Random Variables Recorded under Mutually Exclusive Conditions: Contextuality-by-Default

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Abstract We present general principles underlying analysis of the dependence of random variables (outputs) on deterministic conditions (inputs). Random outputs recorded under mutually exclusive input values are labeled by these values and considered stochastically unrelated, possessing no joint distribution. An input that does not directly influence an output creates a context for the latter. Any constraint imposed on the dependence of random outputs on inputs can be characterized by considering all possible couplings (joint distributions) imposed on stochastically unrelated outputs. The target application of these principles is a quantum mechanical system of entangled particles, with directions of spin measurements chosen for each particle being inputs and the spins recorded outputs. The sphere of applicability, however, spans systems across physical, biological, and behavioral sciences.

Keywords: contextuality; couplings; joint distribution; random outputs.

1 Introduction

This paper pertains to any system, physical, biological, or behavioral, with *random outputs* recorded under varying conditions (*inputs*). A target example for us is a quantum mechanical system of two entangled particles, "Alice's" and "Bob's." Alice measures the spin of her particle in one of two directions, α_1 or α_2 , and Bob measures the spin of his particle in one of two directions, β_1 or β_2 . Here, α and β are inputs, and each trial is characterized by one of four possible *input values* (α_i, β_j). The spins recorded in each trial are realizations of random variables A and B, which, in the simplest case, can attain two values each: a_1 or a_2 for A and b_1 or b_2

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for *B*. One can think of many examples in other domains with similar formal structure, e.g., a psychophysical experiment with an observer responding to stimuli with varying characteristics α (say, intensity) and β (say, shape). These characteristics then constitute inputs, while some characteristics of the responses, such as response time *A* (with a continuum of values) and response correctness *B* (with two possible values), are random outputs.

Accounts of the approach presented in this paper can be found in [5-7], but this paper is the first one focusing entirely on its basic principles. The approach amounts to philosophical rethinking (or at least conceptual tweaking) of the foundations of probability, specifically, of random variables and their joint distributions. Here, it is presented without technical details (that can be reconstructed from [3-6]).

2 Basic Principles

Let all or some of the random outputs of a system form a random variable X,¹ and the totality of all inputs be a variable χ . In our target example, $\chi = (\alpha, \beta)$ with input values $\chi_1 = (\alpha_1, \beta_1), ..., \chi_4 = (\alpha_2, \beta_2)$, whereas X can be (A, B) with values $x_1 = (a_1, b_1), ..., x_4 = (a_2, b_2)$, or A with values a_1, a_2 , or B with values b_1, b_2 . If χ itself is a random variable, so that $\chi_1, \chi_2, ...$ occur with some probabilities, we ignore these probabilities and simply condition the recorded outputs X on values of χ . In other words, we have a distribution of X given that $\chi = \chi_1$, a distribution of X given that $\chi = \chi_2$, etc., irrespective of whether we can control and predict the values of χ , or they occur randomly. Now, this conditioning upon input values means that X is indexed by different values of χ . We obtain thus, "automatically," a set of different random variables in place of what we previously called a random variable X. We have X_{χ_1} (or X_1 , if no confusion is likely) which is X when $\chi = \chi_1$, X_{χ_2} (or X_2) which is X when $\chi = \chi_2$, etc. Let us formulate this simple observation as a formal principle.

Principle 1 Outputs recorded under different (hence mutually exclusive) input values are labeled by these input values and considered different random variables. These random variables are stochastically unrelated, i.e., they possess no joint distribution.

Thus, in our target example, we have four random variables A_{ij} , four random variables B_{ij} , and four random variables $(A,B)_{ij} = (A_{ij},B_{ij})$ corresponding to the four input values $\chi_k = (\alpha_i, \beta_j)$. The principle holds irrespective of how the distribution of X_k depends on χ_k . Thus, the variables A_{i1} and A_{i2} remain different even if their distributions are identical (as they should be if Bob's choice cannot influence Alice's measurements). One must not assume that they are one and the same random variable, $A_i = A_{i1} = A_{i2}$. The latter would mean that A_{i1} and A_{i2} have a joint

¹ Random variables are understood in the broadest sense, so that a vector of random variables (or any set thereof, or a random process) is a random variable too.

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distribution, because of which the probabilities $\Pr[A_{i1} = A_{i2}]$ are well defined, and that these probabilities equal 1. But A_{i1} and A_{i2} do not have a joint distribution. Indeed, two random variables X and Y have a joint distribution only if their values can be thought of as observed "in pairs," i.e., if there is a scheme of establishing correspondence $x_{(i)} \leftrightarrow y_{(i)}$ between observations $x_{(1)}, x_{(2)}, \dots$ of X and $y_{(1)}, y_{(2)}, \dots$ of Y. In our example, the correspondence is defined by the two measurements being simultaneously performed on a given pair of entangled particles. Each such a pair of measurements corresponds to a certain input value, e.g., A_{21} and B_{21} correspond to $\chi = (\alpha_2, \beta_1)$. Therefore, no measurement outputs corresponding to different input values, such as A_{i1} and A_{i2} , or A_{i1} and B_{i2} co-occur in the same sense in which, say, A_{i1} co-occurs with B_{i1} .

However, given any two random variables X and Y, one can impose on them a joint distribution, and create thereby a random variable Z = (X, Y), referred to as a *coupling* for X and Y. By definition, the distribution of a coupling Z agrees with the distributions of X and Y as its marginals.

Principle 2 Stochastically unrelated outputs recorded under mutually exclusive input values can be coupled (imposed a joint distribution upon) arbitrarily. There are no privileged couplings.

Thus, in our target example, the famous Bell-type theorems [1,3,8] implicitly impose on $(A_{11}, B_{11}), \ldots, (A_{22}, B_{22})$ a coupling with $A_{i1} = A_{i2}$ and $B_{1j} = B_{2j}$. This amounts to considering a random variable (A'_1, A'_2, B'_1, B'_2) such that (A'_i, B'_j) is distributed as (A_{ij}, B_{ij}) . The Bell-type theorems show that such a coupling exists if and only if the distributions of the coupled pairs $(A_{11}, B_{11}), \ldots, (A_{22}, B_{22})$ satisfy certain constraints (Bell-type inequalities, known to be violated in quantum mechanics). In our approach, however, except possibly for simplicity considerations, this coupling has no privileged status among all possible coupling for $(A_{11}, B_{11}), \ldots, (A_{22}, B_{22})$. Thus, any distribution of spins satisfying Bell-type inequalities is also compatible with the coupling in which $(A_{11}, B_{11}), \ldots, (A_{22}, B_{22})$ are stochastically independent pairs of random variables, as well as with an infinity of other couplings in which Pr $[A_{i1} = A_{i2}]$ and Pr $[B_{1j} = B_{2j}]$ may be different from 1.

If the distributions of A_{i1} and A_{i2} are not the same for i = 1 or i = 2, the situation is simple: the output A is influenced by both inputs α and β (and analogously for B_{1j} and B_{2j}). If, however, the distributions of A_{i1} and A_{i2} are always the same, and if, moreover, substantive considerations (e.g., laws of special relativity) prevent the possibility of interpreting β as "directly" influencing A, then we can say that β forms a *context* for the dependence of A on α (and analogously for α creating a context for the dependence of B on β). Principle 1 ensures that this contextuality is introduced "automatically," by labeling all outputs by all conditions under which they are recorded. The degree and form of contextuality in a given system (e.g., those with constraints more relaxed than the Bell-type inequalities [2,9]) can be characterized by considering all possible probabilities $\Pr[A_{i1} = A_{i2}]$ and $\Pr[B_{1j} = B_{2j}]$, called *connection probabilities* in [5-7]. This approach allows one to embark on a deeper investigation of the relationship between the classical probability theory and quantum mechanics than in the Bell-type theorems.

3 Apparent Problems with the Approach

Two objections can be raised against our approach. One is that it requires to label random variables by circumstances that cannot possibly be relevant. If reaction time X to a given stimulus is recorded in conjunction with measurements of the temperature on Mars with the values $\chi_1 = low$ and $\chi_2 = high$, would it be meaningful to "automatically" split X into stochastically unrelated X_{low} and X_{high} ? The answer is: it is meaningful. If the temperature on Mars affects the distribution of X, then considering X_{low} and X_{high} as different random variables is clearly useful for understanding of X. If, as we suspect, the temperature on Mars does not affect the distribution of X, then one can impose on (X_{low}, X_{high}) an arbitrary coupling, including one with $X_{low} = X_{high} = X$. The latter choice amounts to ignoring the temperature on Mars altogether.

The other objection is that if we apply Principle 1 systematically, we have to consider different realizations of a random variable X as stochastically unrelated random variables. X occurring in trial 1 as $x_{(1)}$ is labeled X_1 and considered stochastically unrelated to X_2 that occurs in trial 2 as $x_{(2)}$, and so on. But this is perfectly reasonable, and moreover, it is a standard issue in the probabilistic theory of couplings [6]. Once a coupling (e.g., the commonly used iid one) is imposed on X_1, X_2, \ldots , it creates a new random variable $Y = (X_1, X_2, \ldots)$, of which we have a single realization $y = (x_{(1)}, x_{(2)}, \ldots)$. One can then investigate whether this y is statistically plausible in view of the distribution of Y using standard statistical reasoning.

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