On selective influences, marginal selectivity, and Bell/CHSH inequalities

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Abstract

The Bell/CHSH inequalities of quantum physics are identical with the inequalities derived in mathematical psychology for the problem of selective influences in cases involving two binary experimental factors and two binary random variables recorded in response to them. The following points are made regarding cognitive science applications: (1) compliance of data with these inequalities is informative only if the data satisfy the requirement known as marginal selectivity; (2) both violations of marginal selectivity and violations of the Bell/CHSH inequalities are interpretable as indicating that at least one of the two responses is influenced by both experimental factors.

Keywords: CHSH inequality; causal communication constraint; concept combinations; entangled particles; EPR paradigm; marginal selectivity; selective influences; spins.

1. Introduction

There are two independently motivated theoretical developments, one in cognitive science (CS) and one in quantum mechanics (QM), that lead to essentially identical mathematical formalisms.

In CS, the issue is known as that of *selective influences*: given several experimental factors and several random variables recorded in response to them, how can one determine whether each factor influences only those response variables it is designed to influence? The issue was introduced by Sternberg (1969), and for stochastically non-independent responses, by Townsend (1984). A rigorous mathematical theory of selective influences has been developed by Dzhafarov and colleagues (Dzhafarov, 2003; Dzhafarov & Gluhovsky, 2006; Dzhafarov & Kujala, 2010, 2012a-c, in press; Dzhafarov, Schweickert, & Sung, 2004; Kujala & Dzhafarov, 2008).[1]

The "parallel" issue in QM is known as the investigation of the (im)possibility of accounting for quantum entanglement in terms of "hidden variables." It dates back to the papers by Einstein, Podolsky, and Rosen (1935), Bohm and Aharonov (1957), and Bell (1964). In the present paper we use the generalized form of Bell's inequalities developed in Clauser and Horne (1974), often referred to as CHSH inequalities (after Clauser, Horne, Shimony, & Holt, 1969). Quantum entanglement and Bell-type inequalities are mentioned in Wang, Busemeyer, Atmanspacher, & Pothos (in press) among the main conceptual frameworks of QM with a potential of applicability to CS problems. Aerts, Gabora, and Sozzo (in press) present an example of such an application, and our paper is essentially an extended commentary on it.[2]

Our discussion is confined to the simplest experimental design in which: (1) there are two *experimental factors* (in CS) or *measurement procedures* (in QM), denoted α and β and varying on two levels each; (2) for each of the four treatments (combinations of factor levels) two *response variables* (in CS) or *measurement results* (in QM) are recorded, denoted A and B; and (3) for each treatment these A and B are generally *stochastically dependent random variables.*

In CS it is often the case that hypothetically or normatively (e.g., in accordance with the instructions given to the participants), A is supposed to be a response to α only, and B a response to β only. In other words, the *influences* of factors α, β on responses A, B are hypothesized to be *selective*,

$$A \leftarrow \alpha, B \leftarrow \beta, \tag{1}$$

as opposed to at least one of A, B being influenced by both α and β. Thus, in Aerts et al. (in press) experiments, the factors are two requests: α = "choose an animal" and β = "choose an animal sound." The first factor has two variants (levels): one is a = "Horse or Bear?", the other is a' = "Tiger or Cat?" Factor β also has two levels: b = "Growls or Whinnies?" and b' = "Snorts or Meows?" The responses $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are the two choices made in response to α and β. We can denote the values of $A_{\alpha\beta}$ by $+1$ and -1 according as the first or second of the alternatives listed in α is chosen; and we assign $+1/-1$ to $B_{\alpha\beta}$ analogously. For instance, the response pair $(A_{ab'}, B_{ab'})$ denotes two choices made in response to treatment (Horse or Bear?, Snorts or Meows?); and $(A_{ab'} = -1, B_{ab'} = +1)$ means that the two choices are Bear, Snorts.[3]

In QM, an analogue of this experiment is the simplest Bohmian version of the Einstein-Podolsky-Rosen paradigm (EPR/B), involving two spin-1/2 entangled particles, "Alice's" and "Bob's." The measurement procedures (factors) α and β here consist in setting detectors for measuring spins along certain spatial directions, $\alpha = a$ or a' for Alice's particle, and $\beta = b$ or b' for Bob's. The measurement results (responses) are spins along these directions, with possible values, $+1$ = "spin-up" and -1 = "spin-down." To emphasize the analogy, each level of α (a or a') can be presented as the "requests to choose" between spin-up and spin-down along the corresponding direction; and analogously for each level of β. Thus, $(A_{ab'} = -1, B_{ab'} = +1)$ means that Alice's detector is set for direction a and the spin measures down, while Bob's detector is set for direction b' and the spin is up.

In both CS and QM, by repeatedly using the four treatments (α, β) one estimates for each of them the joint probabilities $\Pr(A_{\alpha\beta} = i, B_{\alpha\beta} = j)$, where i and j stand for $+1$ or -1.

2. Selective influences and marginal selectivity

In the following the terms "factor" and "response" will be used generically, to include both the CS and QM meanings. It is easy to show that whatever the factors α, β and responses A, B, one can always find a random variable R and two functions $A'_{\alpha\beta} = f(R, \alpha, \beta)$ and $B'_{\alpha\beta} = g(R, \alpha, \beta)$, such that, for any treatment (α, β), the joint distribution of $(A_{\alpha\beta}, B_{\alpha\beta})$ is the same as that of the pair of random variables $\left(A'_{\alpha\beta}, B'_{\alpha\beta}\right)$. In symbols, denoting "is distributed as" by $\sim$,

$$\begin{pmatrix} A_{\alpha\beta} \\ B_{\alpha\beta} \end{pmatrix} \sim \begin{pmatrix} A'_{\alpha\beta} = f(R, \alpha, \beta) \\ B'_{\alpha\beta} = g(R, \alpha, \beta) \end{pmatrix}. \tag{2}$$

In CS, R can be interpreted as a *common source of randomness* for the responses, accounting for both their stochasticity and non-independence. In the EPR/B paradigm of QM, R represents all "hidden variables" responsible for the joint distributions of the spins.[4] It was proposed in Dzhafarov (2003) that the selectiveness (1) simply means that, referring to (2), β is a dummy argument in f and α a dummy argument in g. That is, (2) acquires the form

$$\left(\begin{array}{c} A_{\alpha\beta} \\ B_{\alpha\beta} \end{array}\right) \sim \left(\begin{array}{c} A'_{\alpha} = f(R,\alpha) \\ B'_{\beta} = g(R,\beta) \end{array}\right), \tag{3}$$

for some random variable R and some functions f, g.[5] In the context of EPR/B, this hypothesis amounts to the possibility of a classical (non-quantum) explanation for the joint distributions of the spins.

An immediate consequence of (3) is *marginal selectivity* (Dzhafarov, 2003; Townsend & Schweickert, 1983): for any α, β, the distribution of $A_{\alpha\beta}$ does not depend on β, nor the distribution of $B_{\alpha\beta}$ on α. In other words,

$$\begin{array}{l} \Pr\left(A_{\alpha,\beta=b} = +1\right) = \Pr\left(A_{\alpha,\beta=b'} = +1\right) = \Pr\left(A'_{\alpha} = +1\right), \\ \Pr\left(B_{\alpha=a,\beta} = +1\right) = \Pr\left(B_{\alpha=a',\beta} = +1\right) = \Pr\left(B'_{\beta} = +1\right). \end{array} \tag{4}$$

In QM marginal selectivity is known under other names, such as *causal communication constraint* (see Cereceda, 2000). It is trivially satisfied if one assumes that the entangled particles are separated by a space-like interval (i.e., either of them may precede the other in time from the vantage point of an appropriately moving observer). In CS marginal selectivity can sometimes be ensured by an appropriate choice of A, B (Dzhafarov & Kujala, 2012a, c), but generally (and in particular, in applications like in Aerts et al., in press) it may very well be violated.

3. Selective influences and CHSH inequality

The CHSH inequality is another consequence of (3): it can be written as

$$\Gamma \leq 2, \tag{5}$$

where, with E denoting expectation,

$$\Gamma = \max\left\{\pm \mathrm{E}\left[A_{ab}B_{ab}\right] \pm \mathrm{E}\left[A_{ab'}B_{ab'}\right] \pm \mathrm{E}\left[A_{a'b}B_{a'b}\right] \pm \mathrm{E}\left[A_{a'b'}B_{a'b'}\right] : \ \#\text{ of } + \text{ signs is odd}\right\}. \tag{6}$$

This inequality follows from but does not imply (3), as we see in the imaginary situation depicted in Table 1.

It is, of course, also possible that marginal selectivity is satisfied but Γ exceeds 2. This is the focal fact in the EPR/B experiments (e.g., Aspect, Grangier, & Roger, 1982). In the pattern of probabilities in Table 2, $\Gamma = 4$ (which is its largest theoretically possible value), violating not only (5) but also its quantum version, with $2\sqrt{2}$ substituting for 2 (Landau, 1987).

4. When marginal selectivity is violated

We see that the two consequences of (3), the CHSH inequality and marginal selectivity, are logically independent. Together, however, they form a *criterion* (a necessary and sufficient condition) for (3). This was first proved by Fine (1982), even if with (4) implied rather than stipulated.[6]

Table 1 Joint probabilities of $A = +1/-1$ and $B = +1/-1$ at four treatments (α, β). †

		B		
	a, b	$+1$	-1	
A	$+1$	.25	.25	.5
	-1	.25	.25	.5
		.5	.5	

		B		
	a, b'	$+1$	-1	
A	$+1$	.25	.5	.75
	-1	.0	.25	.25
		.25	.75	

	a', b	$+1$	-1	
A	$+1$	.25	.35	.6
	-1	.15	.25	.4
		.4	.6	

	a', b'	$+1$	-1	
A	$+1$	.25	.45	.7
	-1	.05	.25	.3
		.3	.7	

† The CHSH inequality (5) is satisfied ($\Gamma = 0$), but marginal selectivity (4) is violated: e.g., $0.5 = \Pr(B_{ab} = +1) \neq \Pr(B_{a'b} = +1) = 0.4$. This rules out a representation (3).

Table 2 Joint probabilities of $A = +1/-1$ and $B = +1/-1$ at four treatments (α, β). †

		B		
	a, b	$+1$	-1	
A	$+1$	.5	0	.5
	-1	0	.5	.5
		.5	.5	

		B		
	a, b'	$+1$	-1	
A	$+1$	.5	0	.5
	-1	0	.5	.5
		.5	.5	

	a', b	$+1$	-1	
A	$+1$	.5	0	.5
	-1	0	.5	.5
		.5	.5	

	a', b'	$+1$	-1	
A	$+1$	0	.5	.5
	-1	.5	0	.5
		.5	.5	

† The CHSH inequality (5) is violated ($\Gamma = 4$) with marginal selectivity satisfied. This rules out a representation (3).

Table 3 Probability estimates from Table 1 of Aerts et al. (in press). † ‡

	a, b	Growls	Whinnies	
A	Horse	.049	.630	.679
	Bear	.259	.062	.321
		.308	.692	

	a, b'	Snorts	Meows	
A	Horse	.593	.025	.618
	Bear	.296	.086	.382
		.889	.111	

	a', b	Growls	Whinnies	
A	Tiger	.778	.086	.864
	Cat	.086	.049	.135
		.864	.135	

	a', b'	Snorts	Meows	
A	Tiger	.148	.086	.234
	Cat	.099	.667	.766
		.247	.753	

† Marginal selectivity is violated, the CHSH inequality is violated too ($\Gamma = 2.420$).

‡ n/treatment = 81.

Consider now Table 3 that presents the experimental results reported in Aerts et al. (in press). Marginal selectivity here is violated in all four cases. Thus,

$$0.135 = P\left(Cat, Growls\right) + P\left(Cat, Whinnies\right) \neq P\left(Cat, Snorts\right) + P\left(Cat, Meows\right) = 0.766.$$

The difference being both large and statistically significant, we conclude that no representation (2) for this experiment reduces to a representation (3). The CHSH inequality here is violated too, $\Gamma = 2.420$, but this does not add ramifications to the rejection of (3).

If a violation of marginal selectivity was obtained in the EPR/B context, it would require a revolutionary explanation, as this would contradict special relativity: a measurement procedure α applied to Alice's particle cannot affect measurement results on Bob's particle, provided they are separated by a space-like interval. But in the context of an experiment like in Aerts et al. (in press), the explanation is both simple and plausible: the choice between animals is influenced not only by animal options but also by sound options; and analogously for the choice between sounds. One is more likely to choose Cat over Tiger if one also faces the choice between Snorts and Meows than if it is between Growls and Whinnies. In fact, the explanation in terms of the lack of selectiveness would be the simplest one even if in Aerts et al. (in press) a value of Γ exceeding 2 was obtained with marginal selectivity satisfied. But an interpretation of the influences exerted by α on B and/or by β on A in this case would be less straightforward, and an invocation of the EPR/B analogy more elucidating.

5. Conclusion

The main points are summarized in the abstract. But perhaps this is not the end of the story. There is a huge gap between representations (2) and (3), and a systematic theory is needed to study *intermediate cases*. We recently began this study (Dzhafarov & Kujala, 2012c), confining it, however, to the cases with marginal selectivity satisfied. It is conceivable that the situations where it is violated could be shown within the framework of a general theory to be structurally different depending on the value of Γ. This would impart a diagnostic value to findings like those reported in Aerts et al. (in press).

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Notes

[1]For a mathematically accessible overview of this development, see Chapter 10 of Schweickert, Fisher, and Sung (2012).

[2]For a detailed overview of the QM-SC parallels related to Bell-type inequalities, see Dzhafarov and Kujala (2012a-b).

[3]For other examples of selective influence problems in behavioral sciences, see Dzhafarov (2003).

[4]It may appear that R should be of enormous complexity, but in fact, when dealing with two binary factors and two binary responses, it never has to be more complex than a discrete random variable with 2^8 values. This follows from an extended version of the Joint Distribution Criterion (Dzhafarov & Kujala, 2012c).

[5]If such a representation exists, R need not be more than a 16-valued random variable. This follows from the Joint Distribution Criterion, as formulated in Fine (1982). See Dzhafarov & Kujala (2012a).

[6]That it was implied is apparent from the notation used: Fine writes all marginal probabilities as depending on one factor only. So if marginal selectivity is violated, Fine's inequalities are simply inapplicable. A proof with explicit derivation of (4) and (5) is obtained as a special case of the Linear Feasibility Criterion described in Dzhafarov and Kujala (2012a).

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