by Larsson [15].\*

CbD is squarely within the framework of the Kolmogorovian probability theory (KPT), although to keep the presentation nontechnical, we avoid using here explicit measure-theoretic formalisms (cf. [11–13] and, especially, [14]). Our position is that there are no empirical or theoretical considerations in quantum mechanics, cognitive science, or anywhere else, that involve random variables but cannot be fully described in the language of KPT. CbD can in fact be viewed as a principle that ensures the universality of the descriptive power of KPT. It is another matter (not elaborated in this paper, cf. [2]) that KPT may not be the most economic, convenient, or useful language for describing quantum phenomena.

\*Non-contextuality can be generalized to be represented by couplings that are as close to identity couplings as it is allowed by the marginal distributions of the variables involved [16, 17]. This allows one to extend the notion of contextuality to signaling systems. We leave this (very recent) development outside the scope of this paper.

## 2. Indexation-by-conditions

The term “conditions” for a random variable  $C$  refers to any variable  $\gamma$  whose values (random, predictable, or controllable at will) are paired with the observed realizations of these random variables. The most familiar pairing employed in empirical sciences is chronological: the values of  $\gamma$  and the values of  $C$  are recorded at the same time, say, in a series of observations.

Consider a toy example. Pat has a monitor that at any given time shows a pair of symbols, 00, 01, 10, or 11. The pairs follow each other in a very long sequence, such as

$$01, 11, 11, 10, 10, 10, 11, 11, 01, \dots \quad (1)$$

Assume, for simplicity, that Pat is unable to record their order,† so she simply counts the occurrences of each of 00, 01, 10, 11. Pat wishes to treat these pairs as four possible values of a random variable  $C$ , with a distribution

$$\begin{array}{cccc} 00 & 01 & 10 & 11 \\ p_{00} & p_{01} & p_{10} & p_{11} \end{array} . \quad (2)$$

So far,  $\gamma$  is an empty notion: it can be viewed as a variable having one and the same single value for each realization of  $C$ .

Assume now that Pat notices that the pairs 00, 01, 10, 11 representing  $C$  are sometimes shown in red color and sometimes in blue. The two colors may alternate randomly or in some regular fashion, e.g., red-blue-red-blue-... In either case, Pat is able to count the occurrences of the four different  $C$  values separately for the blue and for the red colors, and form thereby two random variables

$$\begin{array}{l} C_{red} \sim \begin{array}{cccc} 00 & 01 & 10 & 11 \\ p_{00} & p_{01} & p_{10} & p_{11} \end{array} , \\ C_{blue} \sim \begin{array}{cccc} 00 & 01 & 10 & 11 \\ p'_{00} & p'_{01} & p'_{10} & p'_{11} \end{array} , \end{array} \quad (3)$$

where  $\sim$  stands for “is distributed as”. In this representation, the two random variables have the same possible values, but are distinguished by the condition  $\gamma$  having two values, “blue” and “red”. That they are two different random variables is obvious if  $(p_{00}, p_{01}, p_{10}, p_{11}) \neq (p'_{00}, p'_{01}, p'_{10}, p'_{11})$ , but this is true even if the the two distributions in (3) are the same. One way to see this is to observe that these distributions can always be made different by viewing the conditions under which the variable was recorded as

†Enumerating observed realizations of a random variable amounts to introducing the ordinal number of the observation as a special condition under which the random variable is recorded. This presents no conceptual difficulties but complicates the discussion.

part of its value. That is, we can have

$$\begin{aligned}
 C_{red} &\sim \begin{array}{cccc} red\ 00 & red\ 01 & red\ 10 & red\ 11 \\ p_{00} & p_{01} & p_{10} & p_{11} \end{array}, \\
 C_{blue} &\sim \begin{array}{cccc} blue\ 00 & blue\ 01 & blue\ 10 & blue\ 11 \\ p'_{00} & p'_{01} & p'_{10} & p'_{11} \end{array},
 \end{aligned} \tag{4}$$

where each value has the structure “color  $ij$ ”. Even if  $(p_{00}, p_{01}, p_{10}, p_{11}) = (p'_{00}, p'_{01}, p'_{10}, p'_{11})$ , the two random variables have different distributions, simply because they have different possible values. Such a redefinition is always possible, and even when it is not convenient and one uses (3) instead, this consideration justifies accepting as a general principle that *different conditions always define different random variables*.

Obviously, the color of the symbols in this example can be replaced by any condition that can be systematically associated with the recorded values in sequence (1). Thus, Pat could simply distinguish odd-numbered and even-numbered presentations, or have her window sometimes open and sometimes closed when observing the symbols.

### 3. Joint Distributions and Stochastic Unrelatedness

Suppose now that Pat wishes to treat  $C$  as a vector consisting of two random variables,

$$\begin{aligned}
 A &= f_1(C) = \text{left-hand component of } C, \\
 B &= f_2(C) = \text{right-hand component of } C.
 \end{aligned} \tag{5}$$

Being functions of one and the same random variable  $C$ , the random variables  $A$  and  $B$  are *jointly distributed*, i.e., for every pair of values  $A = i$  and  $B = j$ , Pat can uniquely determine the probability with which these two values *co-occur*. In this case,

$$\begin{aligned}
 \Pr[A = i \text{ and } B = j] &= \Pr[C = ij], \\
 i, j &\in \{0, 1\}.
 \end{aligned} \tag{6}$$

The co-occurrence in this example is chronological:  $i$  and  $j$  occur simultaneously, within a single pair displayed on Pat’s monitor. But the deeper, more general meaning of the co-occurrence is that

- (i) there are function  $A = f_1(C)$  and  $B = f_2(C)$  of one and the same random variable  $C$ ; and
- (ii) the co-occurring values are  $i = f_1(c)$  and  $j = f_2(c)$  for any one value  $c$  of  $C$ .

The joint distribution of  $A$  and  $B$  in any such case is uniquely determined: for any value  $(i, j)$  of  $(A, B)$  one determines the set  $S_{ij}$  of the values  $c$  of  $C$  such that  $i = f_1(c)$  and  $j = f_2(c)$ , and one puts

$$\begin{aligned}
 \Pr[A = i \text{ and } B = j] &= \Pr[c \in S_{ij}], \\
 i, j &\in \{0, 1\}.
 \end{aligned} \tag{7}$$

The reverse of this statement is also true: if the joint distribution of  $(A, B)$  is well-defined, then  $A$  and  $B$  can be presented as functions of one and the same random variable

$C$ . To prove this, put  $C = (A, B)$  with  $\Pr[C = (i, j)]$  defined as  $\Pr[A = i \text{ and } B = j]$ . This amounts to using (??) as the definition of  $C$ . Then  $A$  and  $B$  as functions of  $C$  are defined by (5).

This can be generalized to any set of random variables of arbitrary nature: for any such a set, the random variables comprising it have a joint distribution if and only if they can be presented as functions of one and the same random variable [2, 11, 12, 14]. (Note that any set of jointly distributed random variables is a random variable: the difference between a “single” random variable and, say, a vector of several random variables is entirely superficial, and can always be eliminated by renaming of the values, e.g.,  $(0, 0), (0, 1), (1, 0), (1, 1)$  into  $1, 2, 3, 4$ .)

The situation is very different when we consider random variables recorded under mutually incompatible conditions (i.e., different values of  $\gamma$ ). Thus, unlike  $A$  and  $B$  in the above example, the variables  $C_{red}$  and  $C_{blue}$  never co-occur in the chronological sense, and Pat would not know how to assign probability values to the logically possible pairs

$$\begin{aligned} C_{red} = x \text{ and } C_{blue} = y, \\ x, y \in \{00, 01, 10, 11\}. \end{aligned} \tag{8}$$

Pat is not able to assess this probability by counting the occurrences of the different pairs, because she does not know which value of  $C_{red}$  she should pair with which value of  $C_{blue}$  to form an “observed” value of the hypothetical random variable  $C' = (C_{red}, C_{blue})$ . This situation is described by saying that  $C_{red}$  and  $C_{blue}$  are *stochastically unrelated*. In view of what was said above, it means that  $C_{red}$  and  $C_{blue}$  are not functions of any single random variable.

This does not mean, however, that  $C_{red}$  and  $C_{blue}$  cannot be *imposed* a joint distribution on. The precise meaning of this is as follows.

#### 4. Couplings

Suppose that Pat formed the pairs  $(x, y)$  in (8) according to some arbitrarily chosen scheme. Any such a pairing scheme defines a (probabilistic) *coupling* for  $C_{red}$  and  $C_{blue}$  [18]. Formally, a coupling for  $C_{red}$  and  $C_{blue}$  is a random variable  $Z = (X, Y)$  that satisfies

$$X \sim C_{red}, Y \sim C_{blue}. \tag{9}$$

Calling  $Z = (X, Y)$  a random variable means that  $X$  and  $Y$  are jointly distributed.  $Z$  is a coupling because the marginal distributions of  $X$  and  $Y$ , taken separately, are the same as those of, respectively,  $C_{red}$  and  $C_{blue}$  in (3) or (4). Note that  $Z$  is a random variables different from both  $C_{red}$  and  $C_{blue}$  (and in fact stochastically unrelated to them): by constructing a  $Z$  satisfying (9) one does not make  $C_{red}$  and  $C_{blue}$  jointly distributed or changed in any way.

The coupling  $Z$  is generally non-unique. In the matrix below,

	Y = 00	Y = 01	Y = 10	Y = 11	
X = 00	$p_{0000}$				$p_{00}$
X = 01		...			$p_{01}$
X = 10			...		$p_{10}$
X = 11				$p_{1111}$	$p_{11}$
	$p'_{00}$	$p'_{01}$	$p'_{10}$	$p'_{11}$	

(10)

any of the (generally infinite) fillings of the interior that agrees with the indicated marginal probabilities will define a possible coupling. The agreement with the marginal probabilities means

$$\begin{aligned}
 p_{x00} + p_{x01} + p_{x10} + p_{x11} &= p_x, \\
 p_{00y} + p_{01y} + p_{10y} + p_{11y} &= p'_y, \\
 x, y &\in \{00, 01, 10, 11\},
 \end{aligned}
 \tag{11}$$

which is merely an explicit version of (9). For instance, Pat can form  $Z = (X, Y)$  in such a way that

$$\begin{aligned}
 \Pr[X = x \text{ and } Y = y] &= \Pr[X = x] \Pr[Y = y] = p_x p'_y, \\
 x, y &\in \{00, 01, 10, 11\}.
 \end{aligned}
 \tag{12}$$

This  $Z$  is called an *independent coupling*, and it is universally impossible on any set of pairwise stochastically unrelated random variables (which is the reason stochastic unrelatedness is often confused with stochastic independence, which is a form of stochastic relationship).

Equations (9)-(11), however, rule out certain subclasses of couplings. Thus, Pat may be especially interested in whether she can simply treat  $C_{red}$  and  $C_{blue}$  as “essentially one and the same” random variable. The rigorous meaning of “essentially the same” is the *identity coupling*, defined by the conjunction of (9) and (11) with

$$\Pr[X = Y] = 1,
 \tag{13}$$

or, if Pat uses (4) instead of (3),

$$\Pr \left[ \begin{array}{l} X = red\ x \text{ and } Y = blue\ x \\ \text{for some } x \in \{00, 01, 10, 11\} \end{array} \right] = 1,
 \tag{14}$$

Obviously this identity coupling exists if and only if  $p_x = p'_x$  for all  $x$ , i.e., if and only if  $C_{red}$  and  $C_{blue}$  in representation (3) have the same distribution.

## 5. Same Identity vs Same Distributions

Being identically distributed, however, does not generally guarantee the possibility of the identity coupling. To see this, let us assume that Pat views  $C$  as a pair  $(A, B)$  defined by (5). In accordance with the indexing-by-conditions principle, she has then

$C_{red} = (A_{red}, B_{red})$  and  $C_{blue} = (A_{blue}, B_{blue})$ , i.e., both  $A$  and  $B$ , since they are recorded in conjunction with  $\gamma = \text{red/blue}$ , are to be indexed by these conditions. The distributions of  $C_{red}$  and  $C_{blue}$  are then represented by two joint distributions,

$$\begin{array}{|c|c|c|c|} \hline \gamma = \text{red} & B_{red} = 0 & B_{red} = 1 & \\ \hline A_{red} = 0 & p_{00} & p_{01} & p_{0\cdot} \\ A_{red} = 1 & p_{10} & p_{11} & p_{1\cdot} \\ \hline & p_{\cdot 0} & p_{\cdot 1} & \\ \hline \end{array} \quad (15)$$

$$\begin{array}{|c|c|c|c|} \hline \gamma = \text{blue} & B_{blue} = 0 & B_{blue} = 1 & \\ \hline A_{blue} = 0 & p'_{00} & p'_{01} & p'_{0\cdot} \\ B_{blue} = 1 & p'_{10} & p'_{11} & p'_{1\cdot} \\ \hline & p'_{\cdot 0} & p'_{\cdot 1} & \\ \hline \end{array}$$

Suppose first that Pat is only interested in whether she can treat  $A_{red}$  and  $A_{blue}$  as an “essentially the same” random variable (disregarding  $B$ ). This translates into the question of the existence of the identity coupling for  $A_{red}$  and  $A_{blue}$ , i.e., a random variable  $(X, X')$  with

$$X \sim A_{red}, X' \sim A_{blue}, \text{ and } \Pr[X = X'] = 1. \quad (16)$$

Repeating the reasoning of the previous subsection, Pat comes to the conclusion that such a coupling exists if and only if  $A_{red}$  and  $A_{blue}$  are identically distributed, i.e.,  $p_{0\cdot} = p'_{0\cdot}$ . The situation is analogous for  $B_{red}$  and  $B_{blue}$ : the identity coupling  $(Y, Y')$  for them exists if and only if  $B_{red} \sim B_{blue}$ , i.e.,  $p_{\cdot 0} = p'_{\cdot 0}$ .

Let now both these conditions be satisfied:  $A_{red} \sim A_{blue}$  and  $B_{red} \sim B_{blue}$ , i.e., let Pat deal with the distributions

$$\begin{array}{|c|c|c|c|} \hline \gamma = \text{red} & B_{red} = 0 & B_{red} = 1 & \\ \hline A_{red} = 0 & p_{00} & p_{01} & p_{0\cdot} \\ A_{red} = 1 & p_{10} & p_{11} & p_{1\cdot} \\ \hline & p_{\cdot 0} & p_{\cdot 1} & \\ \hline \end{array} \quad (17)$$

$$\begin{array}{|c|c|c|c|} \hline \gamma = \text{blue} & B_{blue} = 0 & B_{blue} = 1 & \\ \hline A_{blue} = 0 & p'_{00} & p'_{01} & p_{0\cdot} \\ B_{blue} = 1 & p'_{10} & p'_{11} & p_{1\cdot} \\ \hline & p_{\cdot 0} & p_{\cdot 1} & \\ \hline \end{array}$$

Any random variable  $Z = (X, Y, X', Y')$  such that

$$(X, Y) \sim (A_{red}, B_{red}), (X', Y') \sim (A_{blue}, B_{blue}) \quad (18)$$

is a coupling for  $(A_{red}, B_{red})$  and  $(A_{blue}, B_{blue})$ . It is easy to see that  $Z$  is also a coupling for separately taken  $A_{red}, B_{red}, A_{blue}, B_{blue}$ , because (18) implies

$$X \sim A_{red}, X' \sim A_{blue}, Y \sim B_{red}, Y' \sim B_{blue}. \quad (19)$$

Let now the question Pat poses for herself be whether the red/blue difference matters when considering *both*  $A$  and  $B$  together. This question translates into that of the possibility of  $Z$  being an identity coupling satisfying

$$\begin{aligned}\Pr[X = X'] &= 1, \\ \Pr[Y = Y'] &= 1.\end{aligned}\tag{20}$$

Even though  $A_{red} \sim A_{blue}$  and  $B_{red} \sim B_{blue}$ , such a coupling may not exist. It is clear that it exists (generally non-uniquely) if and only if the two joint distributions are identical, which in this case (with  $p_0$  and  $p'_0$  fixed) is equivalent to  $p_{00} = p'_{00}$ . If  $p_{00} \neq p'_{00}$ , then, in any coupling, one or both of the equations in (20) should be violated.

This leads us to the notion of probabilistic contextuality.

## 6. Probabilistic Contextuality

It can be said that when  $(A_{red}, B_{red}), (A_{blue}, B_{blue})$  cannot be coupled by an identity coupling, the color creates a *context* for the probability distributions involved. It can be shown that Pat can always find a value  $p$  such that  $Z = (X, Y, X', Y')$  is a coupling for  $(A_{red}, B_{red})$  and  $(A_{blue}, B_{blue})$  that satisfies

$$\begin{aligned}\Pr[X = X'] &= p, \\ \Pr[Y = Y'] &= p.\end{aligned}\tag{21}$$

Choosing  $p = 1$  means having the identity coupling, and we take this case as representing a *lack of contextuality*. As mentioned earlier, in the distributions described by (17), this is not the case if  $p_{00} \neq p'_{00}$ . In this case  $p$  should be chosen to be less than 1. The minimum possible value of  $1 - p$  can in fact be taken as a measure of *contextuality*, i.e., a measure of deviation of the system from the identity coupling representing lack of contextuality.

We will not, however, pursue the subject of quantitatively measuring contextuality in this paper. We only want to establish the defining aspect of contextuality:

*the contextuality in a system of random variables recorded under various conditions is a deviation of the possible couplings for this system from a specifically chosen identity coupling.*

There can be more than one identity coupling, depending on which of the random variables involved are hypothesized to be “essentially the same” despite being labeled by different conditions. To each specific choice of an identity coupling there corresponds a specific meaning of contextuality.

Let us make this clear on the abstract notion of a system whose inputs are  $\alpha, \beta, \gamma, \dots$  and whose outputs are  $A, B, C, \dots$ . (This is not the most general conceptual set-up, but if one wants to avoid technicalities, it is general enough.) The inputs are simply variables, each having several possible values, while the outputs are random variables with a well-defined joint distribution for each possible combination of the inputs values.



Let  $\phi, \chi, \psi, \dots$  be these possible combinations: we call them *treatments* or *conditions*. By the indexation-by-conditions principle, the outputs are to be labeled

$$(A_\phi, B_\phi, C_\phi, \dots), (A_\chi, B_\chi, C_\chi, \dots), (A_\psi, B_\psi, C_\psi, \dots), \dots, \quad (22)$$

where any random variable is jointly distributed with any identically indexed random variable but stochastically unrelated to any differently indexed one. Any random variable

$$U = (X_\phi, Y_\phi, Z_\phi, \dots, X_\chi, Y_\chi, Z_\chi, \dots, X_\psi, Y_\psi, Z_\psi, \dots) \quad (23)$$

such that

$$\begin{aligned} (X_\phi, Y_\phi, Z_\phi, \dots) &\sim (A_\phi, B_\phi, C_\phi, \dots), \\ (X_\chi, Y_\chi, Z_\chi, \dots) &\sim (A_\chi, B_\chi, C_\chi, \dots), \\ (X_\psi, Y_\psi, Z_\psi, \dots) &\sim (A_\psi, B_\psi, C_\psi, \dots), \\ &\dots \end{aligned} \quad (24)$$

is a coupling for (22).

Assume now that, for whatever reason, one thinks that of the inputs  $\alpha, \beta, \gamma, \dots$  only  $\alpha$  can influence the identity of  $A$  and only  $\beta$  can influence the identity of  $B$ . This means that if, e.g.,  $\phi(\alpha_0), \phi'(\alpha_0), \phi''(\alpha_0), \dots$  denote treatments containing the same value  $\alpha_0$  of  $\alpha$ , then all  $A_{\phi(\alpha_0)}, A_{\phi'(\alpha_0)}, A_{\phi''(\alpha_0)}, \dots$  are “essentially” the same, even if differently labeled. A rigorous formulation is that there exists a coupling  $U'$  in which

$$\Pr [X_{\phi(\alpha_0)} = X_{\phi'(\alpha_0)} = X_{\phi''(\alpha_0)} = \dots] = 1, \quad (25)$$

for every value  $\alpha_0$  of  $\alpha$ . Analogously, the hypothesized relation between  $\beta$  and  $B$  translates into the constraint

$$[Y_{\phi(\beta_0)} = Y_{\phi'(\beta_0)} = Y_{\phi''(\beta_0)} = \dots] = 1, \quad (26)$$

for every value  $\beta_0$  of  $\beta$ . In the sense of being subject to these two sets of constraints,  $U'$  is an identity coupling. If now it can be shown that such a coupling does not exist, then the system is contextual with respect to the identity coupling  $U'$ . The interpretation is that the identity of  $A$  depends not only on  $\alpha$  or/and the identity of  $B$  depends not only on  $\beta$ .

## 7. Alice-Bob EPR/Bohm Paradigm

Let us now illustrate the notion of contextuality on an example well familiar in quantum physics: two entangled spin-half particles, with Alice and Bob measuring spins along two directions each. Let the direction chosen by Alice be denoted  $\alpha$ , with values  $\alpha_1, \alpha_2$ ; the direction chosen by Bob is denoted by  $\beta$ , with values  $\beta_1, \beta_2$ . We treat the four combinations  $(\alpha_i, \beta_j)$  ( $i, j \in \{1, 2\}$ ) of settings by Alice and Bob as conditions under which the spins are recorded,  $A$  in Alice’s particle,  $B$  in Bob’s, both random variables with possible values  $+1$  and  $-1$ .

In accordance with the indexation-by-conditions and unrelatedness principles, we have four stochastically unrelated to each other pairs of random variables  $(A_{ij}, B_{ij})$ ,  $i, j \in \{1, 2\}$ . They are distributed as

$$\begin{array}{|c|c|c|} \hline \alpha_i, \beta_j & B_{ij} = +1 & B_{ij} = -1 \\ \hline A_{ij} = +1 & p_{ij} & q_{ij} \\ A_{ij} = -1 & r_{ij} & s_{ij} \\ \hline & p_{ij} + r_{ij} & \\ \hline \end{array} \quad p_{ij} + q_{ij} \quad | \quad (27)$$

A coupling for these pairs of random variables is a random variable

$$V = (X_{11}, Y_{11}, X_{12}, Y_{12}, X_{21}, Y_{21}, X_{22}, Y_{22}) \quad (28)$$

such that

$$(X_{ij}, Y_{ij}) \sim (A_{ij}, B_{ij}), \quad i, j \in \{1, 2\}. \quad (29)$$

It is taken as a given that a change in Alice's setting,  $\alpha_1 \rightarrow \alpha_2$ , changes the identity of Alice's random variable (and analogously for Bob). This means that  $X_{1j}$  and  $X_{2j}$  in the coupling  $V$  should not be required to be equal to each other, no matter what  $j$  is (and analogously for  $Y_{i1}$  and  $Y_{i2}$ ). It seems, however, reasonable to assume that Bob's settings "have nothing to do" with Alice's measurements, and vice versa. This translates into requiring that

$$\begin{aligned} \Pr[X_{i1} = X_{i2}] &= 1, \\ \Pr[Y_{1j} = Y_{2j}] &= 1, \\ i, j &\in \{1, 2\}. \end{aligned} \quad (30)$$

The coupling  $V$  subject to this requirement can be chosen as the identity coupling of special interest (this is only one of logically possible identity couplings). Equivalently, the requirement is that the four pairs  $(A_{ij}, B_{ij})$  allow for a *reduced coupling*

$$V' = (X'_1, X'_2, Y'_1, Y'_2), \quad (31)$$

such that

$$\begin{aligned} (X'_i, Y'_j) &\sim (A_{ij}, B_{ij}) \\ i, j &\in \{1, 2\}. \end{aligned} \quad (32)$$

This is the closest rigorous formulation for the usually considered "joint distribution of  $A_1, A_2, B_1, B_2$ " [19]. We know that  $V$  subject to (30) exists if and only if the conjunction of the following two conditions is satisfied:

- (i) Marginal selectivity [14, 20, 21] or no-signaling [22, 23],

$$\begin{aligned} p_{i1} + q_{i1} &= p_{i2} + q_{i2} = p_{i\cdot}, \\ p_{1j} + r_{1j} &= p_{2j} + r_{2j} = p_{\cdot j}, \\ i, j &\in \{1, 2\}; \end{aligned} \quad (33)$$

(ii) CH/Fine inequalities [19, 24],

$$-1 \leq p_{11} + p_{12} + p_{21} + p_{22} - (2p_{3-i,3-j} + p_{i\cdot} + p_{\cdot j}) \leq 0, \quad (34)$$

$$i, j \in \{1, 2\}.$$

We can say that if (and only if) these two requirements are jointly met, then the system in question is non-contextual with respect to the identity coupling  $V$  defined by (28)-(29)-(30), or equivalently, (31)-(32). The essence of all Bell-type theorems is to establish conditions for such non-contextuality.

Conversely, we can say that the system exhibits contextuality if and only the two requirements are violated. It may be important or at least useful in many cases to distinguish the following two cases:

**Case 1.** Marginal selectivity is violated, that is, either the distribution of  $B_{ij}$  (the spin recorded by Bob for the direction  $\beta_j$  he chose) changes with  $\alpha_i$  (the direction chosen by Alice), or vice versa. This (perhaps) should be interpreted as a direct influence of Bob's choices on Alice's measurements, i.e., some form of signaling. Note that the way they are written above, the probabilities  $p_{i\cdot}, p_{\cdot j}$  in CH/Fine inequalities are not defined if marginal selectivity (no-signaling) is violated.

**Case 2.** Marginal selectivity is satisfied but CH/Fine inequalities are violated. This can be called the case of "pure contextuality". Bob's settings do not affect the distribution of Alice's recordings, they only determine the way they are grouped into random variables (see the next section). The laws of quantum physics and special relativity dictate this case when the two particles are separated by a space-like interval.

In all our previous publications regarding contextuality [1–5] we only considered Case 2 as that of contextuality, preferring to speak of "direct cross-influences" in Case 1. Intuitively, the two cases must be distinguished, although perhaps not as sharply: direct cross-influences need not prevent the system from also being contextual. It is a challenge for the future conceptual analysis to see whether there is a principled way to define contextuality "on top of" violations of marginal selectivity. There is in fact a principled way to define contextuality "on top of" violations of marginal selectivity [16, 17].

## 8. Why Contextual Indexing Does not Violate Laws of Physics

Let us focus on Case 2 ("pure contextuality") and ask ourselves: if Bob's choices change the identity of Alice's measurements, does not this mean a form of signaling from Bob to Alice? In the case of a space-like separation this would contravene special relativity. The answer is negative, and it can be justified on two levels.

On a most obvious one, Alice can never guess that Bob even exists if the only information she has is the distributions of spins in her particle in response to her choices of settings. The non-identity of  $A_{i1}$  and  $A_{i2}$  is not available to her. It can only be established by someone else, a Charlie who receives the information both about

the settings and about the spin recording from both Alice and Bob. From Charlie's point of view, Alice observes a mixture of  $A_{i1}$  and  $A_{i2}$ , but Alice cannot know that.

On a deeper level, the negative answer is justified because the identity of a random variable is not an objective, physical entity to begin with. The realizations of random variables, such as "spin-down in direction  $\alpha_1$ " are objective, and the probabilities of all such observations (in the classical, frequentist sense) among, say, all spins in direction  $\alpha_1$  are objective too. But which realizations are grouped together to count them and establish their relative frequencies is a matter of choice.

It is analogous to looking at a set of points scattered on a sheet of paper: one can group them in this or that way and create various patterns without ever affecting the objective locations of the points. For Charlie,  $A_{ij}$  and  $B_{ij}$  are double-indexed because he chose to relate their realizations to both Alice's and Bob's settings,  $\alpha_i$  and  $\beta_j$ . CbD prescribes doing this "automatically", because in this way one can gain information (e.g., about quantum correlations between entangled particles), and because in cases when this is redundant one does not lose anything: in those cases differently labeled random variables are simply merged within an identity coupling.

The latter point deserves being emphasized. The requirement to index a random variable by all conditions paired with its realizations may be interpreted as a call for some kind of "all-is-one" holism. What if Charlie notices that the information he gets from Alice and Bob is received either in the morning, or during the daytime, or else in the evening? CbD requires then from Charlie to index the spins as  $A_{ij t}$  and  $B_{ij t}$ , where  $t$  assumes three values (morning, daytime, evening). Does not this "automatic" extra-labeling make the analysis unnecessarily complicated? The answer is: Charlie can always choose not to record the time of the day, but once he chooses to record it, he will either gain valuable knowledge (if the time of the day turns out to affect the joint distributions of the spins), or, in the worst case, he will find out that an identity coupling for  $(A_{ij t}, B_{ij t})$  can be constructed eliminating the need for using  $t$ .

There is yet another, purely formal way of justifying why the identity of random variables has no physical meaning. If we list all the random variables in play, each indexed by the conditions under which it has been recorded, their identities are entirely defined by (and define) the coupling imposed on them. Different couplings correspond to different identities. But couplings for one and the same system of random variables are generally non-unique, and should therefore be viewed as no more than possible (mutually exclusive) mathematical descriptions. The totality of all couplings that are impossible on a given system does characterize the system physically, as detailed in Refs. [1–4], but no single coupling is more "real" than another.

## 9. Concluding Remarks

The position presented in this paper is summarized in the abstract, and need not be repeated.

Commenting on an earlier draft of this paper, Arkady Plotnitsky suggested a

connection between CbD and Bohr’s use of the notion of complementarity in his 1935 reply [6] to the famous EPR paper [7] (for a thorough analysis of this exchange, see Ref. [8]). Plotnitsky notes that Indexation-by-Conditions corresponds to Bohr’s view that each quantum phenomenon is unique, so that to specify it one needs all the conditions under which it occurs; and that the Unrelatedness and Coupling principles reflect Bohr’s notion of complementary quantum phenomena as being mutually exclusive but equally necessary for a rigorous description of these phenomena.

Of the modern approaches to probabilistic contextuality in the literature on foundations of quantum mechanics, Larsson’s [15] comes very close to ours, while Krennikov’s approach [25–27] is in some respects more general. See also Refs. [28–31]. Here, we will briefly discuss two other treatments.

One of them is proposed by Avis, Fischer, Hilbert, and Khrennikov [32]. In Ref. [1] we called it “*conditionalization*” and compared it with CbD. Conditionalization consists in considering different values of  $\gamma$  associated with realizations of a random variables as if  $\gamma$  were a random variable in its own right. For instance, in our introductory toy example, the color of Pat’s random variable  $C$  (with values 00, 01, 10, 11) would be considered a random variable with two values, “red” and “blue”, whether the color changes randomly or alternates in some regular fashion, say, red-blue-red-blue-... The probabilities assigned to the two values have to be nonzero; otherwise they are arbitrary and play no role in characterizing the system being studied. The variable  $C$  is then considered conditioned on the values “red” and “blue”, with the distributions in (3) treated as conditional distributions. This is equivalent to the indexation-by-conditions in CbD: two different random variables are indexed by “red” and “blue”. However, here the conditionalization analysis ends, while the analysis led to by CbD approach only begins at this point: it entails considering various couplings of the two variables and determining, when contextuality is of the main interest, whether they contain identity couplings.

Another approach is based on the use of signed probability measures (sometimes referred to as “negative probabilities”). The approach dates back to Paul Dirac [33], but here it will be presented primarily based on Refs. [34, 35]. In relation to CbD, this approach can be presented in terms as considering identity couplings only, but allowing some of the joint probabilities in them to be negative (and some greater than 1). As these are not then true couplings, they can be called *quasi-couplings*. This approach will not be applicable to our introductory toy example, but it can be illustrated on the EPR/Bohm paradigm. Given  $(A_{ij}, B_{ij})$ ,  $i, j \in \{1, 2\}$ , we construct a quasi-coupling  $G = (E_1, F_1, E_2, F_2)$  such that

$$\Pr[E_i = a, F_j = b] = \Pr[A_{ij} = a, B_{ij} = a], \quad (35)$$

$$i, j \in \{1, 2\}, \quad a, b \in \{-1, +1\}.$$

The quasi-coupling  $G$ , however, is not a conventional random variable, in the sense that the signed probability values

$$p(a_1, b_1, a_2, b_2) = \Pr[E_1 = a_1, F_1 = b_1, E_2 = a_2, F_2 = b_2] \quad (36)$$

while well-defined for all  $a_1, b_1, a_2, b_2 \in \{-1, +1\}$  and summing to 1, may be negative or greater than 1. If the Alice-Bob system allows for the identity coupling in the sense of Cbd, then  $G$  simply coincides with the corresponding reduced coupling  $V'$  in (31), with all probabilities between 0 and 1. If the identity coupling does not exist (i.e., we have a contextual system), then some of the probabilities in (36) will have to be negative. The relationship between Cbd and the signed probability measures is presently under investigation. It should be noted that the approach in question is not applicable if the marginal selectivity condition is violated (cf. the last paragraph of Section 7). In fact the quasi-coupling  $G$  above exists if and only if marginal selectivity (no-signaling) holds [36].

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