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Psychophysics without physics: a purely psychological theory of Fechnerian scaling in continuous stimulus spaces

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Abstract

The theory of Fechnerian scaling, as developed by the present authors, uses "same-different" discrimination probabilities defined on a stimulus set to derive from them a measure of local discriminability (of each stimulus from its neighbors), and by cumulating this measure along special paths in the stimulus space it obtains subjective (Fechnerian) distances among stimuli. Previously the theory has been developed for two kinds of stimulus spaces: (A) "continuous spaces", that were understood as connected regions of Euclidean space (such as the amplitude–frequency space of tones, or the CIE color triangle), and (B) discrete stimulus spaces (such as alphabets or words). In the former case the theory is psycho-physical rather than purely psychological, in the sense that the resulting subjective distances are based not only on discrimination probabilities but also on certain properties provided by physical measurements of stimuli. Thus, the two-dimensionality of the amplitude–frequency space of tones, its vectorial structure, and its Euclidean topology are all physical properties, and Fechnerian computations make use of them. This is an unsatisfactory situation, as the definition of a subjective distance between two stimuli should not critically depend on how these stimuli are measured by physicists. The theory of Fechnerian scaling for discrete stimulus spaces is, in contrast, purely psychological: the discreteness of a stimulus space and all Fechnerian computations can be defined there entirely in terms of discrimination probabilities. In the present work we show how to construct Fechnerian scaling as a purely psychological theory for arcwise connected (intuitively, "continuous") spaces of arbitrary nature, including spaces with infinite-dimensional or nondimensional physical descriptions (such as spaces of pictures or motions). As in the Euclidean special case, this general theory of Fechnerian scaling is based on the defining property of discrimination, called Regular Minimality, and on the idea of regular variation of psychometric differentials, with all previously derived main theorems of Fechnerian scaling remaining valid.

Keywords: Discrimination probability; Subjective dissimilarity; Fechnerian scaling; Fechnerian distance; Self-dissimilarity; Observation area; Regular minimality; Psychometric length; Psychometric order; Same-different comparisons; Stimulus space; Subjective distance

Contents

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1. Introduction

In the world-view of classical psychophysics ("mind-view", perhaps, being a better term), sensory perception is essentially characterized by a collection of unidimensional continua representable by axes of nonnegative real numbers. Each such a continuum corresponds to a certain "sensory quality" (loudness, spatial extent, saturation, etc.) any two values of which, "sensory magnitudes", are comparable in terms of " less than or equal to". Moreover, each such a continuum has a primary "physical correlate", an axis of nonnegative reals representing intensity, or spatiotemporal extent of a particular physical attribute: the sensory attribute is related to its physical correlate monotonically and (with an appropriate choice of physical measures) smoothly, starting from the value of the absolute threshold (see Fig. 1). The subjective distance between two stimulus values according to this point of view is simply the difference between the corresponding sensory magnitudes. This " mind-view" characterized psychophysics at its inception (Fechner, 1860, 1887), has been dominant throughout one and a half century of its development (Stevens, 1975), and is very much well and alive at present (see, e.g., Luce, 2002, 2004).

The theory of multidimensional Fechnerian scaling (MDFS) developed by the present authors (Dzhafarov, 2001a,b; 2002a–d; 2003a–c; Dzhafarov & Colonius, 1999, 2001) has its historical roots in another " mind-view", whose classical implementations can be found in Helmholtz's (1891) and Schrödinger's (1920, 1920/1970, 1926/1970) geometric models of color space. These models lay the foundation of the modern color science (Indow, 1993; Indow & Morrison, 1991; Indow, Robertson, von Grunau, & Fielder, 1992; Izmailov, 1995; Izmailov & Sokolov, 1991, 2003; Wyszecki & Stiles, 1982). Physically, colors are functions relating radiometric energy to wavelength, but even if their representation by means of one of the traditional color diagrams (such as CIE or Munsell) is considered their physical description, and even if the subjective representation of colors is thought of in terms of a finite number of unidimensional attributes (such as, in the case of aperture colors, their hue, saturations, and brightness), the mapping of physical descriptions into subjective ones is clearly that of one multidimensional space into another. In this context the notions of "sensory magnitudes" ordered in terms of "greater-less" and of psychophysical functions like those shown in Fig. 1 become artificial, if applicable at all. The notion of subjective dissimilarity, by contrast, acquires the status of a natural and basic concept, whose applicability allows for but does not presuppose any specific system of color coordinates, either physical or subjective. The natural operationalization of the discrimination of similar colors in this context is their judgment in terms of "same or different", rather than "greater or less". (For a detailed discussion of the "greater-less" versus "same-different" comparisons, see Dzhafarov, 2003a.)

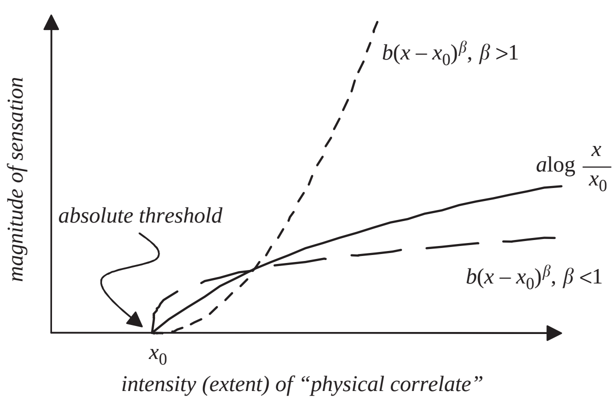

Fig. 1. The view of sensory perception in classical psychophysics.

The reason MDFS includes the adjective "Fechnerian" in its name is in that MDFS, the same as Helmholtz's and Schrödinger's color geometries, borrows from Fechner the fundamental idea of computing subjective dissimilarities among stimuli from the observers' ability to tell apart very similar stimuli. When stimulus space is a unidimensional continuum, this is done by means of cumulating a "local discriminability" (or "distinctiveness") measure as one moves from one stimulus to another through all intermediate positions (see Fig. 2). However, to generalize this basic idea to multidimensional stimulus spaces (open connected

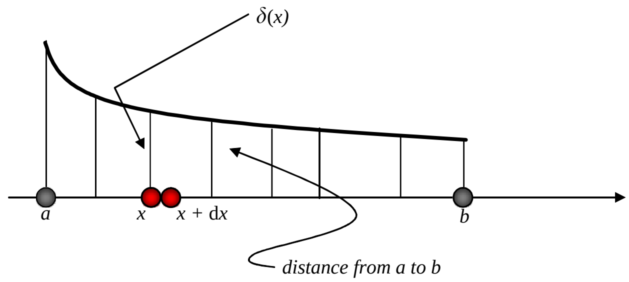

Fig. 2. Fechner's basic idea: the subjective distance from a to b is $\int_a^b \delta(x)\,\mathrm{d}x$, where $\delta(x)$ is the degree of discriminability of $x + \mathrm{d}x$ from x (shown by spikes and the solid curve). In a modern interpretation of Fechner's theory (Dzhafarov, 2001b; Dzhafarov and Colonius, 1999), $\delta(x)$ is computed as the growth rate of $\xi(x, y)$ (the probability of judging y to be greater than x) taken at $\xi(x, y) = \frac{1}{2}$. Fechner approximated $\delta(x)$ by the reciprocal of the "just noticeable difference" (jnd) at x, and he was primarily interested in the case when a is at the absolute threshold.

regions of Re^n, with axes representing independent physical attributes), MDFS has to radically depart from Fechner's original theory in several respects, the two most conspicuous departures being as follows.

1. The local discriminability measure (assigned to every point in stimulus space and every possible direction of change attached to it) is computed from the discrimination probabilities of the " same-different" kind,

$$\psi(\mathbf{x}, \mathbf{y}) = \Pr[\mathbf{x}\text{and } \mathbf{y} \text{ are judged to be different from each other}].$$

   This necessary switch from the "greater-less" to "same-different" judgments does not deprive one from the possibility of dealing with semantically unidimensional attributes of stimuli, such as loudness or brightness: as explained later (Section 3.2), $\psi(\mathbf{x}, \mathbf{y})$ may have a variety of methodological versions, including the one in which the sameness or difference of $\mathbf{x}$ and $\mathbf{y}$ is judged with respect to a single designated quality (say, brightness), ignoring everything else.
2. Any two stimuli $\mathbf{a}, \mathbf{b}$ in Re^n can be generally connected by an infinite number of well-behaved paths, and the integration of the local discriminability measure along them yields the "subjective lengths" of these paths rather than the subjective distance between $\mathbf{a}$ and $\mathbf{b}$. To compute the latter, one has to take the infimum of the " subjective lengths" of all closed loops leading from $\mathbf{a}$ to $\mathbf{b}$ and back. The logic of this computation is explained in Dzhafarov (2002d) and, in much greater detail and greater generality, in the present paper.

Like Fechner's original theory, MDFS is not a model aimed at explaining or predicting specific phenomena. Rather it is a combination of a theoretical language with a measurement procedure: it explicates the intuitions behind the notion of a "subjective distance" and provides a procedure by which subjective distances can be computed from standardized observable judgments ("same" or "different") in response to pairs of stimuli. As stated in Dzhafarov and Colonius (2001), "what motivates this theory [MDFS] is the vague belief that, the discrimination among stimuli being arguably the most basic cognitive function and the probability of discrimination being a universal measure of discriminability, distances computed from discrimination probabilities should have a fundamental status among behavioral measurements".

MDFS is based on certain assumptions about the properties of discrimination probabilities $\psi(\mathbf{x}, \mathbf{y})$, and some of these assumptions or properties derived therefrom can be empirically tested (see Dzhafarov, 2002d; see also Sections 4.4 and 11.6 in the present paper). One property of discrimination probabilities underlying MDFS (Regular Minimality, first formulated in Dzhafarov (2001a, 2002d) and prominently discussed in the present paper) we believe to be the fundamental law of perceptual discrimination: it has, especially when combined with another important property of discrimination, Nonconstant Self-Dissimilarity, far-reaching and rather unexpected consequences for possible shapes of discrimination probability functions and possible ways of modeling the process of perceptual discrimination. MDFS has proved to be useful in addressing and conceptually clarifying several traditional problems of psychophysics, such as the number of jnd's between isosensitivity curves (Dzhafarov, 2001b; Dzhafarov & Colonius, 1999), the so-called "Fechner problem" in unidimensional and multidimensional settings (the hypothesis that discrimination probabilities are monotonically related to subjective distances, see Dzhafarov, 2002b), the definition and properties of perceptually separable stimulus dimensions, (Dzhafarov, 2002c, 2003c), and the representability of perceptual images of stimuli by random entities in a hypothetical perceptual space (Dzhafarov, 2003a,b).

We do not recapitulate here the main points of the Fechnerian analysis of these problems, referring the reader to the literature cited. The purpose of the present paper is far from being an overview: *it is to identify and overcome*, *by means of a more general and comprehensive theoretical construction*, *the main limitation of MDFS*, *the fact that this theory is* "*psycho-physical*" *rather than* "*purely psychological*".

1.1. Psycho-physical versus purely psychological

We use the hyphenated term "*psycho-physical*" to designate a theoretical construct that is based both on subjective judgments about stimuli and on a specific choice of physical measurements for these stimuli. The classical psychophysical functions shown in Fig. 1 are psycho-physical constructs in this sense. Fechner's logarithmic function and Stevens's power function presuppose that one confines stimulus measurements to a certain class, namely, the conventional measures of energy, mass, and spatial or temporal extent (multiplied by constants or raised to powers). This constraint applies irrespective of what kind of subjective judgments the scaling theory utilizes, be it magnitude estimates taken to directly provide scale values, or "greater-less" judgments used to compute jnd's or other local discriminability measures that have to be cumulated to provide scale values. In contrast, multidimensional scaling (MDS; see, e.g., Kruskal & Wish, 1978) is an example of a *purely psychological theory*. Whatever the means by which subjective interstimulus dissimilarities are obtained, the resulting spatial configuration in an MDS solution is invariant under all possible changes of stimulus descriptions.

MDFS belongs to the first of these two classes. Although this theory does not involve the notion of "normative" physical measurements for stimulus attributes, it is still a psycho-physical theory as it utilizes some of the weaker physical properties of stimuli, such as the dimensionality of stimulus space, its topology, and its vectorial structure.

To illustrate, consider a space of visually presented rectangular objects varying in their horizontal and vertical dimensions h and v (see Fig. 3). In MDFS, the Fechnerian distance $G(\mathbf{a}, \mathbf{b})$ between rectangles $\mathbf{a} = (h_1, v_1)$ and $\mathbf{b} = (h_2, v_2)$ is computed in such a way that it is invariant with respect to all possible diffeomorphic transformations (continuously differentiable transformations with continuously differentiable inverses). Thus, assuming the values of h and v fill in some interval (m, M), the distances $G(\mathbf{a}, \mathbf{b})$ remain the same if one redefines stimuli $\mathbf{x} = (h, v)$ as $\mathbf{x} = (h/v, hv)$, or as $\mathbf{x} = (h^{0.3}v^2, h^2v^{0.3})$ (see Fig. 4). This seems to be a desirable property, since the definition of a subjective distance between two rectangles should not depend on whether physicists measure them by their width and height (h, v) or, say, by their aspect ratio and area $(h/v, hv)$.

This logic, however, breaks down when considering nondiffeomorphic transformations, such as the one shown in the left-hand panel of Fig. 5. Even though this transformation is bijective, moreover, homeomorphic (continuous with a continuous inverse), Fechnerian distances computed in accordance with MDFS from these two equivalent representations for the space of rectangles will not generally coincide. The situation worsens when one contemplates nontopological (i.e., discontinuous) transformations. Let, for example, $0<h<1$, $0<v<1$, and let x' for $x \in (0, 1)$ be obtained from x by permuting its first two digits after the decimal point (with some convention eliminating either indefinitely repeated 9's or indefinitely repeated 0's). The overall space of (h', v') vectors remains the same as that

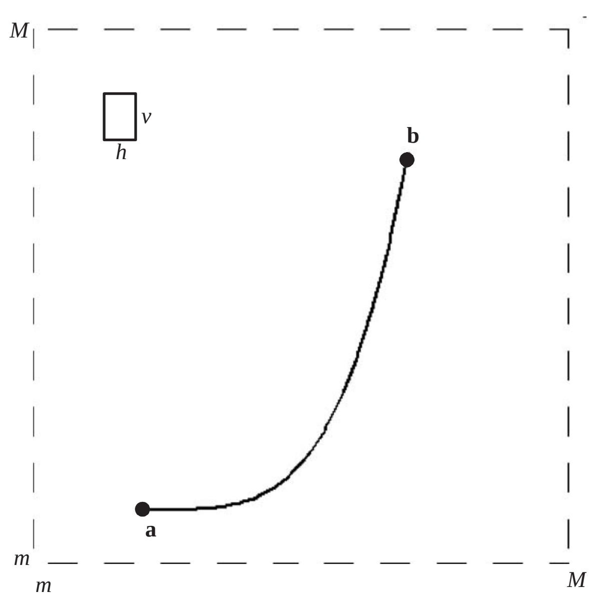

Fig. 3. A stimulus space for visually presented rectangles varying in their horizontal (h) and vertical (v) dimensions ($m<h<M$, $m<v<M$). Two stimuli are shown with a smooth arc connecting them.

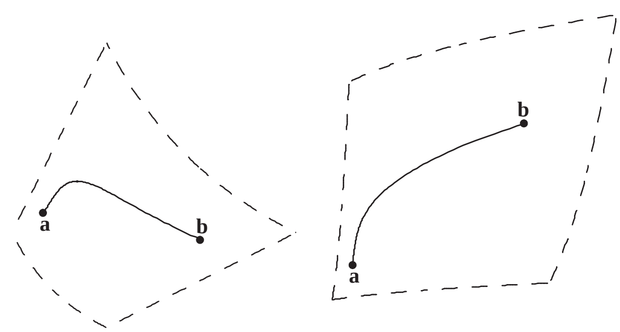

Fig. 4. Diffeomorphic transformations of the space shown in Fig. 3, with the same two points and the arc connecting them. Left: $(h, v) \to (h/v, hv)$. Right: $(h, v) \to (h^{0.3}v^2, h^2v^{0.3})$.

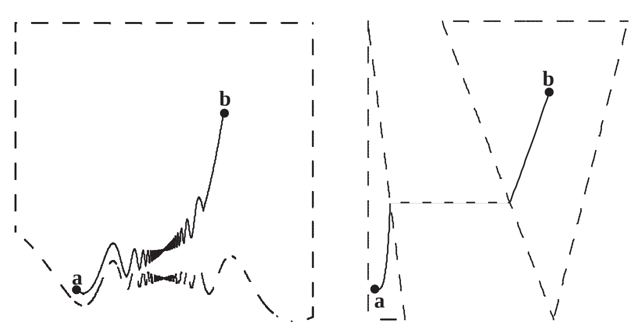

Fig. 5. Non-diffeomorphic bijective transformations of the space shown in Fig. 3, with the same two points and the arc connecting them. Left: $(h, v) \to (h, v \max\{v, c + (c-h)\sin\frac{h}{c-h}\})$, where $c = (m+M)/2$. This transformation is homeomorphic. Right: $(h, v) \to (hP(h+v, 2c, m), v)$, where $P(x, y, z)$ is defined as x if $x>y$ and z if $x \leqslant y$.

of (h, v) vectors (Fig. 3, with $m = 0$, $M = 1$), but the Fechnerian distances between $\mathbf{a} = (h_1, v_1)$ and $\mathbf{b} = (h_2, v_2)$ and between $\mathbf{a} = (h'_1, v'_1)$ and $\mathbf{b} = (h'_2, v'_2)$ will have nothing in common, even though these two representations for rectangles can be uniquely reconstructed from each other (note that an arc in one of these representations will not generally be an arc in another). The transformation of (h, v) shown in the right panel of Fig. 5 provides an example where Fechnerian distances in the transformed space are not even defined in the sense of the "standard" MDFS (Dzhafarov& Colonius, 2001; Dzhafarov, 2002b, d), because the new representation no longer forms a connected region of Re^n.

Granting that none of the three latter examples is a likely candidate for a useful parametrization of varying rectangles, the situation they reveal is clearly unsatisfactory. The requirement "*The definition of a subjective distance between two stimuli should not depend on one's choice of their physical descriptions*" should be expected to apply universally, and not only to diffeomorphic transformations of specially chosen descriptions. This requirement accords with the "invariance principle" formulated by Narens and Mausfeld (1992). Quoting from their paper, this principle "is based on the fact that (a) from the point of view of theoretical physics the stimulus can be characterized in many different but physically equivalent ways, and (b) the physical theory does not depend on which of these equivalent ways are used in the formulation. Thus, from this point of view, it is only a matter of convention which equivalent way is used to characterize the stimulus in psychophysics, and psychological conclusions in psychological settings should be invariant under equivalent physical formulations; that is, one should reach the same conclusion no matter which equivalent formulation of the physical stimulus is used" (p. 468).

To understand why this requirement fails in our examples, one should be reminded the basics of MDFS.

The computation of Fechnerian distances in this theory is critically based on the possibility of connecting any two points $\mathbf{a}, \mathbf{b}$ in a stimulus space by piecewise continuously differentiable paths $\mathbf{x}(t)$, $t \in [0, 1]$, $\mathbf{x}(0) = \mathbf{a}$, $\mathbf{x}(1) = \mathbf{b}$. This notion is purely physical, and is predicated on the possibility of subtracting one stimulus-point from another, in order to form ratios

$$\frac{\mathbf{x}(t+s) - \mathbf{x}(t)}{s}$$

and to evaluate their limits as $s \to 0+$. This subtraction operation is well-defined if the stimulus space in question is vectorial, which is trivially satisfied for subspaces of Re^n, where each stimulus is described by n real components,

$$\mathbf{x} = (x^1, x^2, \ldots, x^n).$$

Moreover, the differentiation requires that the space be endowed with certain topology, so that $\mathbf{x}(t+s) \to \mathbf{x}(t)$ as $s \to 0+$. Again, this notion is uniquely defined in Re^n if the latter is endowed with the usual (Euclidean) topology. This topology allows one to define the stimulus space as a connected region of Re^n, and to utilize the fact that in such a region any two points can be connected by (generally an infinity of) piecewise continuously differentiable paths.

The psychological part of the theory begins with using a discrimination probability function $\psi(\mathbf{x}, \mathbf{y})$ to assign to every line element $(\mathbf{x}(t), \dot{\mathbf{x}}(t))$ on path $\mathbf{x}(t)$ (i.e., a pair consisting of a stimulus on this path and a vector of its change along this path) two local discriminability measures

$$F_1(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = \lim_{s \to 0+} \frac{\Phi[\psi(\mathbf{x}(t), \mathbf{x}(t+s)) - \psi(\mathbf{x}(t), \mathbf{x}(t))]}{s},$$

$$F_2(\mathbf{x}(t), \dot{\mathbf{x}}(t)) = \lim_{s \to 0+} \frac{\Phi[\psi(\mathbf{x}(t+s), \mathbf{x}(t)) - \psi(\mathbf{x}(t), \mathbf{x}(t))]}{s}.$$

(For the sake of brevity we omit here the discussion of the overall psychometric transformation Φ, as well as of Regular Minimality and Points of Subjective Equality upon which this computation is based; these notions will be systematically introduced later, in a much more general framework.)

Fechnerian distance $G(\mathbf{a}, \mathbf{b})$ is defined as the infimum (greatest lower bound) of quantities

$$\int_0^1 F_1(\mathbf{x}(t), \dot{\mathbf{x}}(t))\,\mathrm{d}t + \int_0^1 F_1(\mathbf{y}(t), \dot{\mathbf{y}}(t))\,\mathrm{d}t$$

taken across all piecewise continuously differentiable paths $\mathbf{x}(t)$ and $\mathbf{y}(t)$ connecting, respectively, $\mathbf{x}(0) = \mathbf{a}$ to $\mathbf{x}(1) = \mathbf{b}$, and $\mathbf{y}(0) = \mathbf{b}$ to $\mathbf{y}(1) = \mathbf{a}$. This happens to be a well-defined distance function, and is shown (Dzhafarov, 2002d) to coincide with the infimum of all quantities

$$\int_0^1 F_2(\mathbf{x}(t), \dot{\mathbf{x}}(t))\,\mathrm{d}t + \int_0^1 F_2(\mathbf{y}(t), \dot{\mathbf{y}}(t))\,\mathrm{d}t.$$

As discussed later (Section 11), the equality of these two infima is one of the most important results of the Fechnerian scaling theory (the invariance of Fechnerian metric with respect to " observation area").[1] For now, however, we are focusing on the invariance of this metric with respect to transformations of stimulus space,

[1] This result follows from the identity

$$\int_0^1 F_1(\mathbf{x}(t), \dot{\mathbf{x}}(t))\,\mathrm{d}t + \int_0^1 F_1(\mathbf{y}(t), \dot{\mathbf{y}}(t))\,\mathrm{d}t = \int_0^1 F_2(\mathbf{x}(t), -\dot{\mathbf{x}}(t))\,\mathrm{d}t + \int_0^1 F_2(\mathbf{y}(t), -\dot{\mathbf{y}}(t))\,\mathrm{d}t$$

which holds for all paths $\mathbf{x}(t)$ and $\mathbf{y}(t)$ connecting $\mathbf{a}$ and $\mathbf{b}$ (Dzhafarov, 2002d).

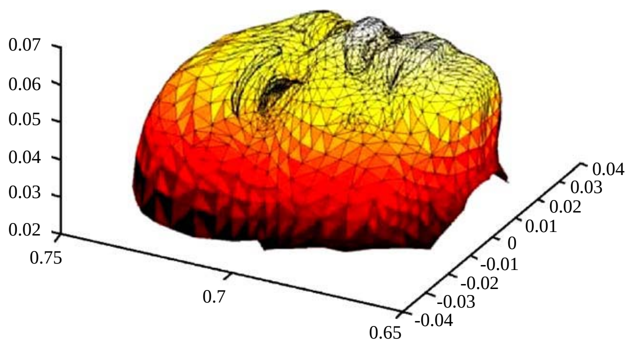

Fig. 6. Human face as an example of stimuli representable by functions $\mathrm{Re}^2 \to \mathrm{Re}$ (taken from Spencer-Smith, Townsend, & Solomon (2000) courtesy of James Townsend). For a theoretical analysis of such stimuli in terms of a Riemannian geometry on infinitely dimensional manifolds, see Townsend, Solomon, and Spencer-Smith (2001).

$\mathbf{x} \to \mathbf{f}(\mathbf{x})$. It is easy to see that the reason why $G(\mathbf{a}, \mathbf{b})$ remains unchanged under all diffeomorphisms of stimulus space (Dzhafarov & Colonius, 2001) is that diffeomorphisms preserve the topology of Re^n, the connectedness of the region constituting the stimulus space, and the continuous differentiability of paths $\mathbf{x}(t)$ in this space. Clearly, transformations that do not have these properties need not preserve Fechnerian distances and may very well lead to spaces where Fechnerian distances are not defined.

The issue is not only in the possible transformations of stimulus spaces representable by regions of Re^n, but also in the possibility of generalizing Fechnerian scaling to spaces of greater complexity, those that cannot be represented in this way. Thus, a natural representation of a human face (or "facial mask", like the one shown in Fig. 6) is by a function $\mathrm{Re}^2 \to \mathrm{Re}$ (elevation of the facial surface versus coordinates of its coronal cross-section). More generally, a set of monocular visual stimuli can be described as a set of functions $\mathrm{Re}^3 \to \mathrm{Re}^3$,

$$\mathbf{c}(\mathbf{x}) = (c^1(x^1, x^2, x^3), c^2(x^1, x^2, x^3), c^3(x^1, x^2, x^3)),$$

mapping, say, the azimuth, elevation, and time coordinates (x^1, x^2, x^3) into, say, a pair of CIE color coordinates (c^1, c^2) and photometric intensity (c^3). A set of binaural auditory stimuli can be analogously described by a set of pairs of functions $p(t) : \mathrm{Re} \to \mathrm{Re}$ (one for each ear) mapping time (t) into air pressure (p). The generalization of Fechnerian scaling from connected regions of Re^n to such spaces of functions or function pairs is neither trivial nor unique.

In view of this variety and complexity of physical descriptions, it is highly desirable to develop a theory that would not have to be adapted separately and specifically to each newly introduced stimulus space or a physical description thereof. This would mean a Fechnerian theory formulated for an *abstract* "continuous" stimulus space, with all its primitives and computations, including the property of its "continuity", being defined solely in terms of discrimination probabilities. The construction of such a Fechnerian theory is the task we undertake in this paper.

1.2. Example: Fechnerian scaling of discrete object sets

A simple prototype for this construction can be found in the theory of Fechnerian scaling of discrete object sets (FSDOS) recently proposed in Dzhafarov and Colonius (submitted). FSDOS applies to stimulus spaces comprised of "isolated entities", such as schematic faces, letters of an alphabet, brands of consumer products, etc. As in MDFS, each pair $(\mathbf{x}, \mathbf{y})$ of such stimuli is assigned a probability $\psi(\mathbf{x}, \mathbf{y})$ with which they are judged to be different from each other. The meaning of "same" and "different" in FSDOS may be different in different contexts: "$\mathbf{x}$ is the same as $\mathbf{y}$" may mean that they appear physically identical (as, e.g., in the case of pairs of Morse codes), or it may mean that they appear to belong to the same category or have the same source (thus, if the objects to be discriminated are writers each of whom is represented by several handwriting samples, the meaning of "same" will be that the two handwritings belong to the same writer).[2] Omitting as before, for brevity, the discussion of Regular Minimality and Points of Subjective Equality, FSDOS is based on local discriminability measures

$$\Psi^{(1)}(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x}, \mathbf{y}) - \psi(\mathbf{x}, \mathbf{x}),$$
$$\Psi^{(2)}(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{y}, \mathbf{x}) - \psi(\mathbf{x}, \mathbf{x}).$$

Fechnerian distance $G(\mathbf{a}, \mathbf{b})$ in FSDOS is defined as the infimum of identical quantities

$$\sum_{i=0}^{k} \Psi^{(1)}(\mathbf{x}_i, \mathbf{x}_{i+1}) + \sum_{i=0}^{l} \Psi^{(1)}(\mathbf{y}_i, \mathbf{y}_{i+1}) = \sum_{i=0}^{k} \Psi^{(2)}(\mathbf{x}_{i+1}, \mathbf{x}_i) + \sum_{i=0}^{l} \Psi^{(2)}(\mathbf{y}_{i+1}, \mathbf{y}_i)$$

computed across all possible finite chains of stimuli

$$\mathbf{a} = \mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{x}_{k+1} = \mathbf{b} = \mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_l, \mathbf{y}_{l+1} = \mathbf{a}$$

connecting $\mathbf{a}$ to $\mathbf{b}$ (the $\mathbf{x}$-chain) and $\mathbf{b}$ to $\mathbf{a}$ (the $\mathbf{y}$-chain).

One can see that in this case we do not make use of any physical measurements of the stimuli, and do not even assume that such measurements exist. The only notion that may at first seem not to be derived from discrimination probabilities alone is the notion of the space being "discrete", comprised of "isolated points".

[2] In this example all different handwriting samples by a given writer are considered to be instances of one and the same stimulus, and $\psi(\mathbf{x}, \mathbf{y})$ is estimated by the proportion of handwriting pairs judged to be written by different writers among all handwriting pairs one of which is by writer $\mathbf{x}$ and another by writer $\mathbf{y}$.

The discreteness property, however, can be defined as follows:

> $\mathbf{x}$ is an isolated point if $\Psi^{(1)}(\mathbf{x},\mathbf{y})$ and $\Psi^{(2)}(\mathbf{x},\mathbf{y})$ computed across all stimuli $\mathbf{y}$ do not fall below some positive quantity; a discrete stimulus space consists of isolated points.

Since $\Psi^{(1)}(\mathbf{x},\mathbf{y})$ and $\Psi^{(2)}(\mathbf{x},\mathbf{y})$ are defined in terms of $\psi(\mathbf{x},\mathbf{y})$, we have an example here of a purely psychological definition, and FSDOS therefore is an example of a purely psychological theory.

1.3. To prevent misunderstanding

A purely psychological theory does not free us from the necessity of using physical measurements for identifying stimuli whose subjective representations (in our case, subjective dissimilarities) are being studied. Unless the stimuli can be identified ostensively, which is only possible if they are finite in number and physically available at every demonstration (as it might be, say, with X-ray films or letters of alphabet), the use of some form of a systematic physical description is unavoidable. Thus, in the foregoing examples with visually presented rectangles, the latter had to be identified in some way, say, by their areas and aspect ratios (preceded by some physical description explicating the specific meaning of a "visually presented rectangle" in a given experimental set-up). In the subsequent discussion we will freely use conventional representations of stimulus sets by intervals of real numbers to provide examples of discrimination probability functions or to describe experimental data. The purely psychological character of our theory, however, ensures that all its notions and computations remain precisely the same if one replaces a given description of stimuli by any other description preserving their identity. Thus, a computational procedure in this theory cannot include instructions like "divide the value of stimulus $\mathbf{a}$ by the value of stimulus $\mathbf{b}$", or "consider a series of stimuli $\mathbf{x}_n$ whose numerical values converge to that of stimulus $\mathbf{a}$", because these instructions will have different outcomes under different physical descriptions of the stimuli (bijectively related to each other). Examples of analogous but purely psychological instructions might be "divide the discrimination probability $\psi(\mathbf{a},\mathbf{b})$ by $\psi(\mathbf{a},\mathbf{c})$", or "consider a sequence of stimuli $\mathbf{x}_n$ such that $\psi(\mathbf{x}_n,\mathbf{a}) \to 0$".

2. Notation and plan

2.1. Notation conventions

Italics and Greek letters designate real-valued quantities.

Boldface lowercase letters $\mathbf{x},\mathbf{y},\mathbf{a},\ldots$ always denote stimuli, or functions mapping into a set of stimuli, as in $\mathbf{x}(t):[a,b]\to\mathfrak{M}$. Note that stimuli are treated as "qualitative", nonnumerical entities, even if their variable attributes can be represented by real numbers or real-component vectors. (Occasionally, in illustrations or descriptions of experiments, the variable real-valued attributes of stimuli $\mathbf{x},\mathbf{y},\mathbf{a},\ldots$ are denoted by lowercase italics, $x,y,a,\ldots$)

All sets (of stimuli, real numbers, functions, etc.) are denoted by Gothic letters $\mathfrak{M},\mathfrak{m},\mathfrak{S},\mathfrak{s},\ldots$, except for the sets of all reals and of all nonnegative reals, denoted by Re and Re^{+}, respectively.

Capital open letters $\mathbb{A},\mathbb{B},\ldots$ are used to denote sets of sets of stimuli.

We use symbol $\to$ in three different meanings, clearly distinguishable by context: to designate mappings (as in $[a,b]\to\mathfrak{M}$), to designate convergence of real numbers (e.g., $\varepsilon_n\to 0$), and to designate convergence of stimuli, as defined in Section 5 (e.g., $\mathbf{x}_n\to\mathbf{x}$).

Greek ι (occasionally also κ) is reserved to represent "observation area" (as defined in Section 4), its value is always 1 or 2. This symbol, as well as its specific values (1 or 2) are used either as superscripts ($\Psi^{(\iota)}$, $\Psi^{(1)}$, $\Psi^{(2)}$, $\mathfrak{B}^{(\iota)}$, $\mathfrak{B}^{(1)}$, $\mathfrak{B}^{(2)}$, etc., parenthesized to distinguish them from exponents) or as subscripts (F_ι, F_1, F_2, G_ι, G_1, G_2, etc.). The difference between superscripts and subscripts in reference to "observation area" is purely decorative.

The logical and set-theoretic symbols are used in a standard way (thus, $\wedge$ denotes conjunction, $\Rightarrow$ implication, etc.).

Script letters are used occasionally, for ad hoc purposes.

2.2. Plan of the paper

The remainder of the paper consists of 11 sections, presenting a systematic development of the topology, analytic properties, and the metric of subjective distances in stimulus sets, all of this being derived from the discrimination probability functions endowed with certain properties, presented in the form of seven axioms.

In Section 3 we introduce the basic distinction between two observation areas (to which the two stimuli $\mathbf{x},\mathbf{y}$ being compared belong) and explain an important distinction: discrimination probabilities $\psi(\mathbf{x},\mathbf{y})$ are always computed across the two observation areas, while the subjective (Fechnerian) distances $G(\mathbf{a},\mathbf{b})$ are computed within each of the observation areas separately.

In Section 4 we define the relation of psychological equality (indistinguishability) among stimuli and formulate what we consider to be the main property of discrimination probabilities, the principle of Regular Minimality. Based on this property we define the notion of psychometric increments $\psi(\mathbf{x},\mathbf{y})-\psi(\mathbf{x},\mathbf{x})$, $\psi(\mathbf{y},\mathbf{x})-$

$\psi(\mathbf{x}, \mathbf{x})$ (of the first and second kind, respectively) used in all Fechnerian computations.

In Section 5 we use the psychometric increments to transform the set of stimuli into a topological space, which turns out to be an arcwise connected Urysohn space,[3] with the discrimination probability function and psychometric increments being continuous in this space.

In Section 6 we define the notion of a smooth arc in the stimulus space, a generalization of the continuously differentiable arcs in Euclidean spaces. The smoothness, however, is now defined entirely in terms of discrimination probabilities. Smooth arcs serve to integrate ''infinitesimally small'' psychometric increments along their lengths.

In Section 7 we postulate that psychometric increments along smooth arcs are all comeasurable in the small (i.e., as they tend to zero, their ratios have finite positive limits), and establish that psychometric increments are regularly varying functions. Asymptotically, these functions differ from each other only in coefficients that depend on infinitesimally small portions of smooth arcs.

A power transformation of these coefficients, as shown in Section 8, can be used as quantities, termed submetric functions, whose integration along any piecewise smooth arc yields the psychometric (''subjective'') length of this arc.

The notion of psychometric length is formally introduced in Section 9, where we also define the notion of oriented Fechnerian distances $G_1(\mathbf{a}, \mathbf{b})$ and $G_2(\mathbf{a}, \mathbf{b})$ (of the first and second kind, according to the kind of the psychometric increments being integrated).

The basic analytic properties of these Fechnerian distances (such as their continuity and differentiability along smooth arcs) are established in Section 10.

In Section 11, we investigate the relationship between Fechnerian distances of the first and second kind, and establish what we call the Second Main Theorem of Fechnerian scaling: Fechnerian distance from **a** to **b** and back is the same for distances of the first and the second kinds (provided that the Points of Subjective Equality, as defined in Section 4, are identically labeled in the two observation areas). This ''to-and-fro'' distance therefore is taken to be the '' true'' (overall) Fechnerian distance $G(\mathbf{a}, \mathbf{b})$ between **a** and **b**.

In the concluding two sections we summarize the main results and discuss some open questions.

The reader may find it convenient, now or as needed, to complement the above set of pointers by referring to Section 12 for preview of the corresponding sections of the paper. The reader may also find useful to consult the index of the main terms and notation at the end of the paper.

[3]All special mathematical terms and their basic properties are briefly explained on or prior to their first being mentioned in the text.

3. Preliminaries

3.1. On the notion of observation area

Two stimuli presented for a comparison can never be physically identical. This is an elementary yet often overlooked fact. Even when we say that two tones or two color patches presented to an observer are physically identical in all parameters, one of these stimuli has to be presented prior to the second, or in another spatial location. That is why $(\mathbf{x}, \mathbf{x})$ can be viewed as a pair of stimuli rather than a single stimulus. Strictly speaking, therefore, we are dealing with (at least) two stimulus sets, $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$, rather than a single one. $\mathfrak{M}_1^*$ may consist of auditory tones or visually presented rectangles described as ''a tone of intensity A dB and frequency F Hz presented first'' or ''a rectangle of height h and width v presented on the left'', whereas elements of $\mathfrak{M}_2^*$ may be described, respectively, as ''a tone of intensity A dB and frequency F Hz presented second'' and ''a rectangle of height h and width v presented on the right''. We refer to $\mathfrak{M}_1^*$ as the set of stimuli belonging to the ''*first observation area*'', and to $\mathfrak{M}_2^*$ as the set of stimuli belonging to the ''*second observation area*''. The adjective '' first'' and ''second'' refer to the ordinal positions of symbols used to designate the stimulus within a pair $(\mathbf{x}, \mathbf{y})$, rather than to their chronological order.

The stimuli in the two observation areas may differ in attributes other than their spatiotemporal location, and in some cases it is the difference in these attributes that is used to define the observation areas as first and second, in addition to or even irrespective of their spatiotemporal location. For example, two sequentially presented tones varying in intensity may have two fixed and different from one another frequencies, say, 1000 and 1200 Hz. In this case the elements of $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ can be described as, respectively, ''a tone of intensity A dB and frequency 1000 Hz'' and ''a tone of intensity A dB and frequency 1200 Hz'', irrespective of their temporal order.

In all these cases the physical descriptions of stimuli in $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ are identical except for the difference in certain properties (such as location) that define the observation area to which a stimulus belongs. When comparing two stimuli, the difference between their observation areas is usually perceptually conspicuous, and the observer is supposed to ignore it. The instruction to ignore this difference need not be explicit: when asked to say whether the stimulus on the left is identical to the stimulus on the right, the observer would normally understand that the judgment must not take into account the difference between the two spatial locations.

In general one can consider a situation involving more than just two observation areas. For example, each of the two stimuli can be presented in one of N perceptually distinct locations, so that the stimulus pair should be encoded as $((\mathbf{x}, a), (\mathbf{y}, b))$, where $\mathbf{x}, \mathbf{y}$ are labels identifying the stimuli in all respects except for their locations, and $a, b \in \{1, \ldots, N\}$ are their locations (with the proviso that $a \neq b$). In this paper, however, we confine our analysis to the situation when stimuli belong to two fixed observation areas, so that $(\mathbf{x}, \mathbf{y})$ always means $((\mathbf{x}, 1), (\mathbf{y}, 2))$.

3.2. On the notion of discrimination

The empirical object of our analysis is a discrimination probability function

$$\psi^* : \mathfrak{M}_1^* \times \mathfrak{M}_2^* \to [0, 1]$$

interpreted as

$$\psi^*(\mathbf{x}, \mathbf{y}) = \Pr[\mathbf{x} \in \mathfrak{M}_1^* \text{ and } \mathbf{y} \in \mathfrak{M}_2^* \text{ are judged to be different}], \quad (1)$$

where $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ represent the sets of stimuli presented to the perceiver in two fixed and distinct observation areas.

The operational meaning of $\psi^*(\mathbf{x}, \mathbf{y})$ given in (1) is not the only possible one. There are other variants:

$$\psi^*(\mathbf{x}, \mathbf{y}) = \Pr[\mathbf{x} \in \mathfrak{M}_1^* \text{ and } \mathbf{y} \in \mathfrak{M}_2^* \text{ are judged to be different, ignoring property } \mathscr{A}] \quad (2)$$

(e.g., "are these two tones different, ignoring their difference in pitch?"), and

$$\psi^*(\mathbf{x}, \mathbf{y}) = \Pr[\mathbf{x} \in \mathfrak{M}_1^* \text{ and } \mathbf{y} \in \mathfrak{M}_2^* \text{ are judged to be different in property } \mathscr{B}, \text{ ignoring everything else}] \quad (3)$$

(e.g., "are these two figures different in shape?"). It should be clear from our discussion of the two observation areas, that the "pure" discriminations in (1) can in fact be viewed as a special case of (2), with $\mathscr{A}$ referring to a subjective representation of the observation area. The analysis presented in this paper applies to all forms and versions of $\psi^*(\mathbf{x}, \mathbf{y})$.

3.3. On the logic of Fechnerian scaling

We have just stipulated that a comparison of two stimuli is always made across two different sets, $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$, and the discrimination probabilities ψ^* are defined on their Cartesian product, $\mathfrak{M}_1^* \times \mathfrak{M}_2^*$. It may be useful to emphasize from the outset, however, that Fechnerian ("subjective") distances eventually derived from ψ^* are computed between stimuli belonging to one and the same observation area, the first or the second one (see Fig. 7). Thus, if stimuli $\mathbf{a}, \mathbf{b}$ are tones characterized by their intensity and frequency, the Fechnerian distance between $\mathbf{a}$ and $\mathbf{b}$ may mean the distance between the two tones presented first ($\mathbf{a}, \mathbf{b} \in \mathfrak{M}_1^*$), or the distance between the two tones presented second ($\mathbf{a}, \mathbf{b} \in \mathfrak{M}_2^*$), but never between $\mathbf{a} \in \mathfrak{M}_1^*$ and $\mathbf{b} \in \mathfrak{M}_2^*$. This stands in a sharp contrast to the fact that in $\psi^*(\mathbf{a}, \mathbf{b})$ the two stimuli $\mathbf{a}, \mathbf{b}$ necessarily belong to $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$, respectively. In particular, while $\psi^*(\mathbf{a}, \mathbf{b})$ and $\psi^*(\mathbf{b}, \mathbf{a})$ are generally different quantities, the overall Fechnerian distance constructed in Section 11 is symmetrical, $G(\mathbf{a}, \mathbf{b}) = G(\mathbf{b}, \mathbf{a})$. Moreover, the Second Main Theorem of Fechnerian Scaling (Theorem 53) tells us that if the stimulus in $\mathfrak{M}_2^*$ which is the least discriminable from any given $\mathbf{a} \in \mathfrak{M}_1^*$ is assigned the same label $\mathbf{a}$ (that this is always possible is guaranteed by the Regular Minimality principle), then $G(\mathbf{a}, \mathbf{b})$ has the same value in both observation areas.

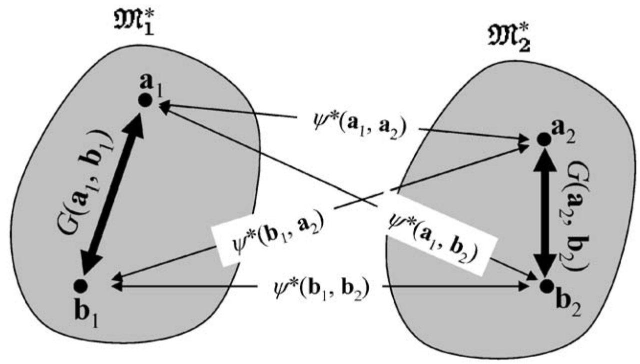

Fig. 7. Important aspect of Fechnerian scaling: discrimination probabilities ψ^* are computed *across* two observations areas, whereas Fechnerian distances G are computed *within* observation areas. Anticipating subsequent development: $G(\mathbf{a}_1, \mathbf{b}_1)$ and $G(\mathbf{a}_2, \mathbf{b}_2)$ are sums of oriented Fechnerian distances, $G_1(\mathbf{a}_1, \mathbf{b}_1) + G_1(\mathbf{b}_1, \mathbf{a}_1)$ and $G_2(\mathbf{a}_1, \mathbf{b}_1) + G_2(\mathbf{b}_1, \mathbf{a}_1)$, respectively; and $G(\mathbf{a}_1, \mathbf{b}_1) = G(\mathbf{a}_2, \mathbf{b}_2)$ when $\mathbf{a}_1$ and $\mathbf{b}_1$ are Points of Subjective Equality for, respectively, $\mathbf{a}_2$ and $\mathbf{b}_2$ (which means that if the Points of Subjective Equality are assigned identical labels, Fechnerian distance is observation-area-invariant).

To reconcile this difference between $\psi^*(\mathbf{a}, \mathbf{b})$ and $G(\mathbf{a}, \mathbf{b})$ with one's intuition, one has to realize that Fechnerian scaling does not presuppose any direct, let alone monotone, relationship between $\psi^*(\mathbf{a}, \mathbf{b})$ and $G(\mathbf{a}, \mathbf{b})$. Rather the entire set of Fechnerian distances $G(\mathbf{a}, \mathbf{b})$ (where $\mathbf{a}, \mathbf{b} \in \mathfrak{M}_1^*$ or $\mathbf{a}, \mathbf{b} \in \mathfrak{M}_2^*$) is computed from the entire set of the discrimination probabilities $\psi^*(\mathbf{x}, \mathbf{y})$ (where $\mathbf{x} \in \mathfrak{M}_1^*$ and $\mathbf{y} \in \mathfrak{M}_2^*$), so that for any given $\mathbf{a}, \mathbf{b}$ the value of $G(\mathbf{a}, \mathbf{b})$ is based generally on the values of $\psi^*(\mathbf{x}, \mathbf{y})$ for all possible pairs $(\mathbf{x}, \mathbf{y})$.

One should note an interesting form of duality here. Intuitively, the Fechnerian distance between $\mathbf{a}, \mathbf{b}$ belonging to $\mathfrak{M}_1^*$, is based on how different are the discrimination probabilities $\psi^*(\mathbf{a}, \mathbf{y})$ and $\psi^*(\mathbf{b}, \mathbf{y})$, taken across all $\mathbf{y} \in \mathfrak{M}_2^*$; while the Fechnerian distance between $\mathbf{a}, \mathbf{b}$ belonging to $\mathfrak{M}_2^*$, is based on how different are the discrimination probabilities $\psi^*(\mathbf{x}, \mathbf{a})$ and $\psi^*(\mathbf{x}, \mathbf{b})$, taken across all $\mathbf{x} \in \mathfrak{M}_1^*$. In particular, if $\psi^*(\mathbf{a}, \mathbf{y})$ and $\psi^*(\mathbf{b}, \mathbf{y})$ are always equal, then $\mathbf{a}$ and $\mathbf{b}$ are psychologically indistinguishable and should be treated as one and the same stimulus in the first observation area (and analogously for the second observation area).

4. Identity of stimuli and the Regular Minimality principle

4.1. Identity of stimuli

Definition 1. For $\mathbf{x}, \mathbf{x}' \in \mathfrak{M}_1^*$, we say that the two stimuli are psychologically equal and write $\mathbf{x} \overset{1}{\approx} \mathbf{x}'$ iff $\psi^*(\mathbf{x}, \mathbf{y}) = \psi^*(\mathbf{x}', \mathbf{y})$ for any $\mathbf{y} \in \mathfrak{M}_2^*$. Analogously, the psychological equality $\mathbf{y} \overset{2}{\approx} \mathbf{y}'$ for $\mathbf{y}, \mathbf{y}' \in \mathfrak{M}_2^*$ is defined by $\psi^*(\mathbf{x}, \mathbf{y}) = \psi^*(\mathbf{x}, \mathbf{y}')$, for any $\mathbf{x} \in \mathfrak{M}_1^*$.

Clearly, $\overset{1}{\approx}$ and $\overset{2}{\approx}$ are equivalence relations, and we can partition $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ into corresponding sets of equivalence classes

$$\tilde{\mathfrak{M}}_1 = \mathfrak{M}_1^* / \overset{1}{\approx}, \quad \tilde{\mathfrak{M}}_2 = \mathfrak{M}_2^* / \overset{1}{\approx}.$$

That is, $\mathfrak{x} \in \tilde{\mathfrak{M}}_1$ iff $\mathfrak{x}$ is the equivalence class for some $\mathbf{x} \in \mathfrak{M}_1^*$, and $\mathfrak{y} \in \tilde{\mathfrak{M}}_2$ iff $\mathfrak{y}$ is the equivalence class for some $\mathbf{y} \in \mathfrak{M}_2^*$.

We can introduce now a new discrimination probability function

$$\tilde{\psi} : \tilde{\mathfrak{M}}_1 \times \tilde{\mathfrak{M}}_2 \to [0, 1]$$

defined as

$$\tilde{\psi}(\mathfrak{x}, \mathfrak{y}) = \psi^*(\mathbf{x}, \mathbf{y}) \quad \text{for any } \mathbf{x} \in \mathfrak{x}, \ \ \mathbf{y} \in \mathfrak{y}.$$

As there is no danger of confusion, we can treat each equivalence class as a single stimulus, and write $\mathbf{x} \in \tilde{\mathfrak{M}}_1$, $\mathbf{y} \in \tilde{\mathfrak{M}}_2$, and $\tilde{\psi}(\mathbf{x}, \mathbf{y})$ instead of $\mathfrak{x} \in \tilde{\mathfrak{M}}_1$, $\mathfrak{y} \in \tilde{\mathfrak{M}}_2$, $\tilde{\psi}(\mathfrak{x}, \mathfrak{y})$. Fig. 8 provides a schematic illustration.

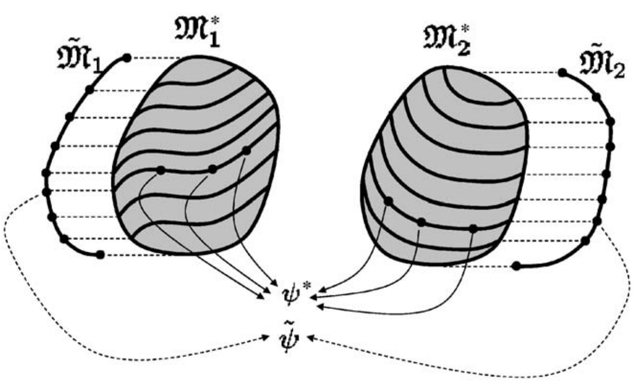

Fig. 8. Schematic demonstration of the transition from stimulus sets $\mathfrak{M}_1^*, \mathfrak{M}_2^*$ to sets $\tilde{\mathfrak{M}}_1, \tilde{\mathfrak{M}}_2$ of the psychological equivalence classes (represented in the picture as "slices"). Each "slice" of $\mathfrak{M}_1^*$ ($\mathfrak{M}_2^*$) is mapped into an element of $\tilde{\mathfrak{M}}_1$ ($\tilde{\mathfrak{M}}_2$). All stimuli within a given "slice" of $\mathfrak{M}_1^*$ have the same probability ψ^* of being judged different from any stimulus within a given "slice" of $\mathfrak{M}_2^*$, and this probability is taken as the value of function $\tilde{\psi}$ for the corresponding elements of $\tilde{\mathfrak{M}}_1, \tilde{\mathfrak{M}}_2$.

The best known example for the transition from $\mathfrak{M}_\iota^*$ to $\tilde{\mathfrak{M}}_\iota$ ($\iota = 1, 2$) and treating equivalence classes of stimuli as individual stimuli, is the replacement of metameric spectral distributions with CIE color coordinates.

4.2. Regular Minimality principle

We introduce now the property of Regular Minimality that we view to be the *defining* property of discrimination, in the sense that a function $\tilde{\psi}(\mathbf{x}, \mathbf{y})$ that violates this property should not be called a discrimination probability function.

Axiom 1 (*Regular Minimality*). *There are functions* $\mathbf{h}: \tilde{\mathfrak{M}}_1 \to \tilde{\mathfrak{M}}_2$ *and* $\mathbf{g}: \tilde{\mathfrak{M}}_2 \to \tilde{\mathfrak{M}}_1$ *such that*

(i) $\tilde{\psi}(\mathbf{x}, \mathbf{h}(\mathbf{x})) < \tilde{\psi}(\mathbf{x}, \mathbf{y})$ *for all* $\mathbf{y} \neq \mathbf{h}(\mathbf{x})$,
(ii) $\tilde{\psi}(\mathbf{g}(\mathbf{y}), \mathbf{y}) < \tilde{\psi}(\mathbf{x}, \mathbf{y})$ *for all* $\mathbf{x} \neq \mathbf{g}(\mathbf{y})$,
(iii) $\mathbf{h} \equiv \mathbf{g}^{-1}$.

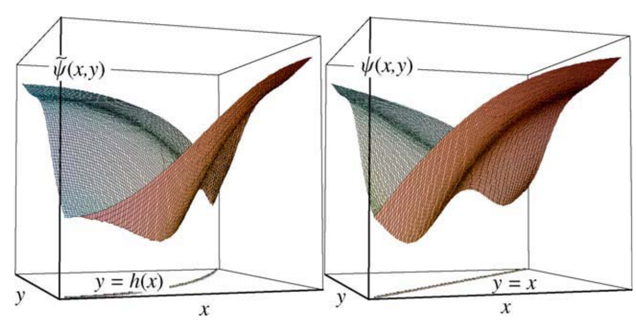

Fig. 9. Left: an example of discrimination probability function $\tilde{\psi}(x, y)$, with $x, y \in (0, 1) = \tilde{\mathfrak{M}}_1 = \tilde{\mathfrak{M}}_2$. The function is $1 - \exp[-B(x, y)/10 + 4]$, where $B(x, y) = 20|x^2 - y| + \sin(4x^2 - 2) + \cos(10y - 5)$. PSE function $y = h(x)$ (here, $y = x^2$) is shown by the thick line in the xy-plane. Right: a canonical transformation $\psi(x, y) = \tilde{\psi}(g(x), y) = \tilde{\psi}(\sqrt{x}, y)$ of this function, with $x, y \in (0, 1) = \mathfrak{M}$; PSE function in the xy-plane transforms into bisector $y = x$. (The functions $\tilde{\psi}, \psi$ in this figure are constructed in accordance with the "uncertainty blobs" model, described in Dzhafarov, 2003b. They satisfy the cross-unbalanced version of the Fechnerian theory, as described in Section 11).

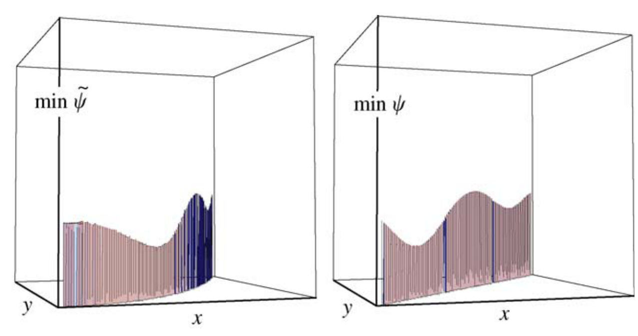

Fig. 10. The PSE lines of the previous figure (the lower contours of the "walls") shown together with the corresponding minimum level functions (the upper contours of the "walls"): $\tilde{\psi}(x, h(x))$, or $\tilde{\psi}(g(y), y)$, on the left and $\psi(x, x)$, or $\psi(y, y)$, on the right.

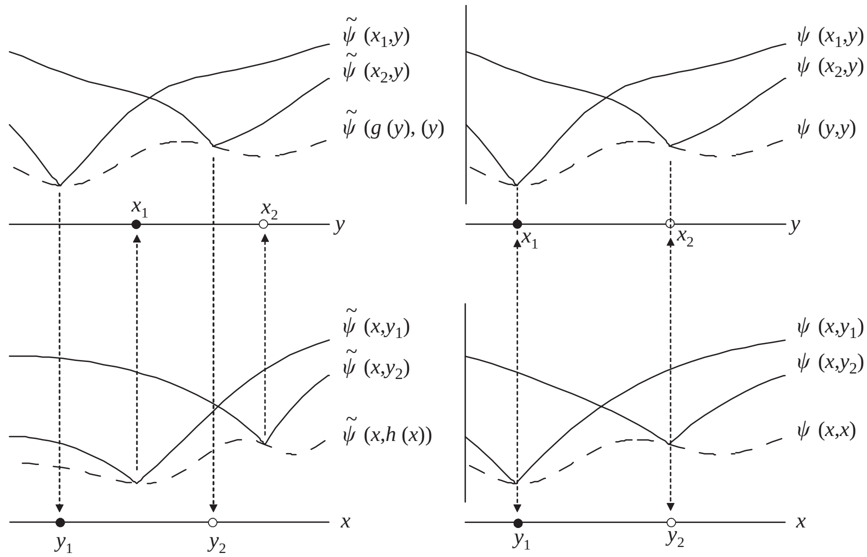

Fig. 11. An illustration of Regular Minimality and Nonconstant Self-Dissimilarity using $\tilde{\psi}(x, y)$ and $\psi(x, y)$ shown in Fig. 9. By solid lines the left upper panel shows two V-shaped cross-sections of $\tilde{\psi}(x, y)$ made at two fixed values x_1, x_2 of x (indicated by dots placed on the y-axis). As y varies, these functions achieve their minima as some points y_1, y_2, and these points are indicated by dots on the x-axis in the left lower panel. Looking now at the V-shaped cross-section of $\tilde{\psi}(x, y)$ at the fixed values y_1, y_2 of y (solid lines in the left lower panel), they achieve their minima precisely at the "starting" values x_1, x_2 of x. The right panels illustrate Regular Minimality in the same way, but here $x_1 = y_1$ and $x_2 = y_2$ due to canonical transformation. Nonconstant self-dissimilarity is apparent in the dashed lines that show the minimum level function, presented as $\tilde{\psi}(g(y), y)$ in the left upper panel and as $\tilde{\psi}(x, h(x))$ in the left lower panel. Here, $h(x) = x^2$, $g(y) = \sqrt{y}$. On the right, the minimum level functions in the two panels are identical: $\psi(y, y)$ (top) and $\psi(x, x)$ (bottom).

The following observation is obvious.

Corollary 1 (*to Axiom 1*). *Mappings* $\mathbf{h}$ *and* $\mathbf{g}$ *are bijective* (*one-to-one and onto*), *and the stimulus sets* $\tilde{\mathfrak{M}}_1 = \mathbf{g}(\tilde{\mathfrak{M}}_2)$ *and* $\tilde{\mathfrak{M}}_2 = \mathbf{h}(\tilde{\mathfrak{M}}_1)$ *have the same cardinality*.

Definition 2. For every $\mathbf{x} \in \tilde{\mathfrak{M}}_1$, its Point of Subjective Equality (PSE) is the stimulus $\mathbf{h}(\mathbf{x}) \in \tilde{\mathfrak{M}}_2$. Analogously, $\mathbf{g}(\mathbf{y}) \in \tilde{\mathfrak{M}}_1$ is the PSE for $\mathbf{y} \in \tilde{\mathfrak{M}}_2$.

According to this definition, the Axiom of Regular Minimality states that every stimulus (either in $\tilde{\mathfrak{M}}_1$ or in $\tilde{\mathfrak{M}}_2$) has its unique PSE (in, respectively, $\tilde{\mathfrak{M}}_2$ and $\tilde{\mathfrak{M}}_1$), and that the relation of "being the PSE of" is symmetrical. Figs. 9–11 (left panels) provide an example of a discrimination probability function that satisfies the Regular Minimality requirement.

4.3. Nonconstant Self-Dissimilarity and Asymmetry

Function $\tilde{\psi}(\mathbf{x}, \mathbf{h}(\mathbf{x}))$, or equivalently, $\tilde{\psi}(\mathbf{g}(\mathbf{y}), \mathbf{y})$, is called the *minimum level function*. It is obtained by confining $\tilde{\psi}(\mathbf{x}, \mathbf{y})$ to pairs $(\mathbf{x}, \mathbf{y})$ of mutual PSEs. The hypothesis that $\tilde{\psi}(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ has one and the same value for all $\mathbf{x} \in \tilde{\mathfrak{M}}_1$ can be called the *Constant Self-Dissimilarity assumption* (the prefix "self" referring to PSE relationship rather than physical identity). The Axiom of Regular Minimality in no way implies that this assumption holds true. In fact it is well documented that in general $\tilde{\psi}(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ does vary with $\mathbf{x}$. We call this important property *Nonconstant Self-Dissimilarity*: $\tilde{\psi}(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ *does not have to be constant for all* $\mathbf{x} \in \tilde{\mathfrak{M}}_1$ (Dzhafarov, 2002d, 2003a,b).[4] This property does not compel $\tilde{\psi}(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ to be different for any two distinct values of $\mathbf{x}$, it merely asserts that there is no law that compels them to be equal. (Clearly, we could have as well formulated this property in terms of $\tilde{\psi}(\mathbf{g}(\mathbf{y}), \mathbf{y})$, $\mathbf{y} \in \tilde{\mathfrak{M}}_2$.) The property is illustrated in Figs. 9–11 (left panels).

The value of $\tilde{\psi}(\mathbf{x}, \mathbf{y})$ is generally different from $\tilde{\psi}(\mathbf{y}, \mathbf{x})$, a property we call Asymmetry, or Order-Unbalance (Dzhafarov, 2002d). Note that in the special case when $\tilde{\psi}(\mathbf{x}, \mathbf{y})$ is symmetrical, the PSE function $\mathbf{h}(\mathbf{x})$ need not be an identity (i.e., the PSE relationship need not imply physical identity), and $\tilde{\psi}(\mathbf{x}, \mathbf{y})$ is still subject to Nonconstant Self-Dissimilarity (i.e. the value of $\tilde{\psi}(\mathbf{x}, \mathbf{h}(\mathbf{x}))$ may vary with $\mathbf{x}$).

Convention 1. *We treat Constant Self-Dissimilarity and Symmetry as special cases of Nonconstant Self-Dissimilarity and Asymmetry. We will say that Nonconstant Self-Dissimilarity* (*or Asymmetry*) *is manifest in stimulus space if there are actual cases when Constant Self-Dissimilarity* (*Symmetry*) *is violated.*

[4] In the cited work the property was referred to as Nonconstant Self-similarity. We find the present version better corresponding to the definition of function $\tilde{\psi}$.

4.4. Empirical evidence

Figs. 13–15 present experimental results illustrating Regular Minimality and Nonconstant Self-Dissimilarity in the format of Fig. 10. Relevant details of experimental set-up and procedures are given in Fig. 12.

Once, for a given participant in a given experiment, the matrix shown in Fig. 12 was filled with estimates of $\tilde{\psi}(x, y)$, we determined the PSEs for the five x-values within the "main stimulus area" ($a - 2\Delta \leqslant x \leqslant a + 2\Delta$) by picking the lowest-valued cells in the corresponding rows. Occasionally, two adjacent cells contained very close values $\tilde{\psi}(x, y)$ and $\tilde{\psi}(x, y + \Delta)$, and plotting of the row suggested that the minimum lied in between: in such cases the PSE for x was set equal to $y + \Delta/2$ and the minimum value of $\tilde{\psi}$ was taken as the smaller of $\tilde{\psi}(x, y)$ and $\tilde{\psi}(x, y + \Delta)$. The PSEs for five y-values within the "main stimulus area" were determined analogously, by scanning the corresponding columns of the matrix. A crude violation of Regular Minimality would have resulted in two at least partially disparate 5-point PSE curves, one for x-values and one for y-values. This is not the case in our plots. Although the data cannot exclude the possibility of subtle violations, they do not contradict the assumption that Regular Minimality is the fundamental property of discrimination probabilities. At the same time the $\tilde{\psi}(x, y)$-values for the PSE pairs in most of our plots clearly manifest Nonconstant Self-Dissimilarity (Figs. 12–15).

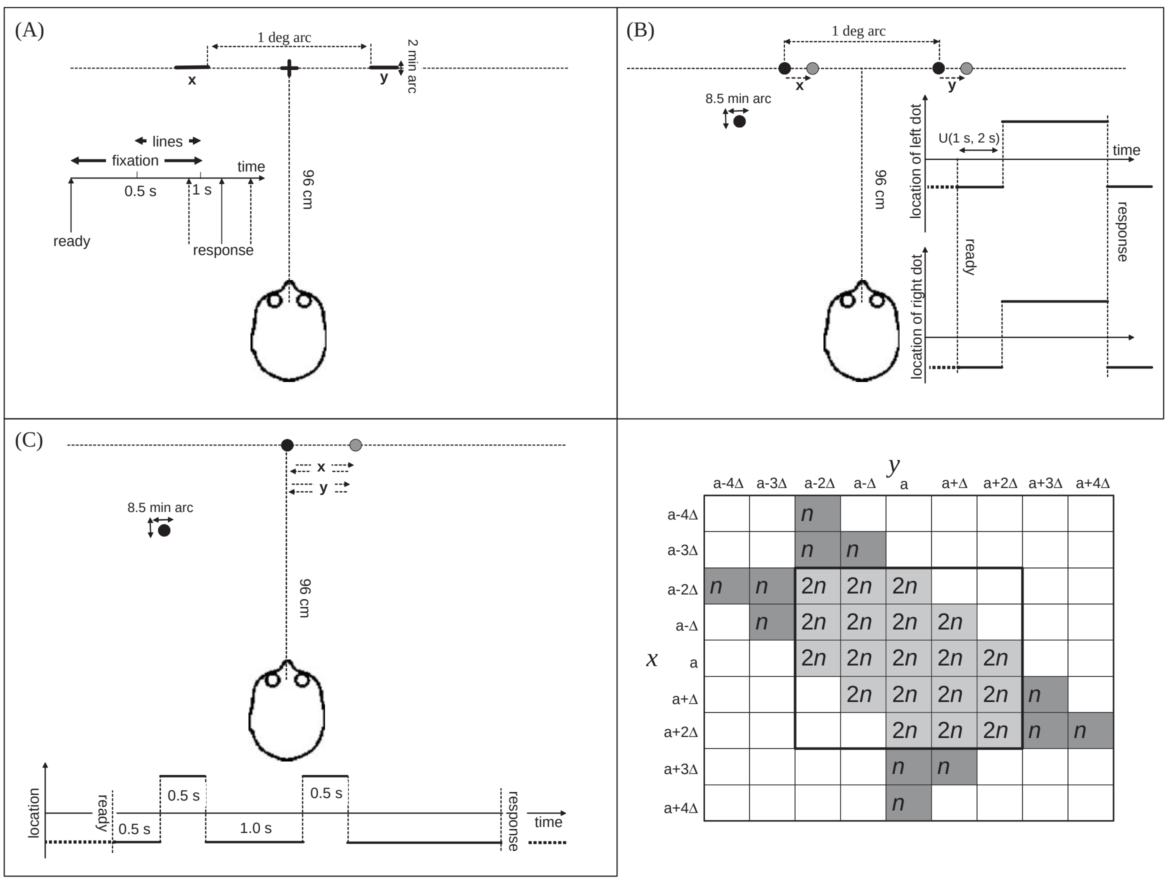

Fig. 12. Experimental set-up and procedures for the data presented in Figs. 13–15, and 30. Stimuli to be judged to be "same" or "different" were pairs of line segments (experiment A), synchronous left-to-right apparent motions of two dots (B), and two successive to-and-fro apparent motions of a single dot (C). Participants initiated trials by pressing a " ready" key, and responded by pressing one of two response keys. In A, the "ready" signal made the fixation cross appear for 1 s and the two lines flash for the second half of this second. In B, the two dots were always present on the screen; the "ready" signal made them shift to new positions after an interval uniformly distributed between 1 and 2 s; a response made them shift back to their initial positions. In C, a dot was always present on the screen; the "ready" signal made it shift (0.5 s later) to a new position and (0.5 s later) back; the second to-and-fro motion followed the first one 1 s later. No feedback was given in any of the experiments. The observers, with normal or corrected vision, viewed the display binocularly, from a chin rest with forehead support, in a dark room. The lines and dots were blue on a dark gray flat screen, the intensity and contrast being fixed throughout at comfortable levels. The variable parameter of the stimuli was length of the lines in A, and amplitude of the position shifts in B and C. The pairs of length/amplitude values (x, y) were chosen from the shaded cells of the matrix shown. The value of a was 15 pixels and Δ was 2 pixels (1 pixel $\approx$ 0.86 min arc), except in the three right-hand panels in Fig. 14 where the respective values were 11 pixels and 1 pixel. $2n$ and n are the numbers of replications per pair. The experiment was divided into 300-trial sessions (c. 25 min), repeated many times over several days; in each session n was 6, the sequence of the pairs being otherwise randomized. The total value of n in one experiment for one participant was between 250 and 300 (except in the data labelled WIN in Fig. 15 where n was about 100).

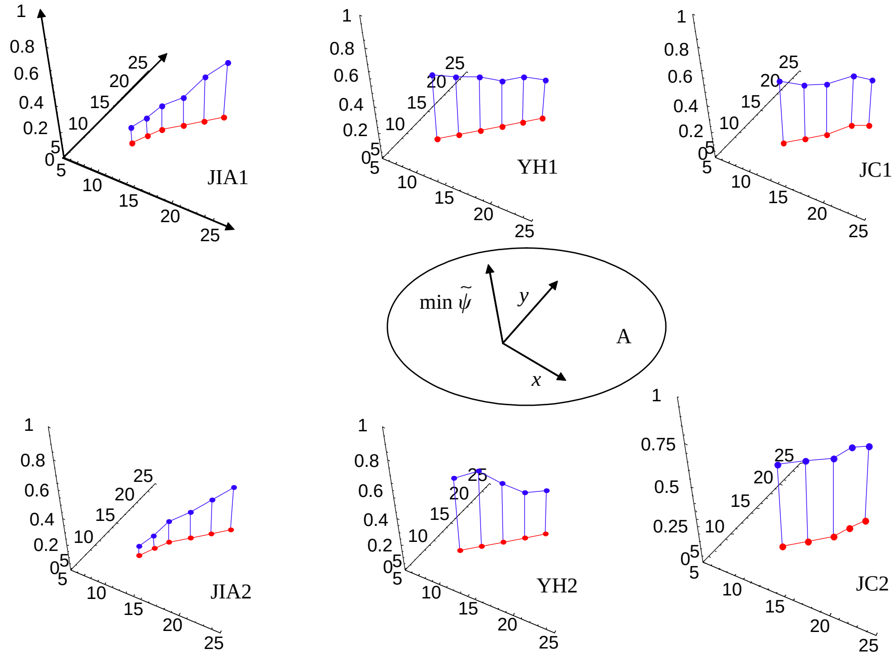

Fig. 13. Experiment A (corresponding to panel A in Fig. 12): x and y are lengths of two line segments, in pixels (1 pixel $\approx$ 0.86 min arc). This is an empirical version of Fig. 10, left. The lower line (in the xy-plane) is the estimated PSE line, the upper one is the minimum level function. The PSE points were estimated for the values of x and y within the main stimulus area, i.e., between $a - 2\Delta$ and $a + 2\Delta$ in reference to the matrix in Fig. 12. More than 5 dots in a PSE line indicate that in some cases the minimum was interpolated between two close-valued cells of the matrix.

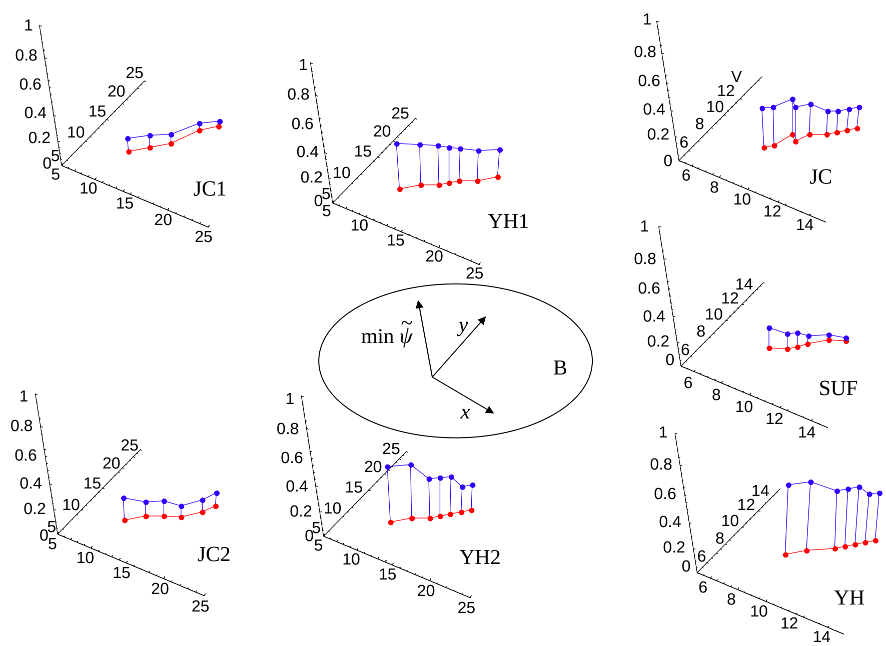

Fig. 14. Experiment B (corresponding to panel B in Fig. 12): x and y are amplitudes of two synchronous apparent motions. The rest as in Fig. 13.

4.5. Canonical transformation (relabeling)

Using the axiom of Regular Minimality we can conveniently redefine the stimulus sets once more, by assigning identical labels to pairs of mutual PSEs, $\mathbf{x} = \mathbf{g}(\mathbf{y}), \mathbf{y} = \mathbf{h}(\mathbf{x})$.

Let $\mathfrak{M}$ be an arbitrary set (called a set of *stimulus labels*) having the same cardinality as $\tilde{\mathfrak{M}}_1$ and $\tilde{\mathfrak{M}}_2$. Let

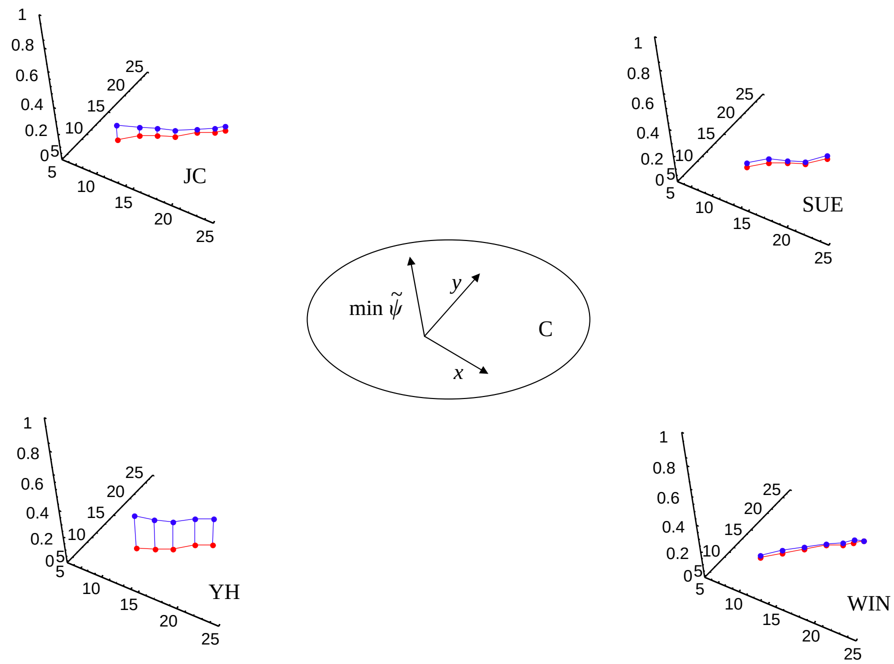

Fig. 15. Experiment C (corresponding to panel C in Fig. 12): x and y are amplitudes of two successive to-and-fro apparent motions. The rest is as in Fig. 13.

$\mathbf{f}_1 : \mathfrak{M} \to \tilde{\mathfrak{M}}_1$ be an arbitrary bijective mapping. Then $\mathbf{f}_2 \equiv \mathbf{h} \circ \mathbf{f}_1$ is a bijective mapping $\mathbf{f}_2 : \mathfrak{M} \to \tilde{\mathfrak{M}}_2$ such that $\mathbf{x} \in \tilde{\mathfrak{M}}_1$ and $\mathbf{y} \in \tilde{\mathfrak{M}}_2$ are mutual PSEs if and only if

$$\mathbf{x} = \mathbf{f}_1(\mathbf{z}), \quad \mathbf{y} = \mathbf{f}_2(\mathbf{z})$$

for some stimulus label $\mathbf{z} \in \mathfrak{M}$. This construction allows us to introduce a new function,

$$\psi : \mathfrak{M} \times \mathfrak{M} \to [0, 1],$$

defined by

$$\psi(\mathbf{x}, \mathbf{y}) = \tilde{\psi}(\mathbf{f}_1(\mathbf{x}), \mathbf{f}_2(\mathbf{y})), \tag{4}$$

where $\mathbf{x}, \mathbf{y}$ now denote elements of $\mathfrak{M}$ (stimulus labels). This function is referred to as a *canonical transformation* of function $\tilde{\psi}(\mathbf{x}, \mathbf{y})$. An example of a canonical transformation is given in the right-hand panels of Figs. 9–11.

It is easy to see that this newly defined function satisfies the Regular Minimality axiom in its simplest version. Namely, function ψ can be treated as a special case of the function $\tilde{\psi}$ in the formulation of the axiom, with $\tilde{\mathfrak{M}}_1 = \tilde{\mathfrak{M}}_2 = \mathfrak{M}$ and the PSE functions $\mathbf{h}$ and $\mathbf{g}$ being identities:

$$\begin{aligned} &\psi(\mathbf{x}, \mathbf{x}) < \psi(\mathbf{x}, \mathbf{y}) \quad \text{for all } \mathbf{y} \neq \mathbf{x} \\ &\psi(\mathbf{x}, \mathbf{x}) < \psi(\mathbf{y}, \mathbf{x}) \quad \text{for all } \mathbf{y} \neq \mathbf{x} \end{aligned} \tag{5}$$

We will say that Regular Minimality holds here in a *canonical form*, and that $\psi(\mathbf{x}, \mathbf{x})$ is the minimum level function in a canonical form.

Two simplest choices of a canonical form are obtained by putting

$$\mathfrak{M} = \tilde{\mathfrak{M}}_1, \quad \mathbf{f}_1 \text{ is identity}, \quad \mathbf{f}_2 \equiv \mathbf{h}, \quad \psi(\mathbf{x}, \mathbf{y}) = \tilde{\psi}(\mathbf{x}, \mathbf{h}(\mathbf{y})),$$

and

$$\mathfrak{M} = \tilde{\mathfrak{M}}_2, \quad \mathbf{f}_1 \equiv \mathbf{g}, \quad \mathbf{f}_2 \text{ is identity}, \quad \psi(\mathbf{x}, \mathbf{y}) = \tilde{\psi}(\mathbf{g}(\mathbf{x}), \mathbf{y})$$

(recall that $\mathbf{g} \equiv \mathbf{h}^{-1}$). In the former construction the stimuli in the first observation area ($\tilde{\mathfrak{M}}_1$) serve as labels for their PSEs in the second observation area ($\tilde{\mathfrak{M}}_2$); in the latter construction the relation is reversed. A specific choice of $\mathbf{f}_1$ in (4), however, is immaterial for the subsequent development.

For brevity sake we use the same terminology for function ψ as we did for $\tilde{\psi}$ and, before that, for ψ^*. We will refer to arguments of $\psi(\mathbf{x}, \mathbf{y})$ (i.e., elements of $\mathfrak{M}$) as *stimuli*, instead of "stimulus labels", and we will call $\psi(\mathbf{x}, \mathbf{y})$ a *discrimination probability function*, with the interpretation

$$\psi(\mathbf{x}, \mathbf{y}) = \Pr[\text{stimuli } \mathbf{x} \text{ and } \mathbf{y} \text{ are judged to be different}]. \tag{6}$$

This should lead to no confusion provided one keeps in mind that the physical identity of $\mathbf{x} \in \mathfrak{M}$ depends on its ordinal position in an ordered pair. Thus, in $(\mathbf{x}, \mathbf{x})$, the first $\mathbf{x}$ and the second $\mathbf{x}$ may be physically distinct, and their physical identities can be uniquely reconstructed as $\mathbf{f}_1^{-1}(\mathbf{x}) \in \tilde{\mathfrak{M}}_1$ and $\mathbf{f}_2^{-1}(\mathbf{x}) \in \tilde{\mathfrak{M}}_2$, respectively (which in their turn, it should be recalled, are psychological equivalence classes of "initial" physical stimuli, elements of $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$). The fact that these two stimuli (equivalence classes) are now identically denoted is a straightforward reflection of the meaning of a PSE. No stimulus in $\tilde{\mathfrak{M}}_2$ is subjectively closer to $\mathbf{f}_1^{-1}(\mathbf{x})$ than its PSE $\mathbf{f}_2^{-1}(\mathbf{x})$, and vice versa, no stimulus in $\tilde{\mathfrak{M}}_1$ is subjectively closer to $\mathbf{f}_2^{-1}(\mathbf{x})$ than its PSE $\mathbf{f}_1^{-1}(\mathbf{x})$; by a canonical transformation, we can say instead that no stimulus in $\mathfrak{M}$ is closer to a given $\mathbf{x} \in \mathfrak{M}$ than $\mathbf{x}$ itself.

In relation to the previous subsection, Nonconstant Self-Dissimilarity in a canonical form means that $\psi(\mathbf{x},\mathbf{x})$ need not be the same for all $\mathbf{x} \in \mathfrak{M}$. Note that $\psi(\mathbf{x},\mathbf{y})$ generally continues to be subject to Asymmetry (Order-Unbalance): $\psi(\mathbf{x},\mathbf{y})$ and $\psi(\mathbf{y},\mathbf{x})$ need not be the same.

4.6. Psychometric increments

We introduce now the notion that plays a central role in the subsequent development.

Definition 3. The quantity

$$\Psi^{(1)}(\mathbf{x},\mathbf{y}) = \psi(\mathbf{x},\mathbf{y}) - \psi(\mathbf{x},\mathbf{x})$$

is called a psychometric increment of the first kind (or, in the second argument). Analogously,

$$\Psi^{(2)}(\mathbf{x},\mathbf{y}) = \psi(\mathbf{y},\mathbf{x}) - \psi(\mathbf{x},\mathbf{x})$$

is called a psychometric increment of the second kind (or, in the first argument).

Due to (5) the psychometric increments of both kinds are nonnegative, and

$$\Psi^{(1)}(\mathbf{x},\mathbf{y}) = 0 \Longleftrightarrow \Psi^{(2)}(\mathbf{x},\mathbf{y}) = 0 \Longleftrightarrow \mathbf{x} = \mathbf{y}. \tag{7}$$

Theorem 1. *There is a bijective correspondence between* $\mathbf{a} \in \mathfrak{M}$ *and any of the functions* $\mathbf{y} \to \Psi^{(1)}(\mathbf{a},\mathbf{y})$, $\mathbf{y} \to \Psi^{(2)}(\mathbf{a},\mathbf{y})$, $\mathbf{x} \to \Psi^{(1)}(\mathbf{x},\mathbf{a})$, $\mathbf{x} \to \Psi^{(2)}(\mathbf{x},\mathbf{a})$.

Proof. What is stated is that

$$\begin{aligned}\mathbf{a}_1 = \mathbf{a}_2 &\Longleftrightarrow [\forall \mathbf{y} : \Psi^{(1)}(\mathbf{a}_1,\mathbf{y}) = \Psi^{(1)}(\mathbf{a}_2,\mathbf{y})] \\ &\Longleftrightarrow [\forall \mathbf{y} : \Psi^{(2)}(\mathbf{a}_1,\mathbf{y}) = \Psi^{(2)}(\mathbf{a}_2,\mathbf{y})] \\ &\Longleftrightarrow [\forall \mathbf{x} : \Psi^{(1)}(\mathbf{x},\mathbf{a}_1) = \Psi^{(1)}(\mathbf{x},\mathbf{a}_2)] \\ &\Longleftrightarrow [\forall \mathbf{x} : \Psi^{(2)}(\mathbf{x},\mathbf{a}_1) = \Psi^{(2)}(\mathbf{x},\mathbf{a}_2)].\end{aligned}$$

The proof is obtained by putting $\mathbf{y} = \mathbf{a}_2$ or $\mathbf{x} = \mathbf{a}_2$ and using (7). □

5. Topology

In this section, we use the psychometric increments $\Psi^{(1)}(\mathbf{x},\mathbf{y})$ and $\Psi^{(2)}(\mathbf{x},\mathbf{y})$ to endow the stimulus set $\mathfrak{M}$ with a topology and to transform thereby this stimulus set into a *stimulus space*.

Recall that a *topology* on $\mathfrak{M}$ is a set $\mathbb{T}$ of *open* subsets of $\mathfrak{M}$, that is, a set of subsets satisfying the following properties:

$$\begin{aligned}&\mathbb{A} \subset \mathbb{T} \Rightarrow \bigcup_{\mathfrak{A} \in \mathbb{A}} \mathfrak{A} \in \mathbb{T} \quad (T1) \\ &(\mathfrak{A} \in \mathbb{T}) \wedge (\mathfrak{B} \in \mathbb{T}) \Rightarrow \mathfrak{A} \cap \mathfrak{B} \in \mathbb{T} \quad (T2) \\ &\varnothing \in \mathbb{T} \wedge \mathfrak{M} \in \mathbb{T} \quad (T3)\end{aligned} \tag{8}$$

A set $\mathbb{B}$ of subsets of $\mathfrak{M}$ is called a *base* for a topology on $\mathfrak{M}$ if

$$\begin{aligned}&\mathfrak{M} = \bigcup_{\mathfrak{B} \in \mathbb{B}} \mathfrak{B} \quad (B1) \\ &(\mathfrak{A} \in \mathbb{B}) \wedge (\mathfrak{B} \in \mathbb{B}) \wedge (\mathbf{x} \in \mathfrak{A} \cap \mathfrak{B}) \\ &\quad \Rightarrow \exists \mathfrak{C} \in \mathbb{B} : \mathbf{x} \in \mathfrak{C} \subset \mathfrak{A} \cap \mathfrak{B} \quad (B2)\end{aligned} \tag{9}$$

The topology for which $\mathbb{B}$ is the base is constructed by taking all possible unions of the base elements,

$$\bigcup_{\mathfrak{A} \in \mathbb{A} \subset \mathbb{B}} \mathfrak{A}.$$

A base $\mathbb{B}_2$ is a *refinement* of a base $\mathbb{B}_1$ if for every $(\mathbf{x}, \mathfrak{A}_1)$ such that $\mathbf{x} \in \mathfrak{A}_1 \in \mathbb{B}_1$ there is a $\mathfrak{A}_2 \in \mathbb{B}_2$ such that $\mathbf{x} \in \mathfrak{A}_2 \subset \mathfrak{A}_1$. The topology based on $\mathbb{B}_2$ is then also a refinement of the topology based on $\mathbb{B}_1$, in the sense that every point of every $\mathbb{B}_1$-open set is contained in a $\mathbb{B}_2$-open set. Two bases are equivalent if they refine each other. Equivalent bases induce one and the same topology.

5.1. Convergence

Axiom 2 (*Convergence*). *As* $n \to \infty$,

$$\Psi^{(1)}(\mathbf{x},\mathbf{x}_n) \to 0 \Longleftrightarrow \Psi^{(2)}(\mathbf{x},\mathbf{x}_n) \to 0.$$

Using the definition of psychometric increments, this means

$$\psi(\mathbf{x},\mathbf{x}_n) \to \psi(\mathbf{x},\mathbf{x}) \Longleftrightarrow \psi(\mathbf{x}_n,\mathbf{x}) \to \psi(\mathbf{x},\mathbf{x}).$$

This axiom allows us to define our first topological notion, that of convergence in stimulus space. It is convenient for this purpose, as well as for further topological considerations, to introduce function

$$\Psi(\mathbf{x},\mathbf{y}) = \min\{\Psi^{(1)}(\mathbf{x},\mathbf{y}), \Psi^{(2)}(\mathbf{x},\mathbf{y})\}. \tag{10}$$

Definition 4. As $n \to \infty$, we say that $\mathbf{x}_n$ converges to $\mathbf{x}$, and write $\mathbf{x}_n \to \mathbf{x}$, if $\Psi(\mathbf{x},\mathbf{x}_n) \to 0$.

Fig. 16 (top) provides a schematic illustration.

5.2. Intrinsic continuity of discrimination probability

To postulate the continuity of the discrimination probability function ψ in the conventional sense one has to have a topology imposed on its domain, $\mathfrak{M} \times \mathfrak{M}$. As we do not have this topology yet, we have to follow the opposite strategy: we will define a property that we call the *intrinsic continuity* of ψ, and then use it to impose the

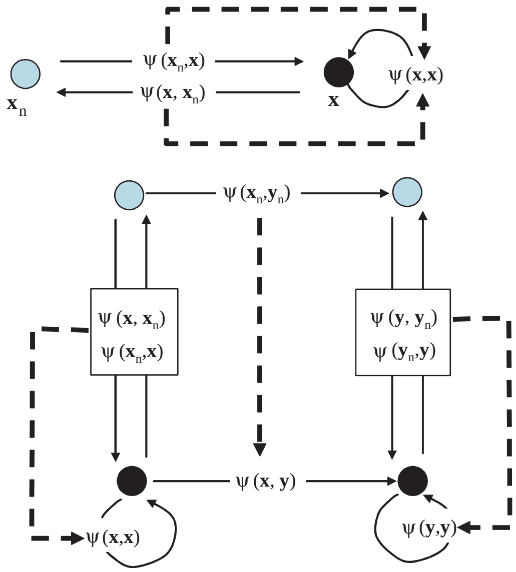

Fig. 16. Convergence relations (dashed lines) among discrimination probabilities (solid lines). Top: numerical convergences $\psi(\mathbf{x},\mathbf{x}_n) \to \psi(\mathbf{x},\mathbf{x})$ and $\psi(\mathbf{x}_n,\mathbf{x}) \to \psi(\mathbf{x},\mathbf{x})$ imply each other and define the convergence $\mathbf{x}_n \to \mathbf{x}$ in stimulus space. Bottom: intrinsic continuity of discrimination probabilities. The numerical convergences that are equivalent to $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{y}_n \to \mathbf{y}$ imply the numerical convergence $\psi(\mathbf{x}_n,\mathbf{y}_n) \to \psi(\mathbf{x},\mathbf{y})$.

topology on $\mathfrak{M}$ (and, by extension, on $\mathfrak{M} \times \mathfrak{M}$). After that, of course, we have to show that with respect to this topology ψ is continuous in the conventional sense. In a more restrictive context of "semi-metrics" a similar construction was used by Blumenthal (1953).

Axiom 3 (*Intrinsic Continuity*). *Discrimination probability function ψ is intrinsically continuous*:

$$(\mathbf{x}_n \to \mathbf{x}) \wedge (\mathbf{y}_n \to \mathbf{y}) \Rightarrow \psi(\mathbf{x}_n,\mathbf{y}_n) \to \psi(\mathbf{x},\mathbf{y}).$$

For an illustration see Fig. 16 (bottom).

Theorem 2. *Psychometric increments are intrinsically continuous*,

$$(\mathbf{x}_n \to \mathbf{x}) \wedge (\mathbf{y}_n \to \mathbf{y}) \Rightarrow \begin{cases} \Psi^{(1)}(\mathbf{x}_n,\mathbf{y}_n) \to \Psi^{(1)}(\mathbf{x},\mathbf{y}), \\ \Psi^{(2)}(\mathbf{x}_n,\mathbf{y}_n) \to \Psi^{(2)}(\mathbf{x},\mathbf{y}), \\ \Psi(\mathbf{x}_n,\mathbf{y}_n) \to \Psi(\mathbf{x},\mathbf{y}). \end{cases}$$

Proof. Follows from the definition of psychometric increments and the fact that, by Axiom 3,

$$\mathbf{x}_n \to \mathbf{x} \Rightarrow (\mathbf{x}_n \to \mathbf{x}) \wedge (\mathbf{x}_n \to \mathbf{x}) \Rightarrow \psi(\mathbf{x}_n,\mathbf{x}_n) \to \psi(\mathbf{x},\mathbf{x}). \quad \square$$

Theorem 3. $\mathbf{x}_n \to \mathbf{x}$ *implies* $\Psi^{(\iota)}(\mathbf{x}_n,\mathbf{x}) \to 0$, *for* $\iota = 1, 2$.

Proof. $\mathbf{x}_n \to \mathbf{x} \Rightarrow (\mathbf{x}_n \to \mathbf{x}) \wedge (\mathbf{x} \to \mathbf{x}) \Rightarrow \Psi^{(\iota)}(\mathbf{x}_n,\mathbf{x}) \to \Psi^{(\iota)}(\mathbf{x},\mathbf{x}) = 0.$ $\square$

In other words, either of the convergences

$$\psi(\mathbf{x},\mathbf{x}_n) \to \psi(\mathbf{x},\mathbf{x}) \quad (\text{for } \iota = 1),$$
$$\psi(\mathbf{x}_n,\mathbf{x}) \to \psi(\mathbf{x},\mathbf{x}) \quad (\text{for } \iota = 2),$$

implies both convergences

$$\psi(\mathbf{x}_n,\mathbf{x}) - \psi(\mathbf{x}_n,\mathbf{x}_n) \to 0 \quad (\text{for } \iota = 1),$$
$$\psi(\mathbf{x},\mathbf{x}_n) - \psi(\mathbf{x}_n,\mathbf{x}_n) \to 0 \quad (\text{for } \iota = 2).$$

Note that the implication does not work in the opposite direction: we cannot conclude from $\Psi^{(\iota)}(\mathbf{x}_n,\mathbf{x}) \to 0$ that $\Psi^{(\iota)}(\mathbf{x},\mathbf{x}_n) \to 0$ (i.e., $\Psi^{(\iota)}(\mathbf{x}_n,\mathbf{x}) \to 0$ does not imply $\mathbf{x}_n \to \mathbf{x}$). It is not difficult to reconcile this asymmetry with one's intuition: $\Psi^{(\iota)}(\mathbf{x}_n,\mathbf{x}) \to 0$ means that $\psi(\mathbf{x}_n,\mathbf{x})$, or $\psi(\mathbf{x},\mathbf{x}_n)$, converges to $\psi(\mathbf{x}_n,\mathbf{x}_n)$, a "moving target". In terms of the relative position of $\mathbf{x}_n$ and $\mathbf{x}$, therefore, $\Psi^{(\iota)}(\mathbf{x}_n,\mathbf{x}) \to 0$ is less definitive than $\Psi^{(\iota)}(\mathbf{x},\mathbf{x}_n) \to 0$.

5.3. Open sets

Theorem 4. *Each of the three sets of neighborhoods*

$$\mathfrak{B}^{(1)}(\mathbf{x},\varepsilon) = \{\mathbf{y} : \Psi^{(1)}(\mathbf{x},\mathbf{y}) < \varepsilon\},$$
$$\mathfrak{B}^{(2)}(\mathbf{x},\varepsilon) = \{\mathbf{y} : \Psi^{(2)}(\mathbf{x},\mathbf{y}) < \varepsilon\},$$
$$\mathfrak{B}(\mathbf{x},\varepsilon) = \{\mathbf{y} : \Psi(\mathbf{x},\mathbf{y}) < \varepsilon\} = \mathfrak{B}^{(1)}(\mathbf{x},\varepsilon) \cup \mathfrak{B}^{(2)}(\mathbf{x},\varepsilon)$$

(*called open balls*), *taken for all* $\mathbf{x} \in \mathfrak{M}$ *and all* $\varepsilon > 0$,[5] *forms a base for a topology on* $\mathfrak{M}$.

Proof. Consider the set of $\mathfrak{B}(\mathbf{x},\varepsilon)$ balls. Property (B1) in (9) being obvious, to demonstrate (B2) one has to show that if $\mathbf{z} \in \mathfrak{B}(\mathbf{x},\varepsilon) \cap \mathfrak{B}(\mathbf{y},\delta)$, then for some $\gamma > 0$, $\mathfrak{B}(\mathbf{z},\gamma) \subset \mathfrak{B}(\mathbf{x},\varepsilon) \cap \mathfrak{B}(\mathbf{y},\delta)$. (In particular, by putting $(\mathbf{x},\varepsilon) = (\mathbf{y},\delta)$, this would prove that if $\mathbf{z} \in \mathfrak{B}(\mathbf{x},\varepsilon)$, then for some $\gamma > 0$, $\mathfrak{B}(\mathbf{z},\gamma) \subset \mathfrak{B}(\mathbf{x},\varepsilon)$.)

Assume the contrary. Then for some sequence $\gamma_n \to 0+$ one could find a sequence of points $\mathbf{z}_n \in \mathfrak{B}(\mathbf{z},\gamma_n)$ that all lie outside $\mathfrak{B}(\mathbf{x},\varepsilon) \cap \mathfrak{B}(\mathbf{y},\delta)$. Then there would be a subsequence $\gamma_{j_n} \to 0+$ such that the corresponding subsequence of $\mathbf{z}_{j_n} \in \mathfrak{B}(\mathbf{z},\gamma_{j_n})$ either lies outside $\mathfrak{B}(\mathbf{x},\varepsilon)$ or outside $\mathfrak{B}(\mathbf{y},\delta)$. Let it be the former. Since $(\mathbf{x} \to \mathbf{x}) \wedge (\mathbf{z}_{j_n} \to \mathbf{z})$, we should have $\Psi(\mathbf{x},\mathbf{z}_{j_n}) \to \Psi(\mathbf{x},\mathbf{z})$. This is impossible, however, because all $\Psi(\mathbf{x},\mathbf{z}_{j_n}) > \varepsilon$ while $\Psi(\mathbf{x},\mathbf{z}) < \varepsilon$. The proof for $\mathfrak{B}^{(1)}$ and $\mathfrak{B}^{(2)}$ balls is essentially identical. $\square$

Now we can form a topology (set of open subsets) in $\mathfrak{M}$ by taking all possible unions of the open balls $\mathfrak{B}(\mathbf{x},\varepsilon)$, or of the open balls $\mathfrak{B}^{(\iota)}(\mathbf{x},\varepsilon)$, for $\iota = 1$ or 2. That the induced topology is one and the same in all three cases is a straightforward demonstration.

[5] It would be sufficient to confine ε to any everywhere dense subset of any interval $(0, E)$. This property of the topological space is called *first countability*.

Theorem 5. *The topological bases formed by open balls* $\mathfrak{B}^{(1)}(\mathbf{x}, \varepsilon)$, $\mathfrak{B}^{(2)}(\mathbf{x}, \varepsilon)$, *and* $\mathfrak{B}(\mathbf{x}, \varepsilon)$, *each taken for all* $\mathbf{x} \in \mathfrak{M}$ *and all* $\varepsilon > 0$, *are equivalent*.

Proof. By restating Axiom 2 in the standard "ε-δ language", the implication $\Psi(\mathbf{x}, \mathbf{x}_n) \to 0 \Rightarrow \Psi^{(1)}(\mathbf{x}, \mathbf{x}_n) \to 0$ means that for any $\varepsilon > 0$ one can find a $\delta > 0$ such that $\mathbf{z} \in \mathfrak{B}(\mathbf{x}, \delta)$ implies $\mathbf{z} \in \mathfrak{B}^{(1)}(\mathbf{x}, \varepsilon)$. Since for every $\mathbf{a} \in \mathfrak{B}^{(1)}(\mathbf{x}, \varepsilon)$ there is a $\mathfrak{B}^{(1)}(\mathbf{a}, \varepsilon') \subset \mathfrak{B}^{(1)}(\mathbf{x}, \varepsilon)$, we have proved that the base of $\mathfrak{B}$-balls refines the base of $\mathfrak{B}^{(1)}$-balls. Other implications are dealt with analogously. □

We do not have analogous results for "reverse balls"

$$\mathfrak{B}^{(1)}(\varepsilon, \mathbf{x}) = \{\mathbf{y} : \Psi^{(1)}(\mathbf{y}, \mathbf{x}) < \varepsilon\},$$

$$\mathfrak{B}^{(2)}(\varepsilon, \mathbf{x}) = \{\mathbf{y} : \Psi^{(2)}(\mathbf{y}, \mathbf{x}) < \varepsilon\}.$$

These sets do not, generally, form topological bases (i.e., an intersection of two reverse balls does not have to contain a reverse ball around each of its points). It is easy to see, however, that reverse balls are open in the topology just constructed, which means that any of the two sets of reverse balls above is *refined* by $\mathfrak{B}$-balls (hence also by $\mathfrak{B}^{(1)}$-balls and $\mathfrak{B}^{(2)}$-balls).

Theorem 6. *For any set* $\mathfrak{B}^{(\iota)}(\varepsilon, \mathbf{x})$ $(\iota = 1, 2)$ *and any* $\mathbf{z} \in \mathfrak{B}^{(\iota)}(\varepsilon, \mathbf{x})$ *there is a ball* $\mathfrak{B}(\mathbf{z}, \gamma) \subset \mathfrak{B}^{(\iota)}(\varepsilon, \mathbf{x})$.

Proof. Assume the contrary. Then for some sequence $\gamma_n \to 0+$ one could find a sequence of points $\mathbf{z}_n \in \mathfrak{B}(\mathbf{z}, \gamma_n)$ that all lie outside $\mathfrak{B}^{(\iota)}(\varepsilon, \mathbf{x})$, that is, $\Psi^{(\iota)}(\mathbf{z}_n, \mathbf{x}) > \varepsilon$. Since $(\mathbf{z}_n \to \mathbf{z}) \wedge (\mathbf{x} \to \mathbf{x})$, we should have $\Psi^{(\iota)}(\mathbf{z}_n, \mathbf{x}) > \varepsilon$ converging to $\Psi^{(\iota)}(\mathbf{z}, \mathbf{x}) < \varepsilon$, which is impossible. □

The topological relations established in this subsection are schematically summarized in Fig. 17.

Intuitively, for small $\varepsilon > 0$, $\mathfrak{B}(\mathbf{x}, \varepsilon)$ is a set of points that are "close to $\mathbf{x}$", and so are $\mathfrak{B}^{(1)}(\mathbf{x}, \varepsilon), \mathfrak{B}^{(2)}(\mathbf{x}, \varepsilon) \subset \mathfrak{B}(\mathbf{x}, \varepsilon)$. The observation below supports the intuition that points close to $\mathbf{x}$ should also be close to each other. For any nonempty $\mathfrak{S} \subset \mathfrak{M}$, let

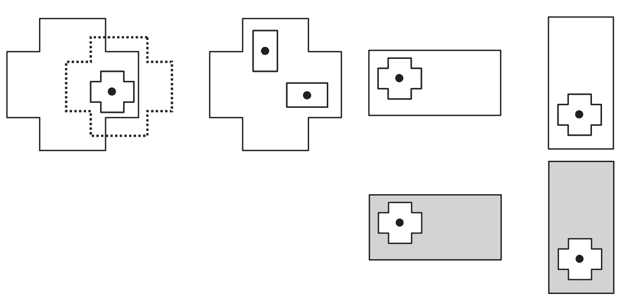

Fig. 17. Schematic relationship among open balls in stimulus space. Wide rectangles represent $\mathfrak{B}^{(1)}$-balls ("reverse" $\mathfrak{B}^{(1)}$-balls if shaded), tall rectangles $\mathfrak{B}^{(2)}$-balls ("reverse" $\mathfrak{B}^{(2)}$-balls if shaded), and crosses (unions of wide and tall rectangles) represent $\mathfrak{B}$-balls. Upper row, from left to right: an intersection of $\mathfrak{B}$-balls contains a $\mathfrak{B}$-ball around any of its points (implying that $\mathfrak{B}$-balls form a topological base); a $\mathfrak{B}$-ball contains $\mathfrak{B}^{(1)}$-balls and $\mathfrak{B}^{(2)}$-balls around all its points; a $\mathfrak{B}^{(1)}$-ball (and a $\mathfrak{B}^{(2)}$-ball) contains $\mathfrak{B}$-balls around all its points. Lower row: a "reverse" $\mathfrak{B}^{(1)}$-ball (and a "reverse" $\mathfrak{B}^{(2)}$-ball) contains $\mathfrak{B}$-balls around all its points.

$$\operatorname{diam} \mathfrak{S} = \sup_{\mathbf{a},\mathbf{b} \in \mathfrak{S}} (\max(\Psi^{(1)}(\mathbf{a}, \mathbf{b}), \Psi^{(2)}(\mathbf{a}, \mathbf{b}))).$$

Theorem 7. *As* $\varepsilon \to 0$,

$$\operatorname{diam} \mathfrak{B}(\mathbf{x}, \varepsilon) \to 0, \quad \operatorname{diam} \mathfrak{B}^{(1)}(\mathbf{x}, \varepsilon) \to 0,$$

$$\operatorname{diam} \mathfrak{B}^{(2)}(\mathbf{x}, \varepsilon) \to 0.$$

Proof. Clearly, $\operatorname{diam} \mathfrak{B}^{(\iota)}(\mathbf{x}, \varepsilon) \leqslant \operatorname{diam} \mathfrak{B}(\mathbf{x}, \varepsilon) \to 0$ $(\iota = 1, 2)$, so we only need to consider $\mathfrak{B}(\mathbf{x}, \varepsilon)$. Assume the contrary. Then, for some sequence $\varepsilon_n \to 0+$, one could find $(\mathbf{a}_n, \mathbf{b}_n)$ such that $\Psi(\mathbf{a}_n, \mathbf{b}_n) > \delta > 0$ for all n. But $\mathbf{a}_n \to \mathbf{x}, \mathbf{b}_n \to \mathbf{x}$, hence $\Psi(\mathbf{a}_n, \mathbf{b}_n) \to 0$. □

5.4. Urysohn and Hausdorff properties

Very briefly: for $\mathfrak{m} \subset \mathfrak{M}$ and $\mathbf{x} \in \mathfrak{M}$, $\mathbf{x}$ is called a *proximate point* for $\mathfrak{m}$ if every open neighborhood of $\mathbf{x}$ intersects with $\mathfrak{m}$ (not necessarily at a point other than $\mathbf{x}$); $\mathbf{x}$ is called an *interior point* of $\mathfrak{m}$ if one of its open neighborhoods lies within $\mathfrak{m}$ (hence an interior point is a proximate point); $\mathfrak{m}$ is open iff all its points are interior points; $\mathfrak{m}$ is *closed* (and $\mathfrak{M} \backslash \mathfrak{m}$ is open) iff $\mathfrak{m}$ contains all its proximate points; the *closure* of $\mathfrak{m}$ is a closed set denoted $\overline{\mathfrak{m}}$ and obtained as the union of $\mathfrak{m}$ and all of its proximate points.

A neighborhood $\mathfrak{B}[\mathbf{x}, \varepsilon] = \{\mathbf{y} : \Psi(\mathbf{x}, \mathbf{y}) \leqslant \varepsilon\}$ is called a *closed ball*. The next theorem says that such balls are closed sets with respect to the topology just constructed.

Theorem 8. *A closed ball* $\mathfrak{B}[\mathbf{x}, \varepsilon]$ *is a closed set that contains the closure* $\overline{\mathfrak{B}}(\mathbf{x}, \varepsilon)$ *of the open ball* $\mathfrak{B}(\mathbf{x}, \varepsilon)$.

Proof. Consider a sequence $\mathbf{x}_n$ belonging to $\mathfrak{B}[\mathbf{x}, \varepsilon]$ and converging to a proximate point $\mathbf{p}$ of $\mathfrak{B}[\mathbf{x}, \varepsilon]$. Since $\Psi(\mathbf{x}, \mathbf{x}_n) \leqslant \varepsilon$ and $(\mathbf{x} \to \mathbf{x}) \wedge (\mathbf{x}_n \to \mathbf{p}) \Rightarrow \Psi(\mathbf{x}, \mathbf{x}_n) \to \Psi(\mathbf{x}, \mathbf{p})$, the latter quantity cannot be greater than ε. Hence $\mathbf{p} \in \mathfrak{B}\,[\mathbf{x}, \varepsilon]$.

That $\overline{\mathfrak{B}}(\mathbf{x}, \varepsilon) \subset \mathfrak{B}[\mathbf{x}, \varepsilon]$ is obvious, since a proximate point for $\mathfrak{B}(\mathbf{x}, \varepsilon)$ is also a proximate point for $\mathfrak{B}[\mathbf{x}, \varepsilon]$. □

Note that it does not follow and we do not postulate the "regularity" property $\overline{\mathfrak{B}}(\mathbf{x}, \varepsilon) = \mathfrak{B}[\mathbf{x}, \varepsilon]$. In other words, one cannot exclude the possibility that $\Psi(\mathbf{x}, \mathbf{y}) = \varepsilon$ but a sufficiently small open ball $\mathfrak{B}(\mathbf{y}, \gamma)$ will not intersect with $\mathfrak{B}(\mathbf{x}, \varepsilon)$.

A topological space $\mathfrak{M}$ is said to be *Hausdorff* if any two distinct points $\mathbf{a}, \mathbf{b} \in \mathfrak{M}$ can be enclosed in nonintersecting open balls $\mathfrak{B}(\mathbf{a}, \varepsilon), \mathfrak{B}(\mathbf{b}, \delta)$. $\mathfrak{M}$ is said to be *Urysohn* if for any two distinct points $\mathbf{a}, \mathbf{b} \in \mathfrak{M}$ one can find a continuous function $f : \mathfrak{M} \to \mathrm{Re}$ (Urysohn function) such that $f(\mathbf{a}) = 0$ and $f(\mathbf{b}) = 1$. An Urysohn space is always a Hausdorff space.

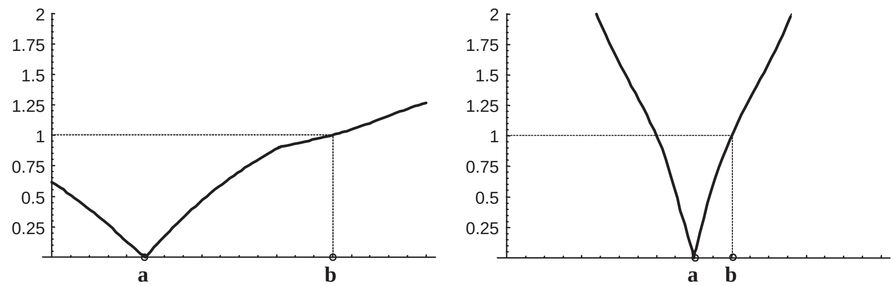

Fig. 18. Demonstration of Theorem 9 on the stimulus space endowed with the discrimination probability function shown in Fig. 9, right. For any two points $\mathbf{a}, \mathbf{b}$ the continuous function $\frac{\Psi(\mathbf{a},\mathbf{x})}{\Psi(\mathbf{a},\mathbf{b})}$ equals zero at $\mathbf{a}$ and 1 at $\mathbf{b}$.

Theorem 9. *The topology based on open balls* $\mathfrak{B}(\mathbf{x}, \varepsilon)$ *is Urysohn (hence also Hausdorff).*

Proof. $\Psi(\mathbf{a}, \mathbf{x})$ is continuous in the second argument, in the conventional sense: if $\Psi(\mathbf{a}, \mathbf{x}) \in (a, b)$, then $\mathbf{x} \in \mathfrak{B}(\mathbf{a}, b) \backslash \mathfrak{B}\,[\mathbf{a}, a]$, which is open because $\mathfrak{B}(\mathbf{a}, b)$ is open while $\mathfrak{B}\,[\mathbf{a}, a]$ is closed. Now, for any given $\mathbf{a}, \mathbf{b}$, take the continuous function $f(\mathbf{x}) = \frac{\Psi(\mathbf{a},\mathbf{x})}{\Psi(\mathbf{a},\mathbf{b})}$ as the Urysohn function. □

Examples of the Urysohn function used in the proof are given in Fig. 18.

5.5. Conventional continuity

The *product topology* on $\mathfrak{M} \times \mathfrak{M}$ is the topology based on the Cartesian products of open balls $\mathfrak{B}(\mathbf{a}, \varepsilon) \times \mathfrak{B}(\mathbf{b}, \delta)$, for all $\mathbf{a}, \mathbf{b} \in \mathfrak{M}$ and positive ε, δ.

Theorem 10. *The discrimination probability function* $\psi(\mathbf{x}, \mathbf{y})$ *is continuous in* $(\mathbf{x}, \mathbf{y})$, *in the conventional sense: if* $\psi(\mathbf{x}, \mathbf{y}) \in (a, b)$, then $(\mathbf{x}, \mathbf{y})$ belongs to an open set in the product topology.

Proof. We prove first that $\{(\mathbf{x}, \mathbf{y}) : \psi(\mathbf{x}, \mathbf{y}) < \varepsilon\}$ is open in the product topology. Let $(\mathbf{a}, \mathbf{b})$ belong to this set, i.e., $\psi(\mathbf{a}, \mathbf{b}) < \varepsilon$. Consider a sequence of open sets $\mathfrak{B}(\mathbf{a}, \varepsilon_n) \times \mathfrak{B}(\mathbf{b}, \varepsilon_n)$ with $\varepsilon_n \to 0+$, and assume that each of them contains a point $(\mathbf{a}_n, \mathbf{b}_n)$ such that $\psi(\mathbf{a}_n, \mathbf{b}_n) \geqslant \varepsilon$. Then we would have

$$(\mathbf{a}_n \to \mathbf{a}) \wedge (\mathbf{b}_n \to \mathbf{b}) \Rightarrow \psi(\mathbf{a}_n, \mathbf{b}_n) \to \psi(\mathbf{a}, \mathbf{b})$$

which is clearly impossible. It follows that for some ε_n, $\psi(\mathbf{a}_n, \mathbf{b}_n) < \varepsilon$ for all $(\mathbf{a}_n, \mathbf{b}_n) \in \mathfrak{B}(\mathbf{a}, \varepsilon_n) \times \mathfrak{B}(\mathbf{b}, \varepsilon_n)$, and this means that $\{(\mathbf{x}, \mathbf{y}) : \psi(\mathbf{x}, \mathbf{y}) < \varepsilon\}$ is open.

We prove next that the set $\{(\mathbf{x}, \mathbf{y}) : \psi(\mathbf{x}, \mathbf{y}) \leqslant \varepsilon\}$ is closed in the product topology. If $(\mathbf{a}, \mathbf{b})$ is its proximate point, then any sequence $\mathfrak{B}(\mathbf{a}, \varepsilon_n) \times \mathfrak{B}(\mathbf{b}, \varepsilon_n)$ with $\varepsilon_n \to 0$ should contain points $(\mathbf{a}_n, \mathbf{b}_n)$ such that $\psi(\mathbf{a}_n, \mathbf{b}_n) \leqslant \varepsilon$. Since $(\mathbf{a}_n \to \mathbf{a}) \wedge (\mathbf{b}_n \to \mathbf{b})$, we conclude that $\psi(\mathbf{a}, \mathbf{b}) \leqslant \varepsilon$.

Then $\{(\mathbf{x}, \mathbf{y}) : \Psi(\mathbf{x}, \mathbf{y}) < b\} \backslash \{(\mathbf{x}, \mathbf{y}) : \Psi(\mathbf{x}, \mathbf{y}) \leqslant a\}$ must be open. □

Corollary 2 (*to Theorem 10*). *Psychometric increments*

$$\Psi^{(1)}(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{x}, \mathbf{y}) - \psi(\mathbf{x}, \mathbf{x}),$$
$$\Psi^{(2)}(\mathbf{x}, \mathbf{y}) = \psi(\mathbf{y}, \mathbf{x}) - \psi(\mathbf{x}, \mathbf{x}),$$
$$\Psi(\mathbf{x}, \mathbf{y}) = \min\{\Psi^{(1)}(\mathbf{x}, \mathbf{y}), \Psi^{(2)}(\mathbf{x}, \mathbf{y})\},$$

are continuous in the conventional sense.

5.6. Arc-connectedness

We consider now the global topological structure of the space $\mathfrak{M}$. We wish to confine our analysis to *arc-connected* spaces $\mathfrak{M}$, in which one can continuously move from any one stimulus to another.

Recall that an *arc* connecting $\mathbf{a} \in \mathfrak{M}$ to $\mathbf{b} \in \mathfrak{M}$ is a homeomorphic function $\mathbf{f} : [a, b] \to \mathfrak{M}$ such that $\mathbf{f}(a) = \mathbf{a}$, $\mathbf{f}(b) = \mathbf{b}$.[6] The variable $t \in [a, b]$ is called the arc's *parameter*. It is convenient to allow for $a = b$ in this definition, ensuring that any stimulus $\mathbf{a}$ is arc-connectable to itself. If $\mathbf{a}$ is arc-connectable to $\mathbf{b}$, then $\mathbf{b}$ is arc-connectable to $\mathbf{a}$. If $\mathbf{a}$ is arc-connectable to $\mathbf{b}$ and $\mathbf{b}$ is arc-connectable to $\mathbf{c}$, then $\mathbf{a}$ is arc-connectable to $\mathbf{c}$. The relation "is arc-connectable to" therefore is an equivalence relation.

Axiom 4 (*Arc-connectedness*). *Stimulus space* $\mathfrak{M}$ *is arc-connected, that is, every two of its points are arc-connectable in* $\mathfrak{M}$.

By the previous remarks, it would have been sufficient to require that there is at least one point in $\mathfrak{M}$ that is arc-connectable to any point in $\mathfrak{M}$.

6. Smooth arcs

In this section, we move from topological to analytic properties of stimulus space $\mathfrak{M}$, by introducing in $\mathfrak{M}$ an analogue for the notion of a piecewise continuously

[6] We could replace "homeomorphic" here with "continuous" and speak of *paths* rather than arcs. In Hausdroff spaces pathwise connectedness is equivalent to arcwise connectedness.

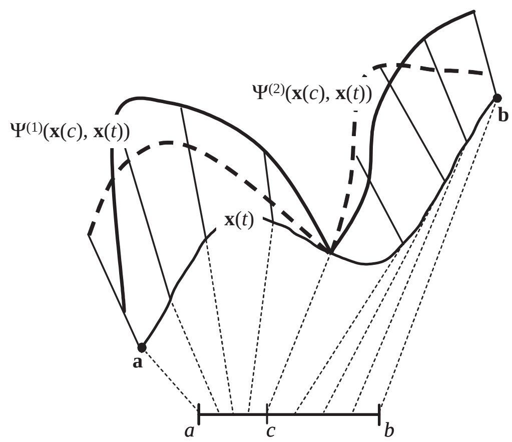

Fig. 19. Smooth arc $\mathbf{x}(t):[a,b]\to\mathfrak{M}:\Psi^{(1)}(\mathbf{x}(c),\mathbf{x}(t))$ (solid thick line) and $\Psi^{(2)}(\mathbf{x}(c),\mathbf{x}(t))$ (dashed thick line) are continuously differentiable below c and above c, and they both increase for a while as t moves away from c in either direction.

differentiable arc in a Euclidean space. Since differentiability is involved, we no longer can deal with minima of psychometric increments and have to consider $\Psi^{(1)}$ and $\Psi^{(2)}$ separately.

Definition 5. An arc $\mathbf{x}(t):[a,b]\to\mathfrak{M}$ is called smooth if, for every $c\in[a,b]$,

(i) $\Psi^{(1)}(\mathbf{x}(c),\mathbf{x}(t))$ and $\Psi^{(2)}(\mathbf{x}(c),\mathbf{x}(t))$ are continuously differentiable in t on $[a,c)\cup(c,b]$; and
(ii) $\mathrm{d}\Psi^{(1)}(\mathbf{x}(c),\mathbf{x}(t))/\mathrm{dt}$ and $\mathrm{d}\Psi^{(2)}(\mathbf{x}(c),\mathbf{x}(t))/\mathrm{dt}$ are negative on $[c-\delta,c)\cap[a,b]$ and positive on $(c,c+\delta]\cap[a,b]$, for some $\delta>0$.

The concept is illustrated in Fig. 19. A few comments.

1. If $c=a$ or $c=b$, the derivatives $\mathrm{d}\Psi^{(1)}(\mathbf{x}(c),\mathbf{x}(t))/\mathrm{dt}$ and $\mathrm{d}\Psi^{(2)}(\mathbf{x}(c),\mathbf{x}(t))/\mathrm{dt}$ should be understood in the unilateral sense.
2. Nothing is assumed about the unilateral derivatives $\mathrm{d}\Psi^{(\iota)}(\mathbf{x}(c),\mathbf{x}(t))/\mathrm{d}t\pm$ at $t=c$: they are allowed to be nonzero, zero, or $\pm\infty$ (they could even be undefined, but this will be ruled out later).
3. Property (ii) implies that $\Psi^{(1)}(\mathbf{x}(c),\mathbf{x}(t))$ and $\Psi^{(2)}(\mathbf{x}(c),\mathbf{x}(t))$ decrease in a left-hand vicinity of c, reach their minima at c, and increase in some right-hand vicinity of c.
4. The definition is formally satisfied by point arcs, with $a=b$.
5. Since

$$\Psi^{(1)}(\mathbf{x}(c),\mathbf{x}(t))=\psi(\mathbf{x}(c),\mathbf{x}(t))-\psi(\mathbf{x}(c),\mathbf{x}(c)),$$
$$\Psi^{(2)}(\mathbf{x}(c),\mathbf{x}(t))=\psi(\mathbf{x}(t),\mathbf{x}(c))-\psi(\mathbf{x}(c),\mathbf{x}(c)),$$

property (i) is equivalent to saying that both $\psi(\mathbf{x}(c),\mathbf{x}(t))$ and $\psi(\mathbf{x}(t),\mathbf{x}(c))$ are continuously differentiable on $[a,c)\cup(c,b]$.

Any arc $\mathbf{x}(t)$ can be *reparametrized* as $\mathbf{y}(\tau)=\mathbf{x}(t(\tau))$, where $t(\tau):[c,d]\to[a,b]$ is a homeomorphism (in particular, a diffeomorphism).

A finite number of pairwise noncrossing arcs,

$$\mathbf{x}_1(t):[a_1,a_2]\to\mathfrak{M},\mathbf{x}_2(t):[a_2,a_3]\to\mathfrak{M},\ldots,$$
$$\mathbf{x}_{n-1}(t):[a_{n-1},a_n]\to\mathfrak{M},$$

with

$$\mathbf{x}_1(a_2)=\mathbf{x}_2(a_2),\mathbf{x}_2(a_3)=\mathbf{x}_3(a_3),\ldots,\mathbf{x}_{n-2}(a_{n-1})$$
$$=\mathbf{x}_{n-1}(a_{n-1})$$

can be concatenated into an arc

$$\mathbf{x}(t):[a_1,a_n]\to\mathfrak{M}.$$

("Noncrossing" means that two arcs may not have common points except for the endpoints at which they are concatenated.) If each of these arcs is smooth, then the resulting arc is called *piecewise smooth*.

Finally, by specializing (i.e., restricting) $\mathbf{x}(t)$ upon a subinterval $[c,d]\subset[a,b]$ one can form a *subarc* of $\mathbf{x}(t)$, denoted $\mathbf{x}_{[c,d]}(t)$. In particular, $\mathbf{x}(t)$ itself can be denoted $\mathbf{x}_{[a,b]}(t)$.

Theorem 11. *If $t(\tau):[u,v]\to[a,b]$ is a positive or negative diffeomorphism (i.e., $t'(\tau)>0$ on $[u,v]$ or $t'(\tau)<0$ on $[u,v]$, respectively), then the reparametrization $\mathbf{y}(\tau)=\mathbf{x}(t(\tau))$ of a (piecewise) smooth arc $\mathbf{x}(t)$ is a (piecewise) smooth arc. A concatenation of a finite number of pairwise noncrossing piecewise smooth arcs is a piecewise smooth arc. A subarc of a (piecewise) smooth arc is a (piecewise) smooth arc.*

Proof. The first statement is obtained by checking

$$\frac{\mathrm{d}\Psi^{(\iota)}(\mathbf{x}(c),\mathbf{y}(\tau))}{\mathrm{d}\tau}=\frac{\mathrm{d}\Psi^{(\iota)}(\mathbf{x}(c),\mathbf{x}(t))}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau},\quad \iota=1,2, \tag{11}$$

against Definition 5. The other two statements hold trivially. $\square$

Definition 6. For $\mathfrak{m}\subset\mathfrak{M}$, if $\mathbf{a}\in\mathfrak{m}$ can be connected to $\mathbf{b}\in\mathfrak{m}$ by a piecewise smooth arc $\mathbf{x}(t):[a,b]\to\mathfrak{m}$, then $\mathbf{a}$ is said to be smoothly connectable to $\mathbf{b}$ in $\mathfrak{m}$. If this is true for any two $\mathbf{a},\mathbf{b}\in\mathfrak{m}$, then the subset $\mathfrak{m}$ is said to be smoothly connected.

Theorem 12. *The relation "is smoothly connectable in $\mathfrak{m}$ to" is an equivalence relation.*

Proof. Reflexivity holds by Comment 4 to Definition 5. Symmetry is obtained by choosing in (11) any negative diffeomorphism $t(\tau):[c,d]\to[a,b]$. To prove the transitivity, assume without loss of generality that $\mathbf{a},\mathbf{b},\mathbf{c}$ are distinct points in $\mathfrak{m}$. Consider $\mathbf{x}(t):[a,b]\to\mathfrak{m}$ and $\mathbf{y}(t):[c,d]\to\mathfrak{m}$ (two piecewise smooth arcs) with $\mathbf{x}(a)=\mathbf{a},\mathbf{x}(b)=\mathbf{y}(d)=\mathbf{b},\mathbf{y}(c)=\mathbf{c}$. If $\mathbf{y}(c)=\mathbf{x}(t_0)$, for some $t_0\in[a,b]$, then the subarc $\mathbf{x}_{[a,t_0]}(t)$ smoothly connects $\mathbf{a}$ to $\mathbf{c}$. If $\mathbf{y}(c)\neq\mathbf{x}(t)$, for all $t\in[a,b]$, then the equality $\mathbf{x}(b)=\mathbf{y}(d)$ implies the existence of $t_1\in[a,b]$ and $t_2\in[c,d]$ such that $\mathbf{x}(t_1) = \mathbf{y}(t_2)$ while the two prior subarcs, $\mathbf{x}_{[a,t_1]}(t)$ and $\mathbf{y}_{[c,t_2]}(t)$, do not cross. Then the concatenation of $\mathbf{x}_{[a,t_1]}(t)$ and $\mathbf{z}_{[t_1,b]}(t)$, where $\mathbf{z}(t) = \mathbf{y}(\frac{t_2-c}{t_1-b}(t-b)+c)$, is a piecewise smooth arc connecting $\mathbf{a}$ to $\mathbf{c}$. $\square$

In consequence of this theorem the relation "is smoothly connectable in $\mathfrak{M}$ to" partitions $\mathfrak{M}$ into equivalence classes $\{\mathfrak{m}_\eta\}_{\eta\in\mathfrak{J}}$ (with some indexing set $\mathfrak{J}$, not necessarily countable). This partitioning, however, consists of a single set ($\mathfrak{M}$ itself) in stimulus spaces possessing the following property.

Definition 7. Stimulus space $\mathfrak{M}$ is called locally smoothly connected if for any $\mathbf{a} \in \mathfrak{M}$ there is a $\delta_\mathbf{a} > 0$ such that $\mathbf{a}$ is smoothly connectable in $\mathfrak{B}(\mathbf{a}, \delta_\mathbf{a})$ to any $\mathbf{b} \in \mathfrak{B}(\mathbf{a}, \delta_\mathbf{a})$.

We do not postulate at this point that stimulus space $\mathfrak{M}$ is locally smoothly connected, because it will follow from a stronger axiom introduced later (Axiom 7).

Theorem 13. *In a locally smoothly connected* $\mathfrak{M}$, *for any* $\mathbf{a} \in \mathfrak{M}$, *any two* $\mathbf{x}, \mathbf{y} \in \mathfrak{B}(\mathbf{a}, \delta_\mathbf{a})$ *are smoothly connectable in* $\mathfrak{B}(\mathbf{a}, \delta_\mathbf{a})$.

Proof. Follows from the fact that the relation "is smoothly connectable to" (in any set) is an equivalence. $\square$

Theorem 14. *Stimuli* $\mathbf{a}, \mathbf{b}$ *in a locally smoothly connected* $\mathfrak{M}$ *are smoothly connectable iff they are arc-connectable.*

Proof. The implication "smoothly" $\to$ "arc" is trivially true. To prove the reverse, let $\mathbf{a}$ be connected to $\mathbf{b}$ by an arc $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$, and let

$$\tau = \inf\{t \in [a,b] : \mathbf{a} \text{ is not smoothly connectable to } \mathbf{x}(t)\}.$$

One can form a sequence $t_n \to \tau-$ such that all $\mathbf{x}(t_n)$ are smoothly connectable to $\mathbf{a}$. By continuity of $\mathbf{x}(t)$, $\mathbf{x}(t_n) \to \mathbf{x}(\tau)$, because of which at some n, $\mathbf{x}(t_n) \in \mathfrak{B}(\mathbf{x}(\tau), \delta_{\mathbf{x}(\tau)})$ (Definition 7). Then $\mathbf{x}(t_n)$ is smoothly connectable to both $\mathbf{a}$ and $\mathbf{x}(\tau)$, hence $\mathbf{a}$ is smoothly connectable to $\mathbf{x}(\tau)$. If $\tau < b$, then, by continuity of $\mathbf{x}(t)$, for a sufficiently small $\alpha > 0$, $\mathbf{x}(\tau+\alpha) \in \mathfrak{B}(\mathbf{x}(\tau), \delta_{\mathbf{x}(\tau)})$, and $\mathbf{x}(\tau+\alpha)$ would therefore be smoothly connectable to $\mathbf{a}$. As this would contradict the definition of τ, we have $\tau = b$, and $\mathbf{b}$ is smoothly connectable to $\mathbf{a}$. $\square$

Corollary 3 (*to Theorem 14*). *A locally smoothly connected stimulus space* $\mathfrak{M}$ *is smoothly connected, that is, any two points of* $\mathfrak{M}$ *are smoothly connectable.*

7. Regular variation of psychometric increments

7.1. Comeasurability im kleinen

Definition 5 imposes no constraints on the possible behavior of

$$\frac{\Psi^{(\iota)}(\mathbf{x}(c), \mathbf{x}(c+\alpha))}{\alpha}, \quad \iota = 1, 2$$

as α approaches zero from the left or from the right. In this section we investigate this behavior in detail, under the assumption that any two psychometric increments, taken along any two smooth arcs, are *comeasurable im kleinen* (meaning "in the small", as they approach the point of vanishing).

Axiom 5 (*Comeasurability im Kleinen*). *For any two smooth arcs* $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ *and* $\mathbf{y}(\tau) : [c,d] \to \mathfrak{M}$,

$$0 < \lim_{\alpha\to 0+} \frac{\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha))}{\Psi^{(1)}(\mathbf{y}(c), \mathbf{y}(c+\alpha))} < \infty, \quad \iota = 1, 2.$$

That is, the limit ratios

$$\lim_{\alpha\to 0+} \frac{\Psi^{(1)}(\mathbf{x}(a), \mathbf{x}(a+\alpha))}{\Psi^{(1)}(\mathbf{y}(c), \mathbf{y}(c+\alpha))}, \quad \lim_{\alpha\to 0+} \frac{\Psi^{(2)}(\mathbf{x}(a), \mathbf{x}(a+\alpha))}{\Psi^{(1)}(\mathbf{y}(c), \mathbf{y}(c+\alpha))}$$

exist as positive quantities, and consequently so do the limit ratios

$$\lim_{\alpha\to 0+} \frac{\Psi^{(1)}(\mathbf{x}(a), \mathbf{x}(a+\alpha))}{\Psi^{(2)}(\mathbf{y}(c), \mathbf{y}(c+\alpha))}, \quad \lim_{\alpha\to 0+} \frac{\Psi^{(2)}(\mathbf{x}(a), \mathbf{x}(a+\alpha))}{\Psi^{(2)}(\mathbf{y}(c), \mathbf{y}(c+\alpha))}.$$

Fig. 20 provides an illustration.

In the axiom's formulation the psychometric increments are confined to left endpoints of arcs, and consequently to positive increments in the arcs' parameters, $\alpha \to 0+$. It is easy to see that this restriction is only apparent.

1. Consider a right-end psychometric increment, say $\Psi^{(1)}(\mathbf{x}(b), \mathbf{x}(b-\alpha))$, $\alpha > 0$. By reparametrization $t(u) = (a+b) - u$, $u \in [a,b]$ we obtain the smooth arc $\mathbf{x}^*(u) = \mathbf{x}(t(u))$ such that

   $$\Psi^{(1)}(\mathbf{x}(b), \mathbf{x}(b-\alpha)) = \Psi^{(1)}(\mathbf{x}^*(a), \mathbf{x}^*(a+\alpha))$$

   for any α. Consequently,

   $$\lim_{\alpha\to 0+} \frac{\Psi^{(1)}(\mathbf{x}(b), \mathbf{x}(b-\alpha))}{\Psi^{(1)}(\mathbf{y}(c), \mathbf{y}(c+\alpha))} = \lim_{\alpha\to 0+} \frac{\Psi^{(1)}(\mathbf{x}^*(a), \mathbf{x}^*(a+\alpha))}{\Psi^{(1)}(\mathbf{y}(c), \mathbf{y}(c+\alpha))}$$

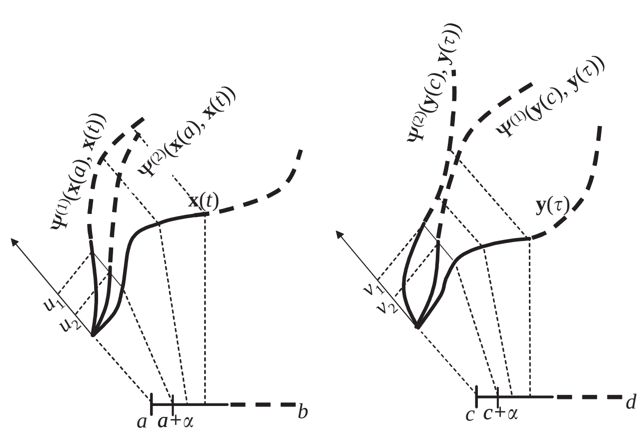

Fig. 20. Small portions of two smooth arcs, $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ and $\mathbf{y}(\tau) : [c,d] \to \mathfrak{M}$. As $\alpha \to 0$, u_1, u_2, v_1, v_2 all tend to zero. Axiom 5 says that none of these four quantites tends to zero infinitely faster than another.

and the axiom applies to right-end increments as well.

2. Consider now any two points $t_1 \in (a,b), \tau_1 \in (c,d)$ and the psychometric increments $\Psi^{(\iota)}(\mathbf{x}(t_1), \mathbf{x}(t_1 \pm \alpha)), \iota = 1,2,$ and $\Psi^{(\kappa)}(\mathbf{y}(\tau_1), \mathbf{y}(\tau_1 \pm \alpha)), \kappa = 1,2.$ To apply the axiom and the previous comment, all one needs is to form subarcs $\mathbf{x}_{[a,t_1]}(t)$ (for $t_1 - \alpha$) and $\mathbf{x}_{[t_1,b]}(t)$ (for $t_1 + \alpha$) of $\mathbf{x}(t)$, and do analogously for $\mathbf{y}(\tau)$.

It is useful to summarize these comments as a general statement.

Corollary 4 (*to Axiom 5*). *For any two smooth arcs* $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ *and* $\mathbf{y}(\tau) : [c,d] \to \mathfrak{M}$, *and any* $t \in [a,b], \tau \in [c,d]$, *any of the psychometric increments* $\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t \pm \alpha)), \iota = 1,2,$ *is comeasurable im kleinen, as* $\alpha \to 0+$, *with any of the psychometric increments* $\Psi^{(\kappa)}(\mathbf{y}(\tau), \mathbf{y}(\tau \pm \alpha)), \kappa = 1,2$ (*with obvious caveats at the endpoints*).

7.2. Codirectionality and arc elements

Clearly, the limit ratio in Axiom 5 does not depend on the entire arcs $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ and $\mathbf{y}(\tau) : [c,d] \to \mathfrak{M}$. It will remain the same if we replace them with their arbitrarily small subarcs originating at points $\mathbf{x}(a)$ and $\mathbf{y}(c)$, respectively. Subsequent considerations therefore require that we deal with "infinitesimally small" subarcs of smooth arcs, and for this purpose we need to develop appropriate language and notation.

Definition 8. Two arcs $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ and $\mathbf{y}(\tau) : [c,d] \to \mathfrak{M}$ are called codirectional if $a = c$ and one of the two arcs is a subarc of another. Given a smooth arc $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ and $t_0 \in [a,b)$, the equivalence class of all arcs codirectional with the subarc $\mathbf{x}_{[t_0,b]}(t)$ is called an arc element and denoted by $(\mathbf{x}(t_0), \mathring{\mathbf{x}}(t_0))$.

Intuitively, arc element $(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$ is a conjunction of a point-stimulus $\mathbf{x}(t)$ and an attached to it direction of stimulus change $\mathring{\mathbf{x}}(t)$, that can be thought of as an "infinitesimally small" arc. In the present treatment we never use a direction $\mathring{\mathbf{x}}(t)$ alone, outside an arc element $(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$.[7]

We will also need the following simple operations on arc elements.

[7]Both the terminology and the notation here are designed to resemble the notion of a line element, $(\mathbf{x}(t), \dot{\mathbf{x}}(t))$, in MDFS (see Introduction). Of course, direction $\mathring{\mathbf{x}}(t)$ cannot be simply identified with tangent $\dot{\mathbf{x}}(t)$: in the general theory the notion of a tangent is not defined, while in differentiable manifolds (assuming continuously differentiable arcs are smooth arcs in the sense of Definition 5) two distinct arc elements $(\mathbf{a}, \mathring{\mathbf{x}}(t))$ and $(\mathbf{a}, \mathring{\mathbf{y}}(t))$ at point $\mathbf{a}$ may share one and the same line element, $(\mathbf{a}, \mathbf{u})$, with $\mathbf{u} = \dot{\mathbf{x}}(t) = \dot{\mathbf{y}}(t)$. The specialization of the general theory to MDFS (not dealt with in this paper) essentially consists in ensuring that arc elements with this property (tangency) behave identically (i.e., are assigned the same value of the submetric function of either kind, as defined in Section 8).

Definition 9. Given a smooth arc $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$,

(i) for any $t_0 \in [a,b)$, $(\mathbf{x}(t_0), k\,\mathring{\mathbf{x}}(t_0))$ $(k > 0)$ denotes the arc element consisting of all arcs codirectional with $\mathbf{y}(\tau) : [t_0, t_0 + \frac{b-t_0}{k}] \to \mathfrak{M}$ defined by $\mathbf{y}(t_0 + \alpha) = \mathbf{x}_{[t_0,b]}(t_0 + k\alpha)$;

(ii) the arc element $(\mathbf{x}(t_0), 0 \cdot \mathring{\mathbf{x}}(t_0))$ is defined as the singleton set containing the point arc $[t_0, t_0] \to \mathbf{x}(t_0)$.

(iii) for any $t_0 \in (a,b]$, $(\mathbf{x}(t_0), -\mathring{\mathbf{x}}(t_0))$ denotes the arc element consisting of all arcs codirectional with $\mathbf{y}(\tau) : [t_0, 2t_0 - a] \to \mathfrak{M}$ defined by $\mathbf{y}(t_0 + \alpha) = \mathbf{x}_{[a,t_0]}(t_0 - \alpha)$.

According to (ii), if $k = 0$, the arc element reduces to a point. This causes no difficulties, as a point on a smooth arc can be viewed as its smooth subarc.

One can combine (i) and (iii) to form arc elements $(\mathbf{x}(t_0), -k\,\mathring{\mathbf{x}}(t_0))$.

7.3. Regular variation

For a comprehensive treatment of regular variation see Bingham, Goldie, and Teugels (1987). The aspects of the theory of regular variation that are relevant in the present context can be found in Dzhafarov (2002a).

We summarize some notions and facts. A function $f(x) : \mathrm{Re}^+ \to \mathrm{Re}^+$ is said to *regularly vary at the origin* (i.e., as $x \to 0+$) *with exponent* μ if

$$f(x) = x^{\mu}\ell(x),$$

where $\ell(x)$ is the *slowly varying component* of $f(x)$, characterized by

$$\lim_{x \to 0+} \frac{\ell(kx)}{\ell(x)} = 1 \quad \text{for every } k > 0.$$

A continuous function $f(x) : \mathrm{Re}^+ \to \mathrm{Re}^+$ regularly varies if and only if

$$0 < \lim_{x \to 0+} \frac{f(kx)}{f(x)} < \infty \quad \text{for every } k > 0.^8$$

Then

$$\lim_{x \to 0+} \frac{f(kx)}{f(x)} = k^{\mu},$$

where μ is the *exponent of regular variation*. As $x \to 0+$,

$$f(x) \to 0 \quad \text{if } \mu > 0,$$
$$f(x) \to \infty \quad \text{if } \mu < 0.$$

All forms of asymptotic behavior are possible if $\mu = 0$ (i.e., if the function is slowly varying).

If $y = f(x)$ is regularly varying with exponent μ and monotone in some vicinity of $x = 0+$, then $x = f^{-1}(y)$ is

[8]If $f(x)$ increases in the right-hand vicinity of zero, then it is sufficient that the finiteness of the limit ratio hold for at least two values of k that are not rational powers of each other.

regularly varying with exponent $1/\mu$ and monotone in some vicinity of $y = 0+$. In this paper we will primarily deal with positive-exponent regularly varying functions $f(x)$ that are continuously differentiable in some vicinity of $x = 0+$, with $\mathrm{d}f(x)/\mathrm{d}x > 0$.

We write $a(x) \sim b(x)$ and say that the two functions are *asymptotically equal* to indicate $\lim(a/b) = 1$ as $x \to 0+$. The term "asymptotic" in the present context will always mean "as $x \to 0+$".

Theorem 15. *For any smooth arc* $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$, *as* $\alpha \to 0+$, $\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha))$, $\iota = 1, 2$, *regularly varies with a nonnegative exponent and a slowly varying component continuously differentiable at sufficiently small* $\alpha > 0$.

Proof. Consider smooth arc $\mathbf{y}(\tau) : [a, a + \frac{b-a}{k}] \to \mathfrak{M}$ defined by $\mathbf{y}(a+\alpha) = \mathbf{x}(a + k\alpha)$, $k > 0, \alpha \in [0, \frac{b-a}{k}]$. By Axiom 5,

$$0 < \lim_{\alpha \to 0+} \frac{\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+k\alpha))}{\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha))} < \infty,$$

which, as we know, implies the existence of some real number μ and some positive slowly varying function $\ell(\alpha)$ such that

$$\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha)) = \alpha^{\mu}\ell(\alpha).$$

Exponent μ must be nonnegative because $\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha)) \to 0$ as $\alpha \to 0+$. Function $\ell(\alpha)$ must be C^1 on some interval $(0, \delta)$ because so is $\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha))$. □

In this theorem both μ and $\ell(\alpha)$ are allowed to be different for different psychometric increments $\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha))$, $\iota = 1, 2$. The next theorem shows that in fact μ is the same for all psychometric increments, and $\ell(\alpha)$ is asymptotically the same, up to scaling coefficients.

Theorem 16. *There exist a constant* $\mu \geqslant 0$ *and a slowly varying function* $\ell(\alpha)$ (*continuously differentiable at sufficiently small* $\alpha > 0$) *such that for any smooth arc* $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ *one can find* $V_\iota(\mathbf{x}(a), \overset{\circ}{\mathbf{x}}(a)) > 0$ *such that, as* $\alpha \to 0+$,

$$\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha)) \sim V_\iota(\mathbf{x}(a), \overset{\circ}{\mathbf{x}}(a))\alpha^{\mu}\ell(\alpha), \quad \iota = 1, 2.$$

Constant μ *is determined uniquely. Slowly varying function* $\ell(\alpha)$ *is determined asymptotically uniquely up to multiplication by a positive constant* (*say*, λ), *and the coefficient* $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ *is determined uniquely up to multiplication by the reciprocal constant* $(1/\lambda)$.

Proof. Let $\Psi_0(\alpha)$ be a psychometric increment (of either kind) taken on an arbitrarily chosen smooth arc. By the previous theorem, $\Psi_0(\alpha) = \alpha^{\mu}\ell(\alpha)$ and $\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha)) = \alpha^{\mu^*}\ell^*(\alpha)$ (where μ^* and ℓ^*, for now, may depend on ι and on the smooth arc $\mathbf{x}_{[a,b]}$). By Axiom 5,

$$\lim_{\alpha \to 0+} \frac{\alpha^{\mu^*}\ell^*(\alpha)}{\alpha^{\mu}\ell(\alpha)} = V_\iota(\mathbf{x}_{[a,b]}) > 0.$$

Ratio $\frac{\ell^*(\alpha)}{\ell(\alpha)}$ is a slowly varying function, for

$$\frac{\ell^*(k\alpha)/\ell(k\alpha)}{\ell^*(\alpha)/\ell(\alpha)} = \frac{\ell^*(k\alpha)/\ell^*(\alpha)}{\ell(k\alpha)/\ell(\alpha)} \to 1.$$

The expression $\alpha^{\mu^*-\mu}\frac{\ell^*(\alpha)}{\ell(\alpha)}$ therefore tends to ∞ if $\mu^* < \mu$ and it tends to 0 if $\mu^* > \mu$. Hence $\mu^* = \mu$. The ratio $\frac{\ell^*(\alpha)}{\ell(\alpha)}$ then tends to $V_\iota(\mathbf{x}_{[a,b]})$, and we have

$$\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha)) \sim V_\iota(\mathbf{x}_{[a,b]})\alpha^{\mu}\ell(\alpha).$$

Obviously, if also

$$\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha)) \sim V_\iota^*(\mathbf{x}_{[a,b]})\alpha^{\mu}\ell^*(\alpha),$$

then for some $\lambda > 0$, $\ell^*(\alpha) \sim \lambda\ell(\alpha)$ and $V_\iota^*(\mathbf{x}_{[a,b]}) = V_\iota(\mathbf{x}_{[a,b]})/\lambda$. Finally, since $V_\iota(\mathbf{x}_{[a,b]})$ is the same for all subarcs of $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ that have the same initial point, $V_\iota(\mathbf{x}_{[a,b]})$ can be written as $V_\iota(\mathbf{x}(a), \overset{\circ}{\mathbf{x}}(a))$. □

Written in a more general (but obviously equivalent) form, the asymptotic decomposition of psychometric increments is

$$\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha)) \sim V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))\alpha^{\mu}\ell(\alpha), \quad \iota = 1, 2. \tag{12}$$

We know that $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ is always positive. The next theorem establishes that $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ has a property resembling Euler homogeneity of order μ.

Theorem 17. *For any arc element* $(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$, *any* $k > 0$, *and for* $\iota = 1, 2$,

$$V_\iota(\mathbf{x}(t), k\overset{\circ}{\mathbf{x}}(t)) = k^{\mu}V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t)).$$

Proof.

$$\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha)) \sim V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))\alpha^{\mu}\ell(\alpha),$$

whence

$$\begin{aligned}\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+k\alpha)) &\sim V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))(k\alpha)^{\mu}\ell(k\alpha)\\ &\sim k^{\mu}V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))\alpha^{\mu}\ell(\alpha).\end{aligned}$$

But

$$\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+k\alpha)) \sim V_\iota(\mathbf{x}(t), k\overset{\circ}{\mathbf{x}}(t))\alpha^{\mu}\ell(\alpha),$$

and the result obtains from

$$k^{\mu}V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))\alpha^{\mu}\ell(\alpha) \sim V_\iota(\mathbf{x}(t), k\overset{\circ}{\mathbf{x}}(t))\alpha^{\mu}\ell(\alpha). \quad \square$$

Two comments.

1. $V_\iota(\mathbf{x}(t), k\overset{\circ}{\mathbf{x}}(t)) = 0$ when $k = 0$. This is clearly implied by $\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t)) = 0$ and
$$V_\iota(\mathbf{x}(t), k\overset{\circ}{\mathbf{x}}(t)) = \lim_{\alpha \to 0+} \frac{\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+k\alpha))}{\alpha^{\mu}\ell(\alpha)}.$$
2. No relationship is postulated or can be derived between $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ and $V_\iota(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t))$. By using

the latter's definition, however, one can see that for any arc element $(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t))$,

$$V_\iota(\mathbf{x}(t), -k\overset{\circ}{\mathbf{x}}(t)) = k^\mu V_\iota(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t)), \quad k>0, \iota = 1,2. \tag{13}$$

Note also that Axiom 5 (Comeasurability im Kleinen) can now be written as stating that for any two arc elements $(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$, $(\mathbf{y}(\tau), \overset{\circ}{\mathbf{y}}(\tau))$,

$$\lim_{\alpha\to 0+} \frac{\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t\pm\alpha))}{\Psi^{(\kappa)}(\mathbf{y}(\tau), \mathbf{y}(\tau+\alpha))} = \frac{V_\iota(\mathbf{x}(t), \pm\overset{\circ}{\mathbf{x}}(t))}{V_\kappa(\mathbf{y}(\tau), \overset{\circ}{\mathbf{y}}(\tau))}, \quad \iota = 1,2, \kappa = 1,2. \tag{14}$$

Definition 10. Constant $\mu \geqslant 0$ in Theorem 16 is called the psychometric order of stimulus space $\mathfrak{M}$.

8. Submetric functions

8.1. Basic properties

The comeasurability im Kleinen axiom does not exclude the possibility of psychometric order $\mu = 0$, which is the possibility that psychometric increments are slowly varying functions. It is excluded, however, by the following assumption.

Axiom 6 (*Double Continuity*). *Coefficient* $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ *in the asymptotic decomposition*

$$\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha)) \sim V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))\alpha^\mu \ell(\alpha), \quad \iota = 1,2$$

has the following properties:

(i) *for any smooth arc* $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$, $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ *is continuous in* $t \in [a,b]$, *for* $\iota = 1,2$;
(ii) $V_\iota(\mathbf{x}(t), k\overset{\circ}{\mathbf{x}}(t)) \to 0$ *as* $k \to 0+$ *at least for one arc element* $(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ *and for* $\iota = 1$ *or* 2.

Strictly speaking, $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ is not defined at $t = b$ (see Definitions 8 and 9). Saying that $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ is continuous in $t \in [a,b]$ therefore is a short version of saying

> $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ is both continuous and bounded on $[a,b)$ and $V_\iota(\mathbf{x}(b), \overset{\circ}{\mathbf{x}}(b))$ is defined as $\lim_{t\to b-} V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$.

Convention 2. *In the following we will tacitly assume that* $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ (*and its continuous transformations*) *are defined on the entire interval* $[a,b]$ *for any smooth arc* $x(t) : [a,b] \to \mathfrak{M}$.

Part (i) of the axiom will play a critical role in constructing Fechnerian distances, as this construction involves Riemann integrals of continuously transformed $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$. Part (ii) is needed to establish the following result.

Theorem 18. *Psychometric order* μ *of stimulus space* $\mathfrak{M}$ *is positive. As a consequence, the following statements hold for all arc elements* $(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$:

(i) $V_\iota(\mathbf{x}(t), k\overset{\circ}{\mathbf{x}}(t)) \to 0$ *as* $k \to 0+$ $(\iota = 1,2)$;
(ii) $V_\iota(\mathbf{x}(t), k\overset{\circ}{\mathbf{x}}(t))/V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ *as a function of* $k>0$ *is not identically* 1.

Proof. Let $V_\iota(\mathbf{x}(t), k\overset{\circ}{\mathbf{x}}(t))$ satisfy part (ii) of Axiom 6 for some arc element $(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ and $\iota = 1$ or 2. Since $V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))>0$, and $V_\iota(\mathbf{x}(t), k\overset{\circ}{\mathbf{x}}(t)) = k^\mu V_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$, the latter expression converges to zero if and only if $\mu>0$. Statements (i) and (ii) of the theorem now follow from the fact that Theorem 17 holds for all arc elements and both $\iota = 1,2$. □

Since function $\ell^{1/\mu}(\alpha)$ with $\mu>0$ is slowly varying provided $\ell(\alpha)$ is, we can rewrite the asymptotic decomposition of psychometric increments, (12), in yet another form, better suited for our purposes: as $\alpha \to 0+$,

$$\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha)) \sim [F_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))R(\alpha)]^\mu, \quad \iota = 1,2, \tag{15}$$

where

$$F_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t)) = V_\iota^{1/\mu}(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$$

and

$$R(\alpha) = \alpha\ell^{1/\mu}(\alpha),$$

a unit-exponent regularly varying function.

For completeness, we should also present here a new form for (13): as $\alpha \to 0+$,

$$\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t-\alpha)) \sim [F_\iota(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t))R(\alpha)]^\mu, \quad \iota = 1,2. \tag{16}$$

Definition 11. Function $R(\alpha)$ is called the characteristic function of space $\mathfrak{M}$.

Definition 12. Let $\mathfrak{S}$ be the set of all arc elements $(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$ that can be formed in space $\mathfrak{M}$. Functions $F_\iota : \mathfrak{S} \to \mathrm{Re}^+\backslash\{0\}$, $\iota = 1,2$, defined in (15) are called (Fechnerian) submetric functions (of the first and second kind).[9]

The following characterizations immediately follows from Theorems 15 and 16.

Corollary 5 (*to Theorems 15 and 16*). *Submetric functions* F_ι $(\iota = 1,2)$ *and characteristic function* $R(\alpha)$ *are determined*, *respectively*, *uniquely and asymptotically*

[9] In previous publications (e.g., Dzhafarov & Colonius, 1999, 2001; Dzhafarov, 2002a,d) F was referred to as *metric function* (or *Fechner–Finsler metric function*). The use of this term requires that one strictly distinguish the metric function and the metric of the stimulus space derived from it (distance function). We think that the present terminology (submetric function) is more convenient and prevents possible confusions. We also drop the adjective "Finsler" because the present development is too far beyond the initial context of Finsler geometry (Dzhafarov & Colonius, 1999).

uniquely, up to multiplications by two reciprocal positive constants.

In other words, one can only replace F_ι (for both $\iota = 1$ and $\iota = 2$) with λF_ι ($\lambda > 0$) and $R(\alpha)$ with $R^*(\alpha)$ such that $R^*(\alpha)/R(\alpha) \to 1/\lambda$, as $\alpha \to 0+$.

Corollary 6 (*to Theorems 15 and 16*). *Characteristic function $R(\alpha)$ vanishes at $\alpha = 0$ and can always be chosen to be continuously differentiable at sufficiently small values of $\alpha > 0$, with $\mathrm{d}R(\alpha)/\mathrm{d}\alpha > 0$.*

Proof. Use Theorem 15 to choose $R(\alpha)$ so that $\Psi^{(\iota)}(\mathbf{x}(a), \mathbf{x}(a+\alpha)) = [F_\iota(\mathbf{x}(a), \mathring{\mathbf{x}}(a))R(\alpha)]^\mu$ for an arbitrarily chosen psychometric increment (of either kind). □

It follows from Theorem 17 and the fact that $\mu > 0$ that an analogue of Euler homogeneity (of the first order) holds for submetric functions.

Corollary 7 (*to Theorem 17 and Axiom 6*). *For any arc element $(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$, any $k > 0$, and for $\iota = 1, 2$,*

$$F_\iota(\mathbf{x}(t), k\,\mathring{\mathbf{x}}(t)) = kF_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t)).$$

We also have the following consequence of Axiom 6 (of Double Continuity). Recall Convention 2 above.

Corollary 8 (*to Axiom 6*). *For any smooth arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$, $F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$ is continuous in t and bounded, for $\iota = 1, 2$.*

8.2. Overall psychometric transformation

The following lemma is needed for the theorem proved next.

Lemma 1. *Let $f(x)$ be a positive-exponent regularly varying function increasing in some vicinity of $x = 0+$. Let $a(x) \to 0+$ and $b(x) \to 0+$ as $x \to 0+$, and let $a(x) \sim b(x)$. Then $f[a(x)] \sim f[b(x)]$.*[10]

Proof. [11]Assume the contrary. Then, for all elements of some sequence $x_n \to 0+$, either

$$\frac{f[a(x_n)]}{f[b(x_n)]} > 1 + \varepsilon$$

or

$$\frac{f[a(x_n)]}{f[b(x_n)]} < 1 - \varepsilon,$$

where $\varepsilon > 0$. Assume the former (the other case being treated analogously), that is,

$$f[a(x_n)] > (1+\varepsilon)f[b(x_n)].$$

As $x_n \to 0+$,

$$\frac{(1+\varepsilon)f[b(x_n)]}{f\left[\left(1+\frac{\varepsilon}{2}\right)^{1/v} b(x_n)\right]} = \frac{1+\varepsilon}{1+\frac{\varepsilon}{2}} \cdot \frac{\left(1+\frac{\varepsilon}{2}\right)f[b(x_n)]}{f\left[\left(1+\frac{\varepsilon}{2}\right)^{1/v} b(x_n)\right]} \to \frac{1+\varepsilon}{1+\frac{\varepsilon}{2}} > 1,$$

where $v > 0$ is the exponent of regular variation for f. Because of this sequence x_n can be redefined (by deleting a finite number of its elements) so that

$$(1+\varepsilon)f[b(x_n)] > f\left[\left(1+\frac{\varepsilon}{2}\right)^{1/v} b(x_n)\right]$$

for all n. It follows then that

$$f[a(x_n)] > f\left[\left(1+\frac{\varepsilon}{2}\right)^{1/v} b(x_n)\right].$$

Since sequence x_n can always be redefined so that $a(x_n)$ and $(1+\varepsilon)^{1/v} b(x_n)$ are all sufficiently small, and since f is increasing at such values, we have

$$a(x_n) > \left(1+\frac{\varepsilon}{2}\right)^{1/v} b(x_n)$$

whence

$$\lim_{x_n \to 0+} \frac{a(x_n)}{b(x_n)} \geqslant \left(1+\frac{\varepsilon}{2}\right)^{1/v} > 1$$

contrary to the premise $a(x) \sim b(x)$. □

We now formulate what in the previous publications was referred to as the "main theorem" of Fechnerian Scaling, and which we now call "the first main theorem", in view of another important result, to be derived later (Theorem 53).

Theorem 19 (*First Main Theorem of Fechnerian Scaling*). *There is a function $\Phi(h) : \mathrm{Re}^+ \to \mathrm{Re}^+$ (called the overall psychometric transformation) vanishing at $h = 0$ and possessing continuous positive derivative at sufficiently small values of $h > 0$, such that for any arc element $(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$,*

$$\Phi[\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))] \sim F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\alpha, \quad \iota = 1, 2.$$

Any other transformation $\Phi^*(h)$ such that

$$\Phi^*[\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))] \sim F^*_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\alpha, \quad \iota = 1, 2,$$

satisfies the relations

$$\Phi^*(h) \sim k\Phi(h),$$

$$F^*_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) = kF_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$$

for some $k > 0$.

Remark. What the theorem says is that there is a well-behaved transformation of psychometric increments (one and the same for all of them, of both kinds) that makes them all comeasurable im kleinen with the arc

[10]Note that this will not hold just for any positive increasing continuous $f(x)$ vanishing as $x \to 0+$. For a counterexample, consider $f(x) = \exp(-1/x)$ (that does not vary regularly at the origin) and asymptotically equal $a(x) = x$ and $b(x) = x + x^2$.

[11]We are grateful to Jun Zhang for identifying a mistake in the original proof.

parameter, α; that the submetric function F_ι is the coefficient of asymptotic proportionality between the transformed psychometric increments and α; and that this transformation is essentially asymptotically unique. Fig. 21 provides an illustration (Γ denotes the "gamma increment" function introduced in the next subsection).

Proof of Theorem 19. By Corollary 6 to Theorems 15 and 16, choose the characteristic function $R(\alpha)$ in (15) to be continuously differentiable at sufficiently small values of $\alpha > 0$, with $\mathrm{d}R(\alpha)/\mathrm{d}\alpha > 0$. Define

$$\Phi(h) = R^{-1}(h^{1/\mu}).$$

This function regularly varies with exponent $1/\mu > 0$, and it is continuously differentiable at sufficiently small values of $h > 0$, with $\mathrm{d}\Phi(h)/\mathrm{d}h > 0$. Applying this function to both sides of (15) and using Lemma 1,

$$\Phi[\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))] \sim R^{-1}[F_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))R(\alpha)],$$

$$\sim F_\iota(\mathbf{x}(t)\, \overset{\circ}{\mathbf{x}}(t))R^{-1}[R(\alpha)] = F_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))\alpha.$$

Now, if

$$\Phi[\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))] \sim F_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))\alpha,$$

$$\Phi^*[\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))] \sim F^*_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))\alpha,$$

then

$$\frac{F^*_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))}{F_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))} \sim \lim_{\alpha\to 0+} \frac{\Phi^*[\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))]}{\Phi[\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))]} = \lim_{h\to 0+} \frac{\Phi^*(h)}{\Phi(h)},$$

whence the uniqueness statement of the theorem. □

It is useful to extract from the proof the following information.

Corollary 9 (*to Theorem 19*). *Overall psychometric transformation is a regularly varying function with exponent reciprocal to the psychometric order* μ *of space* $\mathfrak{M}$. *Specifically*, $\Phi(h) = R^{-1}(h^{1/\mu})$, *where* R *is the characteristic function of space* $\mathfrak{M}$.

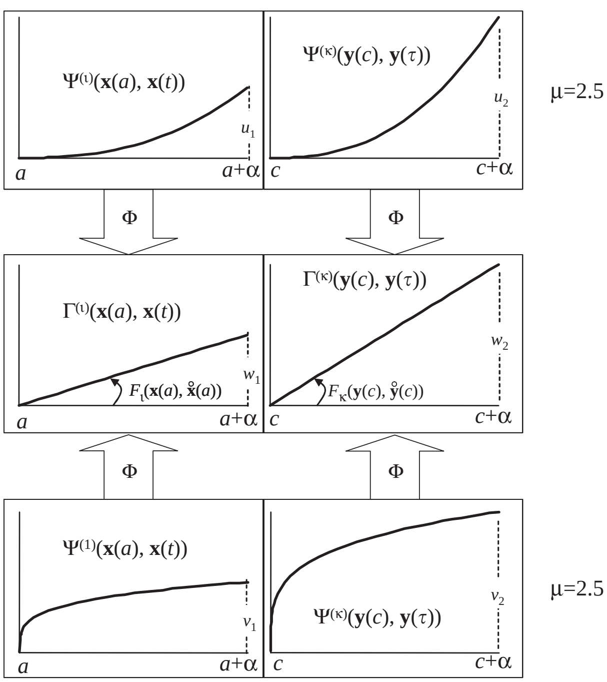

Fig. 21. Schematic illustration for overall psychometric transformation Φ. Shown are very small pieces of the parametric domains of two arcs, $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ and $\mathbf{y}(\tau) : [c,d] \to \mathfrak{M}$, together with the corresponding psychometric increments with respect to left endpoints, of either kind ($\iota = 1,2$; $\kappa = 1,2$). $\mathfrak{M}$ is of psychometric order 2.5 (top) or 0.25 (bottom). As $\alpha \to 0$, both u_1/u_2 and v_1/v_2 tend to positive finite limits in accordance with Axiom 5. The quantities u_1, u_2, v_1, v_2 are not, however, comeasurable with $\alpha \to 0$: the derivatives at the origin equal 0 in the case of $\mu = 2.5$ and equal ∞ in the case of $\mu = 0.25$. Overall psychometric transformation (different for $\mu = 0.25$ and for $\mu = 2.5$) makes these derivatives positive and finite (middle panels, $\Gamma \equiv \Phi \circ \Psi$); their values are taken as those of submetric function, $F_\iota(\mathbf{x}(a), \overset{\circ}{\mathbf{x}}(a))$ and $F_\kappa(\mathbf{y}(c), \overset{\circ}{\mathbf{y}}(c))$. Anticipating subsequent development: the possibility of $\mu > 1$ will be ruled out, and the cross-unbalanced version of the theory, supported by empirical evidence, is only compatible with $\mu = 1$ and with R (characteristic function) being identity (in this case Φ is also identity, and the notions of Ψ and Γ coincide).

8.3. Gamma-increments

The overall psychometric transformation $\Phi(h)$ is only *asymptotically* unique (up to multiplication by a positive constant). When considered on a finite interval $h \in [0, H]$ rather than in an arbitrarily small vicinity of $h = 0$, we can choose different *variants* of Φ. One of the properties of regularly varying functions is that a variant can always be chosen to be arbitrarily smooth (e.g., infinitely differentiable) everywhere except, perhaps, at zero.

Convention 3. *In the following we will tacitly assume that if the domain of* $\Phi(h)$ *is to be extended to an interval* $[0, H]$, *then the term "variant of* Φ*" means a variant that possesses continuous positive derivative on* $(0, H]$.

Lemma 2. *Any variant* $\Phi^*(h)$ *of* $\Phi(h)$ *defined on* $h \in [0, H]$ *satisfies*

$$\Phi^*(h) = f[\Phi(h)],$$

where $f : [0, \Phi(H)] \to [0, \Phi^*(H)]$ *is vanishing at zero and continuously differentiable on* Re^+, *with* $f'(x) > 0$ *for all* $x \in [0, \Phi(H)]$.

Proof. For positive $\Phi(h)$ this follows from the above convention. The existence of $f'(0) > 0$ follows from the uniqueness part of Theorem 19, with $f'(0) = k$. □

Definition 13. For any variant of the overall psychometric transformation $\Phi(h)$, the transformed psychometric increments $\Phi[\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))]$, $\iota = 1, 2$, are called gamma-increments and denoted $\Gamma^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))$.

It is worth emphasizing again that the notion of gamma-increments $\Gamma^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))$ for finite values of α, being dependent on the choice of Φ, is inherently nonunique. But the choice of Φ makes no difference for limit propositions, as $\alpha \to 0+$. In particular, the following properties hold universally.

Theorem 20. *For any smooth arc* $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$, *any* $t \in [a,b)$ *and* $\iota = 1,2$, $\Gamma^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t \pm \alpha))$ *is differentiable at* $\alpha = 0+$, *with*

$$\left.\frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t \pm \alpha))}{\mathrm{d}\alpha}\right|_{\alpha=0+} = F_\iota(\mathbf{x}(t), \pm\mathring{\mathbf{x}}(t)).$$

Proof. This is essentially a restatement of the asymptotic decomposition in Theorem 19. □

Theorem 21. *For any smooth arc* $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ *and every* $c \in [a,b]$, *one can find* $c_* \in [a,c)$ *and* $c^* \in (c,b]$ *such that both* $\Gamma^{(1)}(\mathbf{x}(c), \mathbf{x}(t))$ *and* $\Gamma^{(2)}(\mathbf{x}(c), \mathbf{x}(t))$ *are negative diffeomorphisms on* $[c_*, c]$ *and positive diffeomorphisms on* $[c, c^*]$.

Proof. Follows from Definition 5 and the previous theorem. □

Comparing this result with Definition 5 we see that the only change in switching from $\Psi^{(\iota)}$ to $\Gamma^{(\iota)}$ occurs at point c itself: $\Gamma^{(1)}(\mathbf{x}(c), \mathbf{x}(t))$ and $\Gamma^{(2)}(\mathbf{x}(c), \mathbf{x}(t))$ possess nonzero unilateral derivatives at $t = c$.

Theorem 22. *For any positive diffeomorphic reparametrization of any smooth arc* $\mathbf{x}(t) = \mathbf{x}(t(\tau)) = \mathbf{y}(\tau)$,

$$F_\iota(\mathbf{y}(\tau), \mathring{\mathbf{y}}(\tau)) = F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))t'(\tau), \quad \iota = 1,2.$$

Proof. We have

$$\Gamma^{(\iota)}(\mathbf{y}(\tau), \mathbf{y}(\tau+\beta)) \sim F_\iota(\mathbf{y}(\tau), \mathring{\mathbf{y}}(\tau))\beta,$$

but also

$$\begin{aligned}\Gamma^{(\iota)}(\mathbf{y}(\tau), \mathbf{y}(\tau+\beta)) &= \Gamma^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t + (t'(\tau) + o\{1\})\beta)) \\ &\sim F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))(t'(\tau) + o\{1\})\beta \\ &\sim F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))t'(\tau)\beta.\end{aligned}$$

Hence the statement of the theorem. □

The following immediate consequence of this theorem is of crucial importance for the construction of Fechnerian distances.

Theorem 23. *For any positive diffeomorphic reparametrization of any smooth arc* $\mathbf{x}(t) = \mathbf{x}(t(\tau)) = \mathbf{y}(\tau)$,

$$F_\iota(\mathbf{y}(\tau), \mathring{\mathbf{y}}(\tau))\,\mathrm{d}\tau = F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t, \quad \iota = 1,2,$$

at any point $t = t(\tau)$.

Proof. From the previous theorem,

$$\begin{aligned}F_\iota(\mathbf{y}(\tau), \mathring{\mathbf{y}}(\tau))\,\mathrm{d}\tau &= F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))t'(\tau)\,\mathrm{d}\tau \\ &= F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t. \qquad \square\end{aligned}$$

8.4. Boundedness of growth

To introduce our last axiom we need a new notion. According to Definition 5, if a smooth arc $\mathbf{x}(t)$ originates at $\mathbf{a}$, $\Psi^{(\iota)}(\mathbf{a}, \mathbf{x}(t))$ varies along the arc in a continuously differentiable way. With any variant of the overall psychometric transformation $\Phi(h)$, the same can be said about $\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t)) = \Phi[\Psi^{(\iota)}(\mathbf{a}, \mathbf{x}(t))]$. The rate of its change, $\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))/\mathrm{d}t$, will depend on the parametrization of the arc. If we normalize this rate by the submetric function $F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$, however, we get a parametrization-invariant measure of the change rate.

Definition 14. For any smooth arc $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$, the quantity

$$\Delta^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t)) = \frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))}{F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t}$$

is called the relative growth rate of $\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))$.

Theorem 24. *Relative growth rate is invariant with respect to positive diffeomorphic reparametrizations.*

Proof. Let $\mathbf{y}(\tau) = \mathbf{x}(t(\tau))$, be such a reparametrization, with $\mathbf{y}(c) = \mathbf{x}(a)$. Then

$$\begin{aligned}\Delta^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t)) &= \frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))/\mathrm{d}t}{F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))} \\ &= \frac{\dfrac{\mathrm{d}t}{\mathrm{d}\tau}\,\mathrm{d}\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))/\mathrm{d}t}{\dfrac{\mathrm{d}t}{\mathrm{d}\tau}F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))} \\ &= \frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{y}(c), \mathbf{y}(\tau))/\mathrm{d}\tau}{F_\iota(\mathbf{y}(\tau), \mathring{\mathbf{y}}(\tau))} \\ &= \Delta^{(\iota)}(\mathbf{y}(c), \mathbf{y}(\tau)). \qquad \square\end{aligned}$$

The first part of the next axiom says, in essence, that the relative rate of growth in $\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))$, computed for all smooth arcs within a sufficiently small ball, cannot get arbitrarily large (in absolute value). The second part of the axiom says that two sufficiently close to each other points $\mathbf{a}$ and $\mathbf{b}$ can be connected by a smooth arc on which $\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))$ increases at a sufficiently fast relative rate. These special arcs are analogous to short segments of straight lines in the Fechnerian theory for finite-dimensional Euclidean spaces.

Axiom 7 (*Local boundedness*). *For any* $\mathbf{p} \in \mathfrak{M}$ *one can find positive constants* $\varepsilon_{\mathbf{p}}, C_{\mathbf{p}}, c_{\mathbf{p}}$ *such that,*

(i) *for any smooth arc* $\mathbf{x}(t) : [a,b] \to \mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$ *and for any* $t \in [a,b]$,

$$|\Delta^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))| < C_{\mathbf{p}}, \quad \iota = 1, 2;$$

(ii) *for any* $\mathbf{a}, \mathbf{b} \in \mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$, *with* $\mathbf{a} = \mathbf{p}$ *or* $\mathbf{b} = \mathbf{p}$, *there are smooth arcs (called straight arcs)* $\mathbf{x}^{(\iota)}(t) : [a,b] \to \mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$ $(\iota = 1, 2)$ *with* $\mathbf{a} = \mathbf{x}^{(\iota)}(a), \mathbf{b} = \mathbf{x}^{(\iota)}(b)$, *such that for any* $t \in [a,b]$,

$$\Delta^{(\iota)}(\mathbf{a}, \mathbf{x}^{(\iota)}(t)) > c_{\mathbf{p}}, \quad \iota = 1, 2.$$

A schematic illustration is provided by Fig. 22. A few remarks are due.

1. Clearly, if $\varepsilon_{\mathbf{p}}, C_{\mathbf{p}}, c_{\mathbf{p}}$ conform with this axiom, then it is also satisfied
   (a) for $\varepsilon_{\mathbf{p}}$ taken with any finite $C'_{\mathbf{p}} > C_{\mathbf{p}}$ and any positive $c'_{\mathbf{p}} < c_{\mathbf{p}}$;
   (b) for any $\varepsilon < \varepsilon_{\mathbf{p}}$ taken with some
   $$C_{\mathbf{p}}(\varepsilon) \leqslant C_{\mathbf{p}}, \quad c_{\mathbf{p}}(\varepsilon) \geqslant c_{\mathbf{p}}.$$
2. The definition of a straight arc depends on $\iota = 1, 2$, and on the initial and terminal points $\mathbf{a}, \mathbf{b}$ (one of which is $\mathbf{p}$). One should, therefore, distinguish straight arcs of the first kind and of the second kind, and we impose no upper limit on the number of straight arcs of either kind connecting two given points. Note that by replacing $c_{\mathbf{p}}$ with $c'_{\mathbf{p}} < c_{\mathbf{p}}$ we may add new straight arcs (without losing the "old" ones) for any given $\mathbf{a}, \mathbf{b}$ and $\iota = 1, 2$.
3. One should not assume that a subarc of a straight arc is a straight arc. Nor should one assume in general that a straight arc remains straight if traversed in the opposite direction (i.e., when subjected to a negative diffeomorphic reparametrization).
4. Writing $\mathbf{x}^{(\iota)}(t) : [a^{(\iota)}, b^{(\iota)}] \to \mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$ would be more general but is not necessary, because the two parametric domains can always be made coincide by a diffeomorphic transformation (that does not change relative growth rate).
5. For a straight arc $\mathbf{x}^{(\iota)}(t)$, (i) and (ii) imply boundedness by positive numbers on both sides, $c_{\mathbf{p}} < \Delta^{(\iota)}(\mathbf{a}, \mathbf{x}^{(\iota)}(t)) < C_{\mathbf{p}}$.
6. Anticipating Definition 16 given in the next section, $\Delta^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))$ can be viewed as $\frac{\mathrm{d}}{\mathrm{d}t} \Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))$ computed for $\mathbf{x}(t)$ parametrized by the psychometric length of the arc (of the ιth kind) measured from $\mathbf{x}(a)$ to $\mathbf{x}(t)$. In this parametrization,
   $$t = \int_a^t F_\iota(\mathbf{x}(u), \mathring{\mathbf{x}}(u))\, \mathrm{d}u.$$

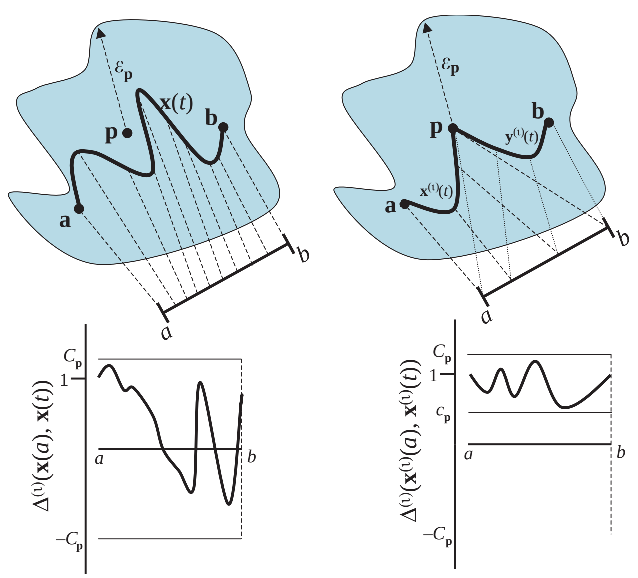

Fig. 22. Schematic illustration for Axiom 7. Left: an open ball $\mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$ shown with points $\mathbf{a}$ and $\mathbf{b}$ connected by an arbitrary smooth arc. The plot below illustrates part (i) of the axiom: relative growth rate along any smooth arc within the ball is bounded. Note that the relative growth rate begins at level 1 (because $F_\iota(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) = \mathrm{d}\Gamma^{(1)}(\mathbf{x}(a), \mathbf{x}(t))/\mathrm{d}t$ at $t = a+$), and the bound $C_{\mathbf{p}} \geqslant 1$. Right: the same open ball with the same points $\mathbf{a}$ and $\mathbf{b}$ connected with $\mathbf{p}$ by straight arcs of the first or second kind. The plot below illustrates part (ii) of the axiom: relative growth rate along a straight arc does not fall below some positive level $c_{\mathbf{p}} \leqslant 1$.

Part (ii) of the axiom implies, of course, that any $\mathbf{p} \in \mathfrak{M}$ can be connected to any point within a sufficiently small ball $\mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$ by a smooth arc, so $\varepsilon_{\mathbf{p}}$ can be taken to be $\delta_{\mathbf{p}}$ of Definition 7. This deserves to be stated formally.

Corollary 10 (*to Axiom 7*). *Stimulus space* $\mathfrak{M}$ *is locally smoothly connected* (*hence smoothly connected*).

As a result, Theorems 13 and 14 of Section 6 can now be considered valid for stimulus space $\mathfrak{M}$ without qualifications.

Definition 14 of $\Delta^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))$ involves function $\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))$ without stipulating $t \to a$. The value of $\Delta^{(\iota)}$ therefore will depend on one's choice of the overall psychometric transformation Φ which we know to be only asymptotically unique. This raises the question of whether Axiom 7 is well-formed, that is, whether its validity can be shown to not depend on the variant of Φ.

Theorem 25. *The validity of Axiom* 7 *does not depend on the variant of* $\Phi(h)$.

Proof. Let two variants Φ and Φ^* be defined on an interval including $[0, H]$. By Theorem 7 and by the first remark following Axiom 7, $\varepsilon_{\mathbf{p}}$ can always be chosen so that $\operatorname{diam} \mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}}) < H$. According to Lemma 2,

$$\Phi^*(h) = f[\Phi(h)],$$

where f' is positive and continuous on $[0, \Phi(H)]$. Then for any smooth arc $\mathbf{x}(t) : [a,b] \to \mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$, with

$\mathbf{a} = \mathbf{x}(a)$,

$$\frac{\mathrm{d}f[\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))]}{\mathrm{d}t} = f'[\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))]\frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))}{\mathrm{d}t}.$$

As a continuous function, $f'(x)$ is bounded between

$$m_{\mathbf{p}} = \min_{x\in[0,\Phi(H)]} f'(x) > 0$$

and

$$M_{\mathbf{p}} = \max_{x\in[0,\Phi(H)]} f'(x) < \infty.$$

By Theorem 19 and Lemma 2, $F^*_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$ corresponding to Φ^* equals $kF_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$, with $k = f'(0)$. As a result,

$$\frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))}{F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t} < C_{\mathbf{p}}$$
$$\Rightarrow \quad \frac{\mathrm{d}f[\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))]}{kF_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t}$$
$$< \frac{M_{\mathbf{p}}\,\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))}{kF_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t} < \frac{M_{\mathbf{p}}C_{\mathbf{p}}}{k},$$

$$\frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))}{F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t} > c_{\mathbf{p}} \quad \Rightarrow \quad \frac{\mathrm{d}f[\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))]}{kF_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t}$$
$$> \frac{m_{\mathbf{p}}\,\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))}{kF_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t} > \frac{m_{\mathbf{p}}c_{\mathbf{p}}}{k}.$$

All statements of Axiom 7 therefore should be as applicable to $f[\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))]$ as they are to $\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))$. $\square$

As remarked earlier, if some $\varepsilon_{\mathbf{p}}, C_{\mathbf{p}}, c_{\mathbf{p}}$ conform with Axiom 7, then it is also satisfied for any $\varepsilon < \varepsilon_{\mathbf{p}}$ with some

$$C_{\mathbf{p}}(\varepsilon) \leqslant C_{\mathbf{p}}, \quad c_{\mathbf{p}}(\varepsilon) \geqslant c_{\mathbf{p}}.$$

We have the following estimate for $C_{\mathbf{p}}(\varepsilon)$ and $c_{\mathbf{p}}(\varepsilon)$.

Theorem 26. *For any* $\mathbf{p} \in \mathfrak{M}$ *and any* $\varepsilon \leqslant \varepsilon_{\mathbf{p}}$, $c_{\mathbf{p}}(\varepsilon) \leqslant 1 \leqslant C_{\mathbf{p}}(\varepsilon)$.

Proof. For any straight arc $\mathbf{x}(t) : [a, b] \to \mathfrak{B}(\mathbf{p}, \varepsilon)$ originating at $\mathbf{p}$,

$$c_{\mathbf{p}}(\varepsilon) \leqslant \Delta^{(\iota)}(\mathbf{x}(a),\ \mathbf{x}(t)) \leqslant C_{\mathbf{p}}(\varepsilon),$$

for all t. But

$$\Delta^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t)) = \frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))/\mathrm{d}t|_{t=a+}}{F_\iota(\mathbf{x}(a), \mathring{\mathbf{x}}(t))} = 1. \qquad \square$$

Note that it cannot be concluded that, as $\varepsilon \to 0+$, $c_{\mathbf{p}}(\varepsilon) \to 1$ or $C_{\mathbf{p}}(\varepsilon) \to 1$. In other words, the inequalities

$$\lim_{\varepsilon\to 0+} c_{\mathbf{p}}(\varepsilon) \leqslant 1$$

and

$$\lim_{\varepsilon\to 0+} C_{\mathbf{p}}(\varepsilon) \geqslant 1$$

may very well be strict. The limits, of course, exist because $c_{\mathbf{p}}(\varepsilon)$ and $C_{\mathbf{p}}(\varepsilon)$ are monotone.

Definition 15. Stimulus space $\mathfrak{M}$ is called quasi-convex if

$$\lim_{\varepsilon\to 0+} C_{\mathbf{p}}(\varepsilon) = \lim_{\varepsilon\to 0+} c_{\mathbf{p}}(\varepsilon) = 1.$$

The term "quasi-convexity" is due to the fact that this property in certain respects generalizes the notion of connected open regions of Re^n endowed with submetric functions with convex indicatrices (Dzhafarov & Colonius, 2001). Most of the results in this paper do not make use of this quasi-convexity assumption.

9. Psychometric lengths and oriented Fechnerian distances

9.1. Lengths

Definition 16. For any piecewise smooth arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$, its psychometric length (of the first kind or second kind) is defined as the Riemann integral

$$\int_a^b F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t = L^{(\iota)}[\mathbf{x}_{[a,b]}], \quad \iota = 1, 2.$$

The definition is valid because $F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$ is a piecewise continuous function on a closed interval. Fig. 23 provides an illustration.

Theorem 27. *Psychometric length* $L^{(\iota)}[\mathbf{x}_{[a,b]}]$ *has the following properties: for* $\iota = 1, 2$,

(i) $L^{(\iota)}[\mathbf{x}_{[a,b]}] \geqslant 0$;
(ii) $L^{(\iota)}[\mathbf{x}_{[a,b]}] = 0$ *if and only if the domain of* $\mathbf{x}(t)$ *is a singleton* (*i.e.*, $a = b$);
(iii) *if* $\mathbf{z}(t) : [a, c] \to \mathfrak{M}$ *is the concatenation of* $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ *and* $\mathbf{y}(t) : [b, c] \to \mathfrak{M}$, *then*
$$L^{(\iota)}[\mathbf{z}_{[a,c]}] = L^{(\iota)}[\mathbf{x}_{[a,b]}] + L^{(\iota)}[\mathbf{y}_{[b,c]}];$$
(iv) $L^{(\iota)}[\mathbf{x}_{[a,b]}]$ *is invariant under all positive diffeomorphic reparametrizations.*

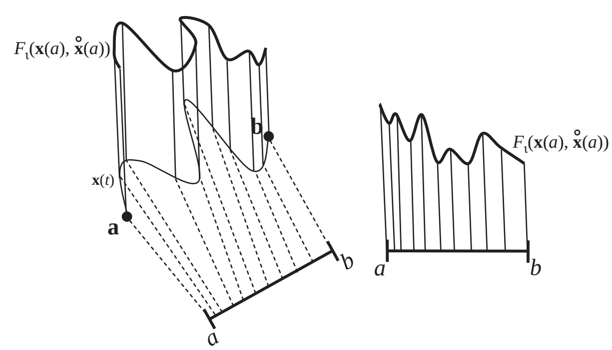

Fig. 23. Submetric function $F_\iota(\mathbf{x}(a), \mathring{\mathbf{x}}(a))$ ($\iota = 1, 2$) plotted against the codomain of a piecewise smooth arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ (left) and the parametric domain of this arc (right). Psychometric length of $\mathbf{x}(t)$ (of the ιth kind) is the area subtended by the right-hand figure.

Proof. (i)–(iii) being obvious, (iv) states that for any positive diffeomorphism $t(\tau) : [c, d] \to [a, b]$,

$$L^{(\iota)}[\mathbf{x}_{[a,b]}] = L^{(\iota)}[\mathbf{y}_{[c,d]}],$$

where $\mathbf{y}(\tau) = \mathbf{x}(t(\tau))$. That is,

$$L^{(\iota)}[\mathbf{y}_{[c,d]}] = \int_c^d F_\iota(\mathbf{y}(\tau), \mathring{\mathbf{y}}(\tau))\,\mathrm{d}\tau = \int_a^b F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t$$
$$= L^{(\iota)}[\mathbf{x}_{[a,b]}].$$

This follows from Theorem 23. □

Psychometric length is a property of *oriented* piecewise smooth arcs: $L^{(\iota)}[\mathbf{x}_{[a,b]}]$ is the length of the piecewise smooth arc *from* $\mathbf{a} = \mathbf{x}(a)$ *to* $\mathbf{b} = \mathbf{x}(b)$. To compute the length "in the opposite direction" one has to apply the definition to a reparametrization of $\mathbf{x}(t)$ by any negative diffeomorphism, say, $\mathbf{y}(\tau) = \mathbf{x}(-\tau + (a + b))$,

$$L^{(\iota)}[\mathbf{y}_{[a,b]}] = \int_a^b F_\iota(\mathbf{y}(t), \mathring{\mathbf{y}}(t))\,\mathrm{d}\tau.$$

Equivalently, using Definition 9, one can write

$$L^{(\iota)}[-\mathbf{x}_{[a,b]}] = \int_a^b F_\iota(\mathbf{x}(t), -\mathring{\mathbf{x}}(t))\,\mathrm{d}t \tag{17}$$

where $-\mathbf{x}(\tau)$ is a natural way of indicating that the length is being computed "in the opposite direction".

9.2. Distances

Recall that a function $d(\mathbf{a}, \mathbf{b})$ is called an *oriented distance* function (or *oriented metric*) if

$d(\mathbf{a}, \mathbf{b}) \geqslant 0$ (nonnegativity),

$d(\mathbf{a}, \mathbf{b}) = 0 \Longleftrightarrow \mathbf{a} = \mathbf{b}$ (zero property),

$d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c}) \geqslant d(\mathbf{a}, \mathbf{c})$ (triangle inequality).

The oriented distance is distance (or metric) proper if, in addition,

$d(\mathbf{a}, \mathbf{b}) = d(\mathbf{b}, \mathbf{a})$ (symmetry).

Definition 17. For any $\mathbf{a}, \mathbf{b} \in \mathfrak{M}$, the (oriented) Fechnerian distance (of the first kind or second kind) from $\mathbf{a}$ to $\mathbf{b}$ is defined as

$$G_\iota(\mathbf{a}, \mathbf{b}) = \inf_{\mathbf{x}_{[a,b]} \in \mathfrak{X}(\mathbf{a}, \mathbf{b})} L^{(\iota)}[\mathbf{x}_{[a,b]}], \quad \iota = 1, 2,$$

where $\mathfrak{X}(\mathbf{a}, \mathbf{b})$ is the set of all piecewise smooth arcs connecting $\mathbf{a}$ to $\mathbf{b}$.

Fechnerian metrics are *intrinsic metrics* in Aleksandrov's sense: distance is defined as infimum of the lengths of specially chosen arcs (see, e.g., Aleksandrov & Zalgaller, 1967). Intrinsic metrics are also called "inner" and " internal" (Dzhafarov, 2002b). The legitimacy of the present construction is demonstrated in the next theorem.

Theorem 28. $G_1(\mathbf{a}, \mathbf{b})$ *and* $G_2(\mathbf{a}, \mathbf{b})$ *are oriented metrics.*

Proof. That $G_\iota(\mathbf{a}, \mathbf{b})$ $(\iota = 1, 2)$ is nonnegative, satisfies the triangle inequality, and that $\mathbf{a} = \mathbf{b}$ implies $G_\iota(\mathbf{a}, \mathbf{b}) = 0$, is shown trivially. The only nontrivial aspect is the implication $\mathbf{a} \neq \mathbf{b} \Rightarrow G_\iota(\mathbf{a}, \mathbf{b}) > 0$. Let $\varepsilon_{\mathbf{a}} > 0$ be any radius conforming with Axiom 7 (putting $\mathbf{p} = \mathbf{a}$), and let δ be any positive number less than $\min\{\Phi(\varepsilon_{\mathbf{a}}), \Gamma^{(\iota)}(\mathbf{a}, \mathbf{b})\}$. For any smooth arc $\mathbf{x}(t)$ connecting $\mathbf{a}$ to $\mathbf{b}$, due to the continuity of $\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))$, there must be a point $c \in (a, b)$ such that $\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(c)) = \delta$ while $\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t)) < \delta$ for all $t < c$. Let $\mathfrak{J}$ be the closure of the set

$$\left\{ t \in [a, c] : \frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))}{\mathrm{d}t} > 0 \right\}.$$

That is, $\mathfrak{J}$ is the subset of $[a, c]$ on which $\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))$ increases. Clearly,

$$\int_{t \in \mathfrak{J}} \mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t)) \geqslant \delta$$

(equality being achieved iff $\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))$ does not decrease between different open subintervals on which it increases). We have

$$L^{(\iota)}[\mathbf{x}_{[a,b]}] > L^{(\iota)}[\mathbf{x}_{[a,c]}] = \int_a^c F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t$$
$$\geqslant \int_{t \in \mathfrak{J}} F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t$$
$$= \int_{t \in \mathfrak{J}} \frac{F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t}{\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))}\,\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))$$
$$\geqslant \int_{t \in \mathfrak{J}} \frac{1}{C_{\mathbf{a}}}\,\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t)) \geqslant \frac{\delta}{C_{\mathbf{a}}} > 0,$$

where we have used the fact that by Axiom 7 (i),

$$\frac{F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t}{\mathrm{d}\Gamma^{(\iota)}(\mathbf{a}, \mathbf{x}(t))} \geqslant \frac{1}{C_{\mathbf{a}}}$$

within $\mathfrak{B}(\mathbf{a}, \varepsilon_{\mathbf{a}})$ wherever it is nonnegative. That is, the length of every arc connecting $\mathbf{a}$ to $\mathbf{b}$ exceeds a positive quantity. Then $G_\iota(\mathbf{a}, \mathbf{b})$, being the infimum of these lengths, cannot fall below this quantity. □

The following results will be used in Section 10 to study topological and analytic properties of metrics $G_\iota(\mathbf{a}, \mathbf{b})$ $(\iota = 1, 2)$; $\varepsilon_{\mathbf{a}}$ in the formulation of these results is any radius conforming with Axiom 7 (putting $\mathbf{p} = \mathbf{a}$).

Theorem 29. *For any* $\mathbf{a} \in \mathfrak{M}$ *and any* $0 < \lambda \leqslant \varepsilon_{\mathbf{a}}$,

$$\begin{array}{l} \Psi^{(\iota)}(\mathbf{a}, \mathbf{b}) \geqslant \lambda \Rightarrow G_\iota(\mathbf{a}, \mathbf{b}) \geqslant \dfrac{\Phi(\lambda)}{C_{\mathbf{a}}(\lambda)} \\ \Psi^{(\iota)}(\mathbf{a}, \mathbf{b}) \leqslant \lambda \Rightarrow G_\iota(\mathbf{a}, \mathbf{b}) \leqslant \dfrac{\Phi(\lambda)}{c_{\mathbf{a}}(\lambda)} \end{array}, \quad \iota = 1, 2.$$

Proof. The first implication immediately follows from the proof of Theorem 28 (on replacing $\varepsilon_{\mathbf{a}}$ with λ).

To prove the second implication, consider $\mathfrak{B}(\mathbf{a},\lambda)$. By part (ii) of Axiom 7 (with $\mathbf{p}=\mathbf{a}$), $\mathbf{a}$ can be connected to any $\mathbf{b}\in\mathfrak{B}(\mathbf{a},\lambda)$ by a straight arc $\mathbf{x}^{(\iota)}(t):[a,b]\to\mathfrak{B}(\mathbf{a},\lambda)$. Its length is

$$\begin{aligned}L^{(\iota)}[\mathbf{x}^{(\iota)}_{[a,b]}] &= \int_a^b F_\iota(\mathbf{x}^{(\iota)}(t),\overset{\circ}{\mathbf{x}}{}^{(\iota)}(t))\,\mathrm{d}t\\ &= \int_a^b \frac{F_\iota(\mathbf{x}(t),\overset{\circ}{\mathbf{x}}(t))\,\mathrm{d}t}{\mathrm{d}\Gamma^{(\iota)}(\mathbf{a},\mathbf{x}(t))}\,\mathrm{d}\Gamma^{(\iota)}(\mathbf{a},\mathbf{x}(t))\\ &\leqslant \int_a^b \frac{1}{c_{\mathbf{a}}(\lambda)}\,\mathrm{d}\Gamma^{(\iota)}(\mathbf{a},\mathbf{x}(t))\\ &\leqslant \frac{\Phi(\lambda)}{c_{\mathbf{a}}(\lambda)}>0,\end{aligned}$$

where we use the fact that, for a straight arc in $\mathfrak{B}(\mathbf{a},\lambda)$,

$$\frac{F_\iota(\mathbf{x}(t),\overset{\circ}{\mathbf{x}}(t))\,\mathrm{d}t}{\mathrm{d}\Gamma^{(\iota)}(\mathbf{a},\mathbf{x}(t))}\leqslant\frac{1}{c_{\mathbf{a}}(\lambda)}$$

and

$$\int_a^b \mathrm{d}\Gamma^{(\iota)}(\mathbf{a},\mathbf{x}(t))\leqslant\Phi(\lambda).$$

$G_\iota(\mathbf{a},\mathbf{b})$, by definition, cannot exceed any $L^{(\iota)}[\mathbf{x}^{(\iota)}_{[a,b]}]$. $\square$

Theorem 30. *For any $\mathbf{a}\in\mathfrak{M}$ one can find a function $\varepsilon(\lambda)$ such that $\varepsilon(\lambda)\to 0+$ as $\lambda\to 0+$, and*

$$\Psi^{(\iota)}(\mathbf{a},\mathbf{b})\leqslant\lambda\Rightarrow G_\iota(\mathbf{b},\mathbf{a})\leqslant\frac{\Phi(\varepsilon(\lambda))}{c_{\mathbf{a}}(\lambda)},\quad \iota=1,2$$

for all $0<\lambda\leqslant\varepsilon_{\mathbf{a}}$.

Proof. By part (ii) of Axiom 7 (with $\mathbf{p}=\mathbf{a}$), any point $\mathbf{b}\in\mathfrak{B}(\mathbf{a},\lambda)$ for $\lambda\leqslant\varepsilon_{\mathbf{a}}$ can be connected to $\mathbf{a}$ by a straight arc $\mathbf{x}^{(\iota)}(t):[u,v]\to\mathfrak{B}(\mathbf{a},\lambda)$. Its length is

$$\begin{aligned}L^{(\iota)}[\mathbf{x}^{(\iota)}_{[u,v]}] &= \int_u^v F_\iota(\mathbf{x}^{(\iota)}(t),\overset{\circ}{\mathbf{x}}{}^{(\iota)}(t))\,\mathrm{d}t\\ &= \int_u^v \frac{F_\iota(\mathbf{x}^{(\iota)}(t),\overset{\circ}{\mathbf{x}}{}^{(\iota)}(t))\,\mathrm{d}t}{\mathrm{d}\Gamma^{(\iota)}(\mathbf{b},\mathbf{x}^{(\iota)}(t))}\,\mathrm{d}\Gamma^{(\iota)}(\mathbf{b},\mathbf{x}^{(\iota)}(t))\\ &\leqslant \frac{1}{c_{\mathbf{a}}(\lambda)}\int_u^v \mathrm{d}\Gamma^{(\iota)}(\mathbf{b},\mathbf{x}^{(\iota)}(t))\\ &= \frac{\Phi(\Psi^{(\iota)}(\mathbf{b},\mathbf{a}))}{c_{\mathbf{a}}(\lambda)}\leqslant\frac{\Phi(\varepsilon(\lambda))}{c_{\mathbf{a}}(\lambda)},\end{aligned}$$

where $\varepsilon(\lambda)=\sup_{\mathbf{b}\in\mathfrak{B}(\mathbf{a},\lambda)}\{\Psi^{(\iota)}(\mathbf{b},\mathbf{a})\}$. $G_\iota(\mathbf{b},\mathbf{a})$, by definition, cannot exceed any $L^{(\iota)}[\mathbf{x}^{(\iota)}_{[u,v]}]$. That $\varepsilon(\lambda)\to 0$ as $\lambda\to 0$ follows from either of Theorems 3 and 7. $\square$

10. Basic properties of oriented Fechnerian distances

10.1. Metrization property and continuity of Fechnerian metrics

It is easy to see that any oriented metric $d:\mathfrak{M}\times\mathfrak{M}\to\mathrm{Re}^+$ induces two topologies, based on the sets of metrics balls

$$\begin{aligned}\mathfrak{D}(\mathbf{x},\varepsilon)&=\{\mathbf{y}:d(\mathbf{x},\mathbf{y})<\varepsilon\},\\ \mathfrak{D}(\varepsilon,\mathbf{x})&=\{\mathbf{y}:d(\mathbf{y},\mathbf{x})<\varepsilon\}.\end{aligned}\tag{18}$$

Lemma 3. *Sets of metric balls $\{\mathfrak{D}(\mathbf{x},\varepsilon)\}_{\mathbf{x}\in\mathfrak{M},\varepsilon\in\mathrm{Re}^+\backslash\{0\}}$ and $\{\mathfrak{D}(\varepsilon,\mathbf{x})\}_{\mathbf{x}\in\mathfrak{M},\varepsilon\in\mathrm{Re}^+\backslash\{0\}}$ are topological bases.*

Proof. Refer to (9), and consider, say, $\{\mathfrak{D}(\varepsilon,\mathbf{x})\}_{\mathbf{x}\in\mathfrak{M},\varepsilon\in\mathrm{Re}^+\backslash\{0\}}$ (the proof for the other set of subsets is essentially identical). Property (B1) being obvious, we have to show that if $\mathbf{c}\in\mathfrak{D}(\varepsilon,\mathbf{a})\cap\mathfrak{D}(\delta,\mathbf{b})$ then there is a $\lambda>0$ such that $\mathfrak{D}(\lambda,\mathbf{c})\subset\mathfrak{D}(\varepsilon,\mathbf{a})\cap\mathfrak{D}(\delta,\mathbf{b})$. Assuming the contrary, there should be a sequence $d(\mathbf{c}_n,\mathbf{c})\to 0$ such that all $d(\mathbf{c}_n,\mathbf{a})>\varepsilon$ or all $d(\mathbf{c}_n,\mathbf{b})>\delta$. Let it be the former. Then $d(\mathbf{c},\mathbf{a})<\varepsilon$, $d(\mathbf{c}_n,\mathbf{c})$ can be made arbitrarily small, but $d(\mathbf{c}_n,\mathbf{a})>\varepsilon$. This contradicts the triangle inequality. $\square$

Note that a priori the two topologies need not coincide, and that if they do not, $d(\mathbf{a},\mathbf{b})$ need not be continuous with respect to either of them. It is, however, continuous in the product topology based on open balls $\mathfrak{D}(\mathbf{x},\varepsilon)\cap\mathfrak{D}(\varepsilon,\mathbf{x})$. Clearly, this is a topological base, and

$$\mathfrak{D}(\mathbf{x},\varepsilon)\cap\mathfrak{D}(\varepsilon,\mathbf{x})=\{\mathbf{y}:\max\{d(\mathbf{x},\mathbf{y}),d(\mathbf{y},\mathbf{x})\}<\varepsilon\}.\tag{19}$$

Lemma 4. *Oriented metric $d(\mathbf{a},\mathbf{b})$ is continuous in the product topology based on $\{\mathfrak{D}(\mathbf{x},\varepsilon)\cap\mathfrak{D}(\varepsilon,\mathbf{x})\}_{\mathbf{x}\in\mathfrak{M},\varepsilon\in\mathrm{Re}^+\backslash\{0\}}$.*

Proof. By triangle inequality,

$$\begin{aligned}&-d(\mathbf{x}_n,\mathbf{a})-d(\mathbf{b},\mathbf{y}_n)\leqslant d(\mathbf{a},\mathbf{b})-d(\mathbf{x}_n,\mathbf{y}_n)\\ &\leqslant d(\mathbf{a},\mathbf{x}_n)+d(\mathbf{y}_n,\mathbf{b})\end{aligned}$$

so if $\max\{d(\mathbf{a},\mathbf{x}_n),d(\mathbf{x}_n,\mathbf{a})\}\to 0$ and $\max\{d(\mathbf{b},\mathbf{y}_n),d(\mathbf{y}_n,\mathbf{b})\}\to 0$ then $d(\mathbf{a},\mathbf{b})-d(\mathbf{x}_n,\mathbf{y}_n)\to 0$. $\square$

This property of oriented metrics generalizes the well-known fact about conventional (symmetric) metrics, that they are continuous in the product topology they induce.

It follows that the oriented Fechnerian metrics $G_1(\mathbf{a},\mathbf{b})$ and $G_2(\mathbf{a},\mathbf{b})$ constructed above induce four topologies, based on open metric balls

$$\begin{aligned}\mathfrak{G}^{(1)}(\mathbf{x},\rho)&=\{\mathbf{y}:G_1(\mathbf{x},\mathbf{y})<\rho\},\\ \mathfrak{G}^{(2)}(\mathbf{x},\rho)&=\{\mathbf{y}:G_2(\mathbf{x},\mathbf{y})<\rho\},\\ \mathfrak{G}^{(1)}(\rho,\mathbf{x})&=\{\mathbf{y}:G_1(\mathbf{y},\mathbf{x})<\rho\},\\ \mathfrak{G}^{(2)}(\rho,\mathbf{x})&=\{\mathbf{y}:G_2(\mathbf{y},\mathbf{x})<\rho\}\end{aligned}\tag{20}$$

and that these two metrics are continuous with respect to the product topologies induced by, respectively, the sets of open balls

$$\mathfrak{G}^{(1)}(\mathbf{x},\rho)\cap\mathfrak{G}^{(1)}(\rho,\mathbf{x}),\quad \mathfrak{G}^{(2)}(\mathbf{x},\rho)\cap\mathfrak{G}^{(2)}(\rho,\mathbf{x}).$$

The question arises: what is the relationship among all these topologies, and what is their relationship with the topology of space $\mathfrak{M}$ constructed in Section 5? Recall that this latter topology is based on open balls

$$\mathfrak{B}(\mathbf{x}, \varepsilon) = \{\mathbf{y} : \Psi(\mathbf{x}, \mathbf{y}) < \varepsilon\},$$

where Ψ is defined by (10). We will denote this topology by $\mathbb{T}$.

Theorem 31 (*Metrization Property*). *The bases* $\{\mathfrak{G}^{(1)}(\mathbf{x}, \rho)\}_{\mathbf{x} \in \mathfrak{M}, \rho \in \mathrm{Re}^+ \setminus \{0\}}$ *and* $\{\mathfrak{G}^{(2)}(\mathbf{x}, \rho)\}_{\mathbf{x} \in \mathfrak{M}, \rho \in \mathrm{Re}^+ \setminus \{0\}}$ *induce topology* $\mathbb{T}$.

Proof. What is to be proved is that every $\mathfrak{B}(\mathbf{a}, \varepsilon)$-ball contains a $\mathfrak{G}^{(1)}(\mathbf{a}, \rho)$-ball and a $\mathfrak{G}^{(2)}(\mathbf{a}, \rho)$-ball; and vice versa, every $\mathfrak{G}^{(\iota)}(\mathbf{a}, \rho)$-ball $(\iota = 1, 2)$ contains a $\mathfrak{B}(\mathbf{a}, \varepsilon)$-ball.

Consider $\mathfrak{B}(\mathbf{a}, \varepsilon)$, and without loss of generality assume that $0 < \varepsilon \leqslant \varepsilon_{\mathbf{a}}$ as defined in Axiom 7. By Theorem 29, for any $\mathbf{b} \notin \mathfrak{B}(\mathbf{a}, \varepsilon)$, $G_\iota(\mathbf{a}, \mathbf{b}) \geqslant \frac{\Phi(\varepsilon)}{C_{\mathbf{a}}}$ $(\iota = 1, 2)$. Hence, if $\rho < \frac{\Phi(\varepsilon)}{C_{\mathbf{a}}}$, then $\mathfrak{G}^{(\iota)}(\mathbf{a}, \rho) \subset \mathfrak{B}(\mathbf{a}, \varepsilon)$.

Consider $\mathfrak{G}^{(\iota)}(\mathbf{a}, \rho)$, for $\iota = 1$ or 2, and choose any $0 < \varepsilon \leqslant \min\{\Phi^{-1}(\rho c_{\mathbf{a}}), \varepsilon_{\mathbf{a}}\}$. Then both $\varepsilon \leqslant \varepsilon_{\mathbf{a}}$ and $\rho \geqslant \frac{\Phi(\varepsilon)}{c_{\mathbf{a}}}$, whence, by Theorem 29, $\mathfrak{B}(\mathbf{a}, \varepsilon) \subset \mathfrak{G}^{(\iota)}(\mathbf{a}, \rho)$. $\square$

The name of the theorem (Metrization Property) is due to the fact that a metric is said to *metrize* a topology if it induces this topology. In the case of an oriented metric one should specify which of the two topologies it induces is meant. Both G_1 and G_2 metrize topology $\mathbb{T}$ if one considers the topology based on $\mathfrak{G}^{(1)}(\mathbf{x}, \varepsilon)$ and $\mathfrak{G}^{(2)}(\mathbf{x}, \varepsilon)$ balls. Put equivalently,

$$\mathbf{x}_n \to \mathbf{x} \Longleftrightarrow G_1(\mathbf{x}, \mathbf{x}_n) \to 0 \Longleftrightarrow G_2(\mathbf{x}, \mathbf{x}_n) \to 0. \tag{21}$$

For the topologies based on "reverse metric balls" $\mathfrak{G}^{(1)}(\varepsilon, \mathbf{x})$ and $\mathfrak{G}^{(1)}(\varepsilon, \mathbf{x})$ these equivalences do not hold and should be replaced with a single one-way implication,

$$\mathbf{x}_n \to \mathbf{x} \Rightarrow \begin{cases} G_1(\mathbf{x}_n, \mathbf{x}) \to 0, \\ G_2(\mathbf{x}_n, \mathbf{x}) \to 0. \end{cases} \tag{22}$$

Theorem 32. *Topology* $\mathbb{T}$ *refines the topologies based on* $\{\mathfrak{G}^{(1)}(\rho, \mathbf{x})\}_{\mathbf{x} \in \mathfrak{M}, \rho \in \mathrm{Re}^+ \setminus \{0\}}$ *and on* $\{\mathfrak{G}^{(2)}(\rho, \mathbf{x})\}_{\mathbf{x} \in \mathfrak{M}, \rho \in \mathrm{Re}^+ \setminus \{0\}}$.

Proof. We should prove that every $\mathfrak{G}^{(\iota)}(\rho, \mathbf{a})$-ball $(\iota = 1, 2)$ contains a $\mathfrak{B}(\mathbf{a}, \lambda)$-ball. By Theorem 30, it is sufficient to choose $\lambda \leqslant \varepsilon_{\mathbf{a}}$ so that $\varepsilon(\lambda) < \Phi^{-1}(\rho c_{\mathbf{a}})$. This is always possible since $\varepsilon(\lambda) \to 0+$ as $\lambda \to 0+$. $\square$

Fig. 24 summarizes these results.

We can now state the main theorem of this subsection. Recall that $\mathbb{T} \times \mathbb{T}$ is the product topology induced by $\mathbb{T}$ (i.e., the topology based on Cartesian products of open $\mathfrak{B}$-balls).

Theorem 33. *Fechnerian metrics* $G_1(\mathbf{a}, \mathbf{b})$ *and* $G_2(\mathbf{a}, \mathbf{b})$ *are continuous in the product topology* $\mathbb{T} \times \mathbb{T}$.

Proof. What we have to prove is that

$$(\mathbf{x}_n \to \mathbf{a}) \wedge (\mathbf{y}_n \to \mathbf{b}) \Rightarrow G_\iota(\mathbf{x}_n, \mathbf{y}_n) \to G_\iota(\mathbf{a}, \mathbf{b}), \quad \iota = 1, 2.$$

This follows from

$$\begin{aligned} -G_\iota(\mathbf{x}_n, \mathbf{a}) - G_\iota(\mathbf{b}, \mathbf{y}_n) &\leqslant G_\iota(\mathbf{a}, \mathbf{b}) - G_\iota(\mathbf{x}_n, \mathbf{y}_n) \\ &\leqslant G_\iota(\mathbf{a}, \mathbf{x}_n) + G_\iota(\mathbf{y}_n, \mathbf{b}) \end{aligned}$$

(a consequence of triangle inequality) and

$$\begin{aligned} &\mathbf{x}_n \to \mathbf{a} \Rightarrow [G_\iota(\mathbf{a}, \mathbf{x}_n) \to 0] \wedge [G_\iota(\mathbf{x}_n, \mathbf{a}) \to 0], \\ &\mathbf{y}_n \to \mathbf{b} \Rightarrow [G_\iota(\mathbf{b}, \mathbf{y}_n) \to 0] \wedge [G_\iota(\mathbf{y}_n, \mathbf{b}) \to 0]. \end{aligned}$$ $\square$

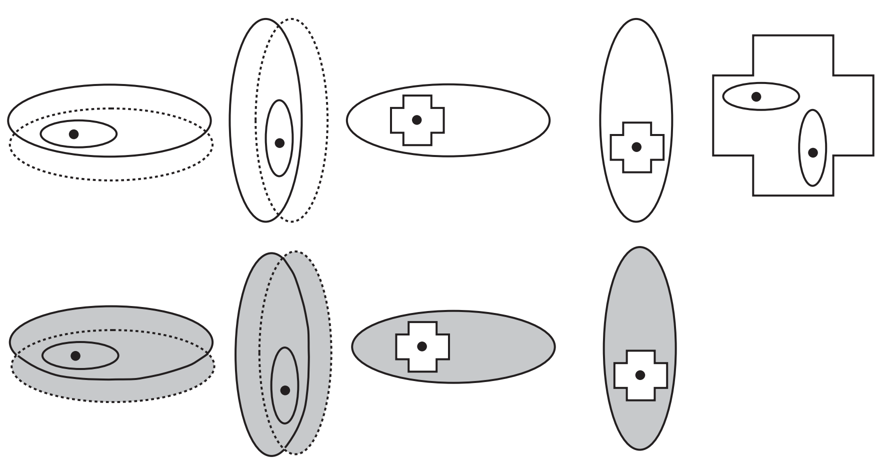

Fig. 24. Schematic relationship among $\mathfrak{G}$ and $\mathfrak{B}$ balls in stimulus space. Wide ellipses represent $\mathfrak{G}^{(1)}$-balls ("reverse" $\mathfrak{G}^{(1)}$-balls if shaded), tall ellipses $\mathfrak{G}^{(2)}$-balls ("reverse" $\mathfrak{G}^{(2)}$-balls if shaded). Crosses represent $\mathfrak{B}$-balls (see Fig. 17). Left four figures: an intersection of $\mathfrak{G}$-balls of any kind ($\mathfrak{G}^{(1)}$, $\mathfrak{G}^{(2)}$, "normal" or "reverse") contains a $\mathfrak{G}$-ball of the same kind around any of its points (implying that $\mathfrak{G}$-balls of any kind form a topological base). Middle four figures: a $\mathfrak{G}$-ball of any kind contains $\mathfrak{B}$-balls around all its points. Rightmost figure: a $\mathfrak{B}$-ball contains $\mathfrak{G}^{(1)}$-balls and $\mathfrak{G}^{(2)}$-balls (but not "reverse" ones) around all its points.

Note that due to Theorem 31, $(\mathbf{x}_n \to \mathbf{a}) \wedge (\mathbf{y}_n \to \mathbf{b})$ in the proof of the last theorem means both

$$[\Psi^{(\iota)}(\mathbf{a}, \mathbf{x}_n) \to 0] \wedge [\Psi^{(\iota)}(\mathbf{b}, \mathbf{y}_n) \to 0]$$

and

$$[G_\iota(\mathbf{a}, \mathbf{x}_n) \to 0] \wedge [G_\iota(\mathbf{b}, \mathbf{y}_n) \to 0].$$

10.2. Differentiability of Fechnerian metrics

Here, we consider the differentiability of functions $G_1(\mathbf{a}, \mathbf{b})$ and $G_2(\mathbf{a}, \mathbf{b})$ with respect to parameters of smooth arcs along which we move the points $\mathbf{a}, \mathbf{b}$. In other words, we ask whether functions $G_\iota(\mathbf{x}(t), \mathbf{y}(\tau))$ $(\iota = 1, 2)$ are differentiable in t and in τ.

Recall that the (right-hand) upper and lower Dini derivatives of $f(x)$ at $x = a+$ are defined as, respectively,

$$\begin{aligned} \mathrm{D}^+ f(x)|_{x=a} &= \lim_{h \to 0+} \sup \frac{f(a+h) - f(a)}{h} \\ &= \inf_{h \to 0+} \sup \left\{ \frac{f(a+g) - f(a)}{g} : 0 < g \leqslant h \right\}, \\ \mathrm{D}_+ f(x)|_{x=a} &= \lim_{h \to 0+} \inf \frac{f(a+h) - f(a)}{h} \\ &= \sup_{h \to 0+} \inf \left\{ \frac{f(a+g) - f(a)}{g} : 0 < g \leqslant h \right\}. \end{aligned}$$

Using $h \to 0-$ in place of $h \to 0+$ defines left-hand (upper and lower) Dini derivatives, $\mathrm{D}^- f(x)|_{x=a}$ and $\mathrm{D}_- f(x)|_{x=a}$. A Dini derivative always exists, as a finite number or $\pm\infty$.

In the following, the quantities $\varepsilon_\mathbf{a}$ and $C_\mathbf{a}(\lambda)$, for $\lambda \leqslant \varepsilon_\mathbf{a}$, have the same meaning as in Theorem 29.

Theorem 34. *For a smooth arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$, any $t \in [a, b)$, and for $\iota = 1, 2$,*

$$\begin{aligned} &\frac{1}{C^*_{\mathbf{x}(t)}} F_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t)) \leqslant \mathrm{D}_+ G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))|_{\alpha=0} \\ &\quad \leqslant \mathrm{D}^+ G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))|_{\alpha=0} \leqslant F_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t)), \end{aligned}$$

where

$$C^*_{\mathbf{x}(t)} = \lim_{\lambda \to 0+} C_{\mathbf{x}(t)}(\lambda).$$

Proof. Let $\Gamma^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha)) = \Phi(\lambda_\alpha)$. Since $\Gamma^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))$ is continuous, we can choose $\beta \in (0, b-t]$ so that $\lambda_\alpha \leqslant \varepsilon_{\mathbf{x}(t)}$ for all $\alpha \in [0, \beta]$. Then, by Theorem 29,

$$G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha)) \geqslant \frac{\Phi(\lambda_\alpha)}{C_{\mathbf{x}(t)}(\lambda_\alpha)}$$

and

$$\begin{aligned} &\mathrm{D}_+ G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))|_{\alpha=0} \\ &\quad = \lim_{\alpha \to 0+} \inf \frac{G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))}{\alpha} \\ &\quad \geqslant \lim_{t \to a+} \inf \frac{\Phi(\lambda_\alpha)}{C_{\mathbf{x}(t)}(\lambda_\alpha)\alpha} \\ &\quad = \lim_{\alpha \to 0+} \frac{1}{C_{\mathbf{x}(t)}(\lambda_\alpha)} \lim_{\alpha \to 0+} \frac{\Gamma^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha))}{\alpha} \\ &\quad = \frac{1}{C^*_{\mathbf{x}(t)}} F_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t)). \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))}{\alpha} &\leqslant \frac{\int_t^{t+\alpha} F_\iota(\mathbf{x}(u), \overset{\circ}{\mathbf{x}}(u))\,\mathrm{d}u}{\alpha} \\ &= F_\iota(\mathbf{x}(t+\xi_\alpha\alpha), \overset{\circ}{\mathbf{x}}(t+\xi_\alpha\alpha)), \end{aligned}$$

where $0 < \xi_\alpha < 1$. It follows that:

$$\begin{aligned} &\mathrm{D}^+ G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))|_{\alpha=0} \\ &\quad = \lim_{\alpha \to 0+} \sup \frac{G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))}{\alpha} \\ &\quad \leqslant \lim_{\alpha \to 0+} F_\iota(\mathbf{x}(t+\xi_\alpha\alpha), \overset{\circ}{\mathbf{x}}(t+\xi_\alpha\alpha)) \\ &\quad = F_\iota(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t)). \qquad \square \end{aligned}$$

By a trivial modification of the argument, we extend this result to the convergence from the left.

Theorem 35. *For a smooth arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$, any $t \in (a, b]$, and for $\iota = 1, 2$,*

$$\begin{aligned} \frac{1}{C^*_{\mathbf{x}(t)}} F_\iota(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t)) &\leqslant \mathrm{D}_+ G_\iota(\mathbf{x}(t), \mathbf{x}(t-\alpha))|_{\alpha=0} \\ &\leqslant \mathrm{D}^+ G_\iota(\mathbf{x}(t), \mathbf{x}(t-\alpha))|_{\alpha=0} \\ &\leqslant F_\iota(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t)), \end{aligned}$$

where $C^*_{\mathbf{x}(t)}$ is the same as in Theorem 34.

In keeping with the previous development, this result is stated in terms of positive α being subtracted rather than negative α being added. Left-hand Dini derivatives proper, $\mathrm{D}_- G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))|_{\alpha=0}$ and $\mathrm{D}^- G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))|_{\alpha=0}$, present no separate problem, as

$$\begin{aligned} \mathrm{D}_- G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))|_{\alpha=0} &= -\mathrm{D}^+ G_\iota(\mathbf{x}(t), \mathbf{x}(t-\alpha))|_{\alpha=0} \\ \mathrm{D}^- G_\iota(\mathbf{x}(t), \mathbf{x}(t+\alpha))|_{\alpha=0} &= -\mathrm{D}_+ G_\iota(\mathbf{x}(t), \mathbf{x}(t-\alpha))|_{\alpha=0}. \end{aligned}$$

In quasi-convex stimulus spaces (Definition 15), $\lim_{\lambda \to 0+} C_\mathbf{a}(\lambda) = 1$ at all points, and we have the following strengthening of the previous results.

Corollary 11 (*to Theorems 34 and 35*). *If stimulus space $\mathfrak{M}$ is quasi-convex, then $G_1(\mathbf{x}(t), \mathbf{x}(t \pm \alpha))$ and $G_2(\mathbf{x}(t), \mathbf{x}(t \pm \alpha))$ are differentiable at $\alpha = 0+$, with the derivatives equal, respectively, $F_1(\mathbf{x}(t), \pm\overset{\circ}{\mathbf{x}}(t))$ and $F_2(\mathbf{x}(t), \pm\overset{\circ}{\mathbf{x}}(t))$.*

This fact allows one to view the property of quasi-convexity as a generalization of the convexity of indicatrices in stimulus spaces that are regions of Re^n endowed with Euclidean topology (Dzhafarov & Colonius, 2001).

Analogous theorems for Dini derivatives of $G_\iota(\mathbf{x}(t\pm\alpha),\mathbf{x}(t))$ do not hold, but we have the following weaker result.

Theorem 36. *For a smooth arc* $\mathbf{x}(t):[a,b]\to\mathfrak{M}$, *any* $t\in(a,b)$, *and for* $\iota=1,2$,

$$\begin{aligned}0&\leqslant \mathrm{D}_+G_\iota(\mathbf{x}(t\pm\alpha),\mathbf{x}(t))|_{\alpha=0}\\&\leqslant \mathrm{D}^+G_\iota(\mathbf{x}(t\pm\alpha),\mathbf{x}(t))|_{\alpha=0}\\&\leqslant F_\iota(\mathbf{x}(t),\mp\overset{\circ}{\mathbf{x}}(t)).\end{aligned}$$

Proof. The left bound being obvious, the right bound is obtained in the same way as in Theorems 34 and 35, by observing that

$$\frac{G_\iota(\mathbf{x}(t+\alpha),\mathbf{x}(t))}{\alpha}\leqslant\frac{\int_t^{t+\alpha}F_\iota(\mathbf{x}(u),-\overset{\circ}{\mathbf{x}}(u))\,\mathrm{d}u}{\alpha},$$
$$\frac{G_\iota(\mathbf{x}(t-\alpha),\mathbf{x}(t))}{\alpha}\leqslant\frac{\int_{t-\alpha}^{t}F_\iota(\mathbf{x}(u),\overset{\circ}{\mathbf{x}}(u))\,\mathrm{d}u}{\alpha}$$

and utilizing the mean value theorem. □

Interestingly, the asymmetry between the first and the second argument of the Fechnerian distance functions all but disappears when we investigate the behavior of $G_\iota(\mathbf{x}(t),\mathbf{y}(\tau))$ $(\iota=1,2)$, with $\mathbf{x}(t)$ and $\mathbf{y}(\tau)$ being generally different points on different smooth arcs.

Lemma 5. *Given smooth arcs* $\mathbf{x}(t):[a,b]\to\mathfrak{M}$ *and* $\mathbf{y}(\tau):[c,d]\to\mathfrak{M}$, *any* $t\in(a,b)$, $\tau\in(c,d)$, *and* $\iota=1,2$,

$$\begin{aligned}-&F_\iota(\mathbf{y}(\tau),\mp\overset{\circ}{\mathbf{y}}(\tau))\\&\leqslant \mathrm{D}_+G_\iota(\mathbf{x}(t),\mathbf{y}(\tau\pm\alpha))|_{\alpha=0}\\&\leqslant \mathrm{D}^+G_\iota(\mathbf{x}(t),\mathbf{y}(\tau\pm\alpha))|_{\alpha=0}\\&\leqslant F_\iota(\mathbf{y}(\tau),\pm\overset{\circ}{\mathbf{y}}(\tau)),\\-&F_\iota(\mathbf{x}(t),\pm\overset{\circ}{\mathbf{x}}(t))\\&\leqslant \mathrm{D}_+G_\iota(\mathbf{x}(t\pm\alpha),\mathbf{y}(\tau))|_{\alpha=0}\\&\leqslant \mathrm{D}^+G_\iota(\mathbf{x}(t\pm\alpha),\mathbf{y}(\tau))|_{\alpha=0}\\&\leqslant F_\iota(\mathbf{x}(t),\mp\overset{\circ}{\mathbf{x}}(t)).\end{aligned}$$

Proof. By triangle inequality,

$$\begin{aligned}-\frac{G_\iota(\mathbf{y}(\tau\pm\alpha),\mathbf{y}(\tau))}{\alpha}&\leqslant\frac{G_\iota(\mathbf{x}(t),\mathbf{y}(\tau\pm\alpha))-G_\iota(\mathbf{x}(t),\mathbf{y}(\tau))}{\alpha}\\&\leqslant\frac{G_\iota(\mathbf{y}(\tau),\mathbf{y}(\tau\pm\alpha))}{\alpha}.\end{aligned}$$

On applying $\lim_{\alpha\to0+}\sup$ and $\lim_{\alpha\to0+}\inf$ to these three ratios, we get

$$\begin{aligned}-\mathrm{D}_+G_\iota(\mathbf{y}(\tau\pm\alpha),\mathbf{y}(\tau))|_{\alpha=0}&\leqslant \mathrm{D}^+G_\iota(\mathbf{x}(t),\mathbf{y}(\tau\pm\alpha))|_{\alpha=0}\\&\leqslant \mathrm{D}^+G_\iota(\mathbf{y}(\tau),\mathbf{y}(\tau\pm\alpha))|_{\alpha=0}\end{aligned}$$

and

$$\begin{aligned}-\mathrm{D}^+G_\iota(\mathbf{y}(\tau\pm\alpha),\mathbf{y}(\tau))|_{\alpha=0}&\leqslant \mathrm{D}_+G_\iota(\mathbf{x}(t),\mathbf{y}(\tau\pm\alpha))|_{\alpha=0}\\&\leqslant \mathrm{D}_+G_\iota(\mathbf{y}(\tau),\mathbf{y}(\tau\pm\alpha))|_{\alpha=0}.\end{aligned}$$

But by Theorems 34 and 35,

$$\begin{aligned}\mathrm{D}_+G_\iota(\mathbf{y}(\tau),\mathbf{y}(\tau\pm\alpha))|_{\alpha=0}&\leqslant \mathrm{D}^+G_\iota(\mathbf{y}(\tau),\mathbf{y}(\tau\pm\alpha))|_{\alpha=0}\\&\leqslant F_\iota(\mathbf{y}(\tau),\pm\overset{\circ}{\mathbf{y}}(\tau))\end{aligned}$$

and

$$\begin{aligned}-\mathrm{D}_+G_\iota(\mathbf{y}(\tau\pm\alpha),\mathbf{y}(\tau))|_{\alpha=0}&\geqslant -\mathrm{D}^+G_\iota(\mathbf{y}(\tau\pm\alpha),\mathbf{y}(\tau))|_{\alpha=0}\\&\geqslant -F_\iota(\mathbf{y}(\tau),\mp\overset{\circ}{\mathbf{y}}(\tau)).\end{aligned}$$

This proves the first statement of the theorem. The second statement is proved analogously, by using

$$\begin{aligned}-\frac{G_\iota(\mathbf{x}(t),\mathbf{x}(t\pm\alpha))}{\alpha}&\leqslant\frac{G_\iota(\mathbf{x}(t\pm\alpha),\mathbf{y}(\tau))-G_\iota(\mathbf{x}(t),\mathbf{y}(\tau))}{\alpha}\\&\leqslant\frac{G_\iota(\mathbf{x}(t\pm\alpha),\mathbf{x}(t))}{\alpha}\end{aligned}$$

as a departure point. □

Theorem 37. *Given smooth arcs* $\mathbf{x}(t):[a,b]\to\mathfrak{M}$ *and* $\mathbf{y}(\tau):[c,d]\to\mathfrak{M}$, *Fechnerian distance* $G_\iota(\mathbf{x}(t),\mathbf{y}(\tau))$ $(\iota=1,2)$ *is differentiable in t and in* τ *almost everywhere on, respectively,* $[a,b]$ *and* $[c,d]$. *Moreover,*

$$\left|\frac{\partial G_\iota(\mathbf{x}(t),\mathbf{y}(\tau))}{\partial\tau}\right|\leqslant\min\{F_\iota(\mathbf{y}(\tau),\overset{\circ}{\mathbf{y}}(\tau)),F_\iota(\mathbf{y}(\tau),-\overset{\circ}{\mathbf{y}}(\tau))\},$$
$$\left|\frac{\partial G_\iota(\mathbf{x}(t),\mathbf{y}(\tau))}{\partial t}\right|\leqslant\min\{F_\iota(\mathbf{x}(t),\overset{\circ}{\mathbf{x}}(t)),F_\iota(\mathbf{x}(t),-\overset{\circ}{\mathbf{x}}(t))\},$$

wherever these derivatives exist.

Proof. Follows from Lemma 5, on invoking the fact (see, e.g., Saks, 1937) that if two Dini derivatives of a function on the same side (i.e., right or left) are finite on an interval, then the function is differentiable on this interval almost everywhere (with respect to the Lebesgue measure). □

10.3. Psychometric length as Burkill integral

In this subsection we establish a simple but important fact: psychometric length of a smooth arc $\mathbf{x}(t)$ can be approximated to any degree of precision by a sum of gamma increments $\Gamma^{(\iota)}(\mathbf{x}(t_{j-1}),\mathbf{x}(t_j))$ computed for a chain of points $\mathbf{x}(t_0),\ldots,\mathbf{x}(t_k)$ taken on the arc. This fact bridges continuous stimulus spaces with discrete stimulus spaces (not considered in this paper; see Dzhafarov & Colonius, submitteda,b).

Consider a smooth arc $\mathbf{x}(t):[a,b]\to\mathfrak{M}$. For any finite segmentation of $[a,b]$,

$$\mathfrak{I}=\{a=t_0,t_1,\ldots,t_{k-1},t_k=b\},$$

define

$$\Lambda^{(\iota)}(\mathfrak{I}) = \sum_{j=1}^{k} \Gamma^{(\iota)}(\mathbf{x}(t_{j-1}), \mathbf{x}(t_j)) \tag{23}$$

and

$$\Delta(\mathfrak{I}) = \max_j \, [t_j - t_{j-1}]. \tag{24}$$

The *Burkill integral* (see Saks (1937) for the general theory) of function $\Gamma^{(\iota)}$ over $\mathbf{x}(t)$ is defined as

$$B^{(\iota)}[\mathbf{x}_{[a,b]}] = \lim_{\Delta(\mathfrak{I}) \to 0} \Lambda^{(\iota)}(\mathfrak{I}), \tag{25}$$

where the limit is taken over all possible segmentations of $[a,b]$. This limit may or may not exist, and if it does, it may be zero, infinity, or a positive number. In this subsection we prove that $B^{(\iota)}[\mathbf{x}_{[a,b]}]$ equals the psychometric length $L^{(\iota)}[\mathbf{x}_{[a,b]}]$.

Lemma 6. *For $\iota = 1$ or 2, and any fixed segmentation $\{a = s_0, s_1, \ldots, s_{n-1}, s_n = b\}$, if*

$$B^{(\iota)}[\mathbf{x}_{[s_{j-1}, s_j]}] = b_j \in (0, \infty)$$

for $j = 1, \ldots, n$, then Burkill integral $B^{(\iota)}[\mathbf{x}_{[a,b]}]$ exists and equals $\sum_{j=1}^{n} b_j$.

Proof. Follows from the observation that any segmentation $\mathfrak{I} = \{a = t_0, t_1, \ldots, t_{n-1}, t_k = b\}$ can be refined by forming its ordered union with $\{a = s_0, s_1, \ldots, s_{n-1}, s_n = b\}$. □

Lemma 7. *Let $\mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$ be as in Axiom 7, and let $\mathbf{x}(t)$ be any smooth arc $[a,b] \to \mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$. Then*

$$\frac{|\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(b))|}{b-a} \leqslant \text{const} < \infty, \quad \iota = 1, 2.$$

Proof.

$$\begin{aligned}\frac{\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(b))}{b-a} &= \frac{1}{b-a} \int_a^b \frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))}{\mathrm{d}t} \, \mathrm{d}t \\ &= \left. \frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))}{\mathrm{d}t} \right|_{t=\tau},\end{aligned}$$

where $a \leqslant \tau \leqslant b$. By Axiom 7,

$$\left| \left. \frac{\mathrm{d}\Gamma^{(\iota)}(\mathbf{x}(a), \mathbf{x}(t))}{\mathrm{d}t} \right|_{t=\tau} \right| < C_{\mathbf{p}} F_\iota(\mathbf{x}(\tau), \mathring{\mathbf{x}}(\tau)) \leqslant M C_{\mathbf{p}},$$

where $M = \max_\tau F_\iota(\mathbf{x}(\tau), \mathring{\mathbf{x}}(\tau))$, which must exist as F_ι is a continuous function on a closed interval. □

Lemma 8. *For any smooth arc $\mathbf{x}(t) : [a,b] \to \mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$ and any $t \in (a,b)$,*

$$\lim_{\substack{\alpha \to 0+ \\ \beta \to 0+}} \frac{\Gamma^{(\iota)}(\mathbf{x}(t-\alpha), \mathbf{x}(t+\beta))}{\alpha + \beta} = F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t)), \quad \iota = 1, 2.$$

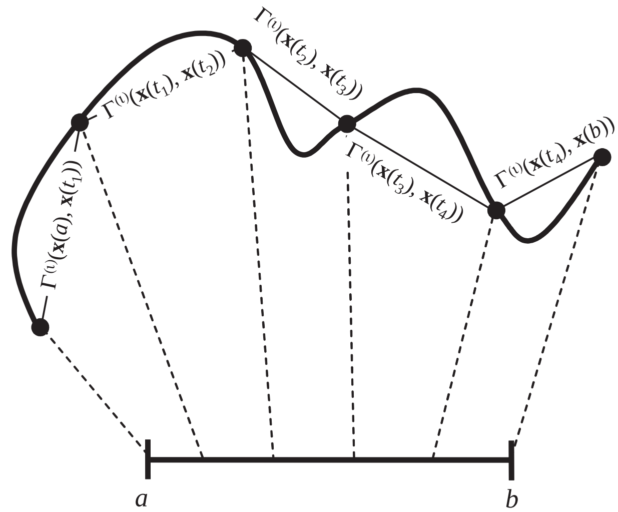

Fig. 25. Psychometric length of a piecewise smooth arc as Burkill integral. As segmentation of $[a,b]$ gets progressively finer, the sum of the gamma-increments gets progressively closer to the psychometric length of the arc.

Proof.

$$\begin{aligned}\frac{\Gamma^{(\iota)}(\mathbf{x}(t-\alpha), \mathbf{x}(t+\beta))}{\alpha+\beta} &\sim F_\iota(\mathbf{x}(t-\alpha), \mathring{\mathbf{x}}(t-\alpha)) \\ &\sim F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t)),\end{aligned}$$

where we have used the continuity of F_ι. □

Our main theorem is illustrated in Fig. 25.

Theorem 38. *For any smooth arc $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$,*

$$B^{(\iota)}[\mathbf{x}_{[a,b]}] = L^{(\iota)}[\mathbf{x}_{[a,b]}], \quad \iota = 1, 2.$$

Proof. We have to show that the Burkill integral $B^{(\iota)}[\mathbf{x}_{[a,b]}]$ as defined by (25) exists and equals the Riemann integral

$$\int_a^b F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) \, \mathrm{d}t.$$

Using Lemma 6, note first that it suffices to prove this for an arc whose codomain is within $\mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$, where $\mathbf{p}$ is any point. Indeed, given $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$, for any $t \in [a,b]$ one can find a δ_t such that

$$u \in (t - \delta_t, t + \delta_t) \cap [a,b] \Rightarrow \mathbf{x}(u) \in \mathfrak{B}(\mathbf{x}(t), \varepsilon_{\mathbf{x}(t)}).$$

Interval $[a,b]$ being compact, one can choose $t_1 < \cdots < t_l$ such that

$$[a,b] \subset \bigcup_{j=1}^{l} (t_j - \delta_{t_j}, t_j + \delta_{t_j}).$$

Assuming, with no loss of generality, that none of the intervals $(t_j - \delta_{t_j}, t_j + \delta_{t_j})$ is a subset of another, $\mathbf{x}(t)$ can then be partitioned into subarcs

$$\mathbf{x}_{[a, t_1+\delta_{t_1}]}(t), \mathbf{x}_{[t_1+\delta_{t_1}, t_2+\delta_{t_2}]}(t), \ldots, \mathbf{x}_{[t_{l-1}+\delta_{t_{l-1}}, b]}(t)$$

which map their domains into, respectively,

$\mathfrak{B}(\mathbf{x}(t_1), \varepsilon_{\mathbf{x}(t_1)}), \mathfrak{B}(\mathbf{x}(t_2), \varepsilon_{\mathbf{x}(t_2)}), \ldots, \mathfrak{B}(\mathbf{x}(t_{l-1}), \varepsilon_{\mathbf{x}(t_{l-1})})$.

So, let $\mathbf{x}(t) : [a,b] \to \mathfrak{B}(\mathbf{p}, \varepsilon_{\mathbf{p}})$. Consider a sequence of segmentations

$$\mathfrak{I}_\eta = \{a = t_{0\eta}, t_{1\eta}, \ldots, t_{k_\eta - 1,\eta}, t_{k_\eta \eta} = b\}, \quad \eta = 1, 2, \ldots,$$

with $\Delta(\mathfrak{I}_\eta) \to 0$ as $\eta \to \infty$. For each such a segmentation $\mathfrak{I}_\eta$ we define on interval $[a,b]$ the function

$$f_\eta^{(\iota)}(t) = \frac{\Gamma^{(\iota)}(\mathbf{x}(t_{j-1,\eta}), \mathbf{x}(t_{j\eta}))}{t_{j\eta} - t_{j-1,\eta}} \quad \text{for } t_{j-1,\eta} \leqslant t < t_{j\eta},$$

$j = 1, \ldots, k_\eta$.

Clearly,

$$\int_a^b f_\eta^{(\iota)}(t)\,\mathrm{d}t = \sum_{j=1}^{k_\eta} \Gamma^{(\iota)}(\mathbf{x}(t_{j-1,\eta}), \mathbf{x}(t_{j\eta})) = \Lambda^{(\iota)}(\mathfrak{I}_\eta).$$

By Lemma 8,

$$f_\eta^{(\iota)}(t) \to F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t)),$$

as $\eta \to \infty$, and because by Lemma 7 $f_\eta^{(\iota)}(t)$ is bounded on a closed interval, we invoke Lebesgue's dominated convergence theorem to conclude

$$\Lambda^{(\iota)}(\mathfrak{I}_\eta) \to \int_a^b F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t = L^{(\iota)}[\mathbf{x}_{[a,b]}]$$

as $\eta \to \infty$. $\square$

10.4. Internal consistency of Fechnerian metrics

We now replicate the construction of the previous subsection but with G_ι-distances replacing gamma-increments. Consider a smooth arc $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$. For any finite segmentation of $[a,b]$,

$$\mathfrak{I} = \{a = t_0, t_1, \ldots, t_{k-1}, t_k = b\},$$

define

$$\lambda^{(\iota)}(\mathfrak{I}) = \sum_{j=1}^{k} G_\iota(\mathbf{x}(t_{j-1}), \mathbf{x}(t_j))$$

and

$$g^{(\iota)}[\mathbf{x}_{[a,b]}] = \lim_{\Delta(\mathfrak{I}) \to 0} \lambda^{(\iota)}(\mathfrak{I}),$$

where the limit is taken over all possible segmentations of $[a,b]$. This is the Burkill integral of function G_ι over $\mathbf{x}(t)$.

Theorem 39. $g^{(\iota)}[\mathbf{x}_{[a,b]}]$ $(\iota = 1, 2)$ *exists as a finite nonnegative quantity for any smooth arc* $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$. *It equals zero if and only if* $a = b$.

Proof. Due to triangle inequality, if segmentation $\mathfrak{I}'$ refines segmentation $\mathfrak{I}$,

$$\lambda^{(\iota)}(\mathfrak{I}') \geqslant \lambda^{(\iota)}(\mathfrak{I}).$$

Since any two segmentations have a common refinement (namely, any refinement of the ordered union of the two segmentations), $g^{(\iota)}[\mathbf{x}_{[a,b]}]$ is either a finite nonnegative quantity or ∞. But

$$\lambda^{(\iota)}(\mathfrak{I}) \leqslant L^{(\iota)}[\mathbf{x}_{[a,b]}] = \int_a^b F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t$$

for any segmentation $\mathfrak{I} = \{a = t_0, t_1, \ldots, t_{k-1}, t_k = b\}$, because

$$G_\iota(t_{j-1}, t_j) = \inf \int_{t_{j-1}}^{t_j} F_\iota(\mathbf{y}(t), \mathring{\mathbf{y}}(t))\,\mathrm{d}t,$$

where the infimum is taken over all piecewise smooth arcs connecting $\mathbf{x}(t_{j-1})$ to $\mathbf{x}(t_j)$. Hence

$$g^{(\iota)}[\mathbf{x}_{[a,b]}] \leqslant L^{(\iota)}[\mathbf{x}_{[a,b]}].$$

This proves the first statement of the theorem. By triangle inequality,

$$G_\iota(\mathbf{x}(a), \mathbf{x}(b)) \leqslant \lambda^{(\iota)}(\mathfrak{I}),$$

whence we get

$$G_\iota(\mathbf{x}(a), \mathbf{x}(b)) \leqslant g^{(\iota)}[\mathbf{x}_{[a,b]}] \leqslant L^{(\iota)}[\mathbf{x}_{[a,b]}]$$

and the second statement of the theorem. $\square$

This result leads to an important question. Denoting by $\mathfrak{x}$ the set of all piecewise smooth arcs connecting $\mathbf{a}$ to $\mathbf{b}$, what is the relationship between

$$\inf_{\mathbf{x}(t) \in \mathfrak{x}} g^{(\iota)}[\mathbf{x}_{[a,b]}]$$

and $G_\iota(\mathbf{a}, \mathbf{b})$? If these two quantities could be different, we would have what might be considered an internal

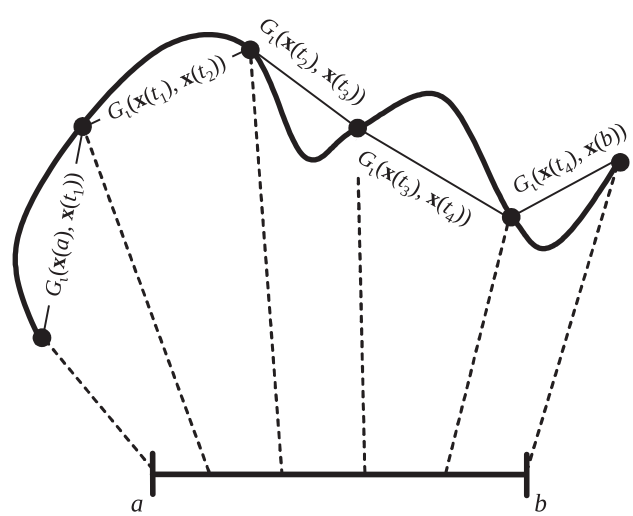

Fig. 26. Internal consistency of Fechnerian metric: as segmentation of $[a,b]$ gets progressively finer, the sum of the G_ι-distances $(\iota = 1, 2)$ converges to a quantity below the psychometric length of the piecewise smooth arc (of the ιth kind). The infimum of these quantities across all piecewise smooth arcs connecting $\mathbf{a}$ to $\mathbf{b}$, however, equals the Fechnerian distance $G_\iota(\mathbf{a}, \mathbf{b})$.

inconsistency in the construction of Fechnerian distances. It is easy to show, however, that this is not the case (see Fig. 26).

Theorem 40. *Let $\mathfrak{x}$ be the set of all piecewise smooth arcs connecting* **a** *to* **b**. *Then*

$$\inf_{\mathbf{x}(t)\in\mathfrak{x}} g^{(\iota)}[\mathbf{x}_{[a,b]}] = G_\iota(\mathbf{a},\mathbf{b}), \quad \iota = 1,2.$$

Proof. From

$$G_\iota(\mathbf{a},\mathbf{b}) \leqslant g^{(\iota)}[\mathbf{x}_{[a,b]}] \leqslant L^{(\iota)}[\mathbf{x}_{[a,b]}]$$

we conclude

$$G_\iota(\mathbf{a},\mathbf{b}) \leqslant \inf_{\mathbf{x}(t)\in\mathfrak{x}} g^{(\iota)}[\mathbf{x}_{[a,b]}] \leqslant \inf_{\mathbf{x}(t)\in\mathfrak{x}} L^{(\iota)}[\mathbf{x}_{[a,b]}] = G_\iota(\mathbf{a},\mathbf{b}),$$

whence the result. □

This is a generalization of what Dzhafarov and Colonius (2001) called the Busemann–Mayer identity (based on Busemann & Mayer, 1941).

11. Overall Fechnerian metric

We have now constructed a theory for two kinds of Fechnerian metrics, $G_1(\mathbf{a},\mathbf{b})$ and $G_2(\mathbf{a},\mathbf{b})$, computed for stimuli in the first observation area and in the second observation area, respectively. The investigation below will reveal their mutual relationships. This investigation, moreover, will lead us to a definition of the "overall" Fechnerian metric, the metric $G(\mathbf{a},\mathbf{b})$ that will be

1. symmetrical, $G(\mathbf{a},\mathbf{b}) = G(\mathbf{b},\mathbf{a})$, and
2. independent of the observation area.

11.1. Decomposition of the minimum level function

Recall that due to the Axiom of Regular Minimality and the canonical representation of discrimination probability functions,

$$\arg\min_{\mathbf{y}\in\mathfrak{M}} \psi(\mathbf{x},\mathbf{y}) = \arg\min_{\mathbf{y}\in\mathfrak{M}} \psi(\mathbf{y},\mathbf{x}) = \psi(\mathbf{x},\mathbf{x}), \tag{26}$$

which is just another way of putting (5). Recall that $\psi(\mathbf{x},\mathbf{x})$ is the minimum level function (in the canonical form).

Recall also that we assume *Nonconstant Self-Dissimilarity*: $\psi(\mathbf{x},\mathbf{x})$ need not be the same for all $\mathbf{x}\in\mathfrak{M}$ (see Section 4.3, especially Convention 1).

Consider changes in the minimum level value $\psi(\mathbf{x},\mathbf{x})$ as $\mathbf{x}$ moves along a smooth arc $\mathbf{x}(t):[a,b]\to\mathfrak{M}$,

$$\begin{cases} \Omega_\alpha\mathbf{x}(t) = \psi(\mathbf{x}(t+\alpha),\mathbf{x}(t+\alpha)) - \psi(\mathbf{x}(t),\mathbf{x}(t)), \\ \Omega_{-\alpha}\mathbf{x}(t) = \psi(\mathbf{x}(t-\alpha),\mathbf{x}(t-\alpha)) - \psi(\mathbf{x}(t),\mathbf{x}(t)), \end{cases} \quad \alpha>0. \tag{27}$$

Theorem 41. [12] *As $\alpha\to 0+$,*

$$\Omega_\alpha\mathbf{x}(t) = \begin{cases} [F_1^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t)) - F_2^\mu(\mathbf{x}(t),-\mathring{\mathbf{x}}(t))]R^\mu(\alpha) + o\{R^\mu(\alpha)\} \\ [F_2^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t)) - F_1^\mu(\mathbf{x}(t),-\mathring{\mathbf{x}}(t))]R^\mu(\alpha) + o\{R^\mu(\alpha)\} \end{cases}$$

and

$$\Omega_{-\alpha}\mathbf{x}(t) = \begin{cases} [F_1^\mu(\mathbf{x}(t),-\mathring{\mathbf{x}}(t)) - F_2^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t))]R^\mu(\alpha) + o\{R^\mu(\alpha)\} \\ [F_2^\mu(\mathbf{x}(t),-\mathring{\mathbf{x}}(t)) - F_1^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t))]R^\mu(\alpha) + o\{R^\mu(\alpha)\} \end{cases}.$$

Proof. As $\alpha\to 0+$, $\Omega_\alpha\mathbf{x}(t)$ can be decomposed in two ways:

$$\begin{aligned} \Omega_\alpha\mathbf{x}(t) &= [\psi(\mathbf{x}(t),\mathbf{x}(t+\alpha)) - \psi(\mathbf{x}(t),\mathbf{x}(t))] \\ &\quad + [\psi(\mathbf{x}(t+\alpha),\mathbf{x}(t+\alpha)) - \psi(\mathbf{x}(t),\mathbf{x}(t+\alpha))] \\ &= \Psi^{(1)}(\mathbf{x}(t),\mathbf{x}(t+\alpha)) - \Psi^{(2)}(\mathbf{x}(t+\alpha),\mathbf{x}(t+\alpha-\alpha)) \\ &= [F_1^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t))R^\mu(\alpha) + o\{R^\mu(\alpha)\}] \\ &\quad - [F_2^\mu(\mathbf{x}(t+\alpha),-\mathring{\mathbf{x}}(t+\alpha))R^\mu(\alpha) + o\{R^\mu(\alpha)\}] \\ &= F_1^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t))R^\mu(\alpha) \\ &\quad - F_2^\mu(\mathbf{x}(t),-\mathring{\mathbf{x}}(t))R^\mu(\alpha) + o\{R^\mu(\alpha)\} \end{aligned}$$

and

$$\begin{aligned} \Omega_\alpha\mathbf{x}(t) &= [\psi(\mathbf{x}(t+\alpha),\mathbf{x}(t)) - \psi(\mathbf{x}(t),\mathbf{x}(t))] \\ &\quad + [\psi(\mathbf{x}(t+\alpha),\mathbf{x}(t+\alpha)) - \psi(\mathbf{x}(t+\alpha),\mathbf{x}(t))] \\ &= \Psi^{(2)}(\mathbf{x}(t),\mathbf{x}(t+\alpha)) - \Psi^{(1)}(\mathbf{x}(t+\alpha),\mathbf{x}(t+\alpha-\alpha)) \\ &= [F_2^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t))R^\mu(\alpha) + o\{R^\mu(\alpha)\}] \\ &\quad - [F_1^\mu(\mathbf{x}(t+\alpha),-\mathring{\mathbf{x}}(t+\alpha))R^\mu(\alpha) + o\{R^\mu(\alpha)\}] \\ &= F_2^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t+))R^\mu(\alpha) \\ &\quad - F_1^\mu(\mathbf{x}(t),-\mathring{\mathbf{x}}(t))R^\mu(\alpha) + o\{R^\mu(\alpha)\}. \end{aligned}$$

We made use in these derivations of the asymptotic decompositions (15)–(16), and the continuity of submetric functions. The decomposition of $\Omega_{-\alpha}\mathbf{x}(t)$ is established analogously. □

Theorem 42. *For any arc element* $(\mathbf{x}(t),\mathring{\mathbf{x}}(t))$,

$$\begin{aligned} &F_1^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t)) - F_2^\mu(\mathbf{x}(t),-\mathring{\mathbf{x}}(t)) \\ &\quad = F_2^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t)) - F_1^\mu(\mathbf{x}(t),-\mathring{\mathbf{x}}(t)) \end{aligned}$$

or equivalently,

$$\begin{aligned} &F_1^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t)) + F_1^\mu(\mathbf{x}(t),-\mathring{\mathbf{x}}(t)) \\ &\quad = F_2^\mu(\mathbf{x}(t),\mathring{\mathbf{x}}(t)) + F_2^\mu(\mathbf{x}(t),-\mathring{\mathbf{x}}(t)). \end{aligned}$$

[12] We are grateful to Jun Zhang for pointing out a mistake in the original formulation of this theorem.

Proof. Immediately follows from equating the two asymptotic decompositions of $\Omega_\alpha \mathbf{x}(t)$ in the previous theorem. □

Definition 18. Stimulus space $\mathfrak{M}$ is called cross-unbalanced if for some arc elements $(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))$,

$$F_1(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t)) \neq F_2(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t)).$$

Otherwise, $\mathfrak{M}$ is called cross-balanced.

We first consider the cross-unbalanced case.

11.2. Cross-unbalanced case: local properties

Theorem 43. *If stimulus space $\mathfrak{M}$ is cross-unbalanced, then for every arc element at which $F_1(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t)) \neq F_2(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t))$, as $\alpha \to 0+$,*

$$\Omega_\alpha \mathbf{x}(t) \sim \begin{cases} [F_1^\mu(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t)) - F_2^\mu(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t))]R^\mu(\alpha) \\ [F_2^\mu(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t)) - F_1^\mu(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t))]R^\mu(\alpha) \end{cases}$$

and

$$\Omega_{-\alpha} \mathbf{x}(t) \sim \begin{cases} [F_1^\mu(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t)) - F_2^\mu(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))]R^\mu(\alpha) \\ [F_2^\mu(\mathbf{x}(t), -\overset{\circ}{\mathbf{x}}(t)) - F_1^\mu(\mathbf{x}(t), \overset{\circ}{\mathbf{x}}(t))]R^\mu(\alpha) \end{cases} \sim -\Omega_\alpha \mathbf{x}(t).$$

Proof. This is an immediate consequence of Theorem 41. □

Theorem 44. *If stimulus space $\mathfrak{M}$ is cross-unbalanced, its psychometric order $\mu = 1$ and its characteristic function $R(\alpha) = \alpha$.*

Proof. Let $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ be a smooth arc with $F_1(\mathbf{x}(a), \overset{\circ}{\mathbf{x}}(a)) \neq F_2(\mathbf{x}(a), -\overset{\circ}{\mathbf{x}}(a))$. Without loss of generality, let

$$F_1(\mathbf{x}(a), \overset{\circ}{\mathbf{x}}(a)) > F_2(\mathbf{x}(a), -\overset{\circ}{\mathbf{x}}(a)).$$

By continuity of the submetric functions, then

$$F_1(\mathbf{x}(a+t), \overset{\circ}{\mathbf{x}}(a+t)) > F_2(\mathbf{x}(a+t), -\overset{\circ}{\mathbf{x}}(a+t))$$

on a sufficiently small interval $t \in [0, \tau]$. By Theorem 43,

$$\Omega_\alpha \mathbf{x}(a+t) \sim [F_1^\mu(\mathbf{x}(a+t), \overset{\circ}{\mathbf{x}}(a+t)) - F_2^\mu(\mathbf{x}(a+t), -\overset{\circ}{\mathbf{x}}(a+t))]R^\mu(\alpha).$$

By definition,

$$\Omega_\alpha \mathbf{x}(a+t) = \psi(\mathbf{x}(a+t+\alpha), \mathbf{x}(a+t+\alpha)) - \psi(\mathbf{x}(a+t), \mathbf{x}(a+t)),$$

$$\Omega_{t+\alpha} \mathbf{x}(a) = \psi(\mathbf{x}(a+t+\alpha), \mathbf{x}(a+t+\alpha)) - \psi(\mathbf{x}(a), \mathbf{x}(a)),$$
$$\Omega_t \mathbf{x}(a) = \psi(\mathbf{x}(a+t), \mathbf{x}(a+t)) - \psi(\mathbf{x}(a), \mathbf{x}(a)),$$

whence

$$\Omega_\alpha \mathbf{x}(a+t) = \Omega_{t+\alpha} \mathbf{x}(a) - \Omega_t \mathbf{x}(a).$$

So

$$\Omega_{t+\alpha} \mathbf{x}(a) - \Omega_t \mathbf{x}(a) \sim [F_1^\mu(\mathbf{x}(a+t), \overset{\circ}{\mathbf{x}}(a+t)) - F_2^\mu(\mathbf{x}(a+t), -\overset{\circ}{\mathbf{x}}(a+t))]R^\mu(\alpha) > 0.$$

It follows that for every $t \in [0, \tau]$ and all sufficiently small $\alpha > 0$, $\Omega_{t+\alpha} \mathbf{x}(a) - \Omega_t \mathbf{x}(a) > 0$, which means that $\Omega_t \mathbf{x}(a)$ is strictly increasing in t on $t \in [0, \tau]$.

Now,

$$\frac{\Omega_{t+\alpha} \mathbf{x}(a) - \Omega_t \mathbf{x}(a)}{\alpha} \sim [F_1^\mu(\mathbf{x}(a+t), \overset{\circ}{\mathbf{x}}(a+t)) - F_2^\mu(\mathbf{x}(a+t), -\overset{\circ}{\mathbf{x}}(a+t))]\frac{R^\mu(\alpha)}{\alpha}.$$

By Lebesgue's theorem, an increasing function has a finite derivative almost everywhere, because of which

$$\lim_{\alpha \to 0+} \frac{\Omega_{t+\alpha} \mathbf{x}(a) - \Omega_t \mathbf{x}(a)}{\alpha} = \frac{\mathrm{d}\Omega_t \mathbf{x}(a)}{\mathrm{d}t+}$$

should be a finite number almost everywhere. This implies that

$$\lim_{\alpha \to 0+} \frac{R^\mu(\alpha)}{\alpha} = \lim_{\alpha \to 0+} \alpha^{\mu-1} \ell(\alpha)$$

is a finite number. Obviously, this limit cannot be zero, because then we would have $\mathrm{d}\Omega_t \mathbf{x}(a)/\mathrm{d}t+ = 0$ everywhere, contrary to $\Omega_t \mathbf{x}(a)$ being increasing. But we know that

$$\lim_{\alpha \to 0+} \alpha^{\mu-1} \ell(\alpha) = \begin{cases} \infty & \text{if } \mu < 1, \\ 0 & \text{if } \mu > 1, \end{cases}$$

whence we have to conclude $\mu = 1$. We have now

$$\frac{\mathrm{d}\Omega_t \mathbf{x}(a)}{\mathrm{d}t+} = [F_1(\mathbf{x}(a+t), \overset{\circ}{\mathbf{x}}(a+t)) - F_2(\mathbf{x}(a+t), -\overset{\circ}{\mathbf{x}}(a+t))] \lim_{\alpha \to 0+} \ell(\alpha),$$

where $\lim_{\alpha \to 0+} \ell(\alpha)$ must be a positive number, say, k. Hence $R(\alpha) \sim k\alpha$, $k > 0$. By Corollary 5 to Theorems 15 and 16, characteristic function $R(\alpha)$ and submetric functions F_1, F_2 can be multiplied by reciprocal positive constants. We can therefore put $k = 1$ and write $R(\alpha) \sim \alpha$. Since the uniqueness of $R(\alpha)$ is only asymptotic, we can simply put $R(\alpha) = \alpha$. □

On recalling, from the First Main Theorem of Fechnerian Scaling (see Corollary 9 to Theorem 19) that the overall psychometric transformation is

$$\Phi(h) = R^{-1}(h^{1/\mu}),$$

which in our case means

$$\Phi(h) = h,$$

we obtain the following result.

Corollary 12 (*to Theorem 44*). *If stimulus space* $\mathfrak{M}$ *is cross-unbalanced, gamma-increments coincide with psychometric increments,*

$$\Gamma^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha)) = \Phi[\Psi^{(\iota)}\mathbf{x}(t), \mathbf{x}(t+\alpha))] = \Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha)), \quad \iota = 1,2,$$

and the asymptotic decomposition in the First Main Theorem of Fechnerian Scaling assumes the form

$$\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t+\alpha)) \sim F_\iota(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\alpha, \quad \iota = 1,2.$$

Corollary 13 (*to Theorem 44*). *If stimulus space* $\mathfrak{M}$ *is cross-unbalanced,*

$$F_1(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) + F_1(\mathbf{x}(t), -\mathring{\mathbf{x}}(t)) = F_2(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) + F_2(\mathbf{x}(t), -\mathring{\mathbf{x}}(t)).$$

Proof. Immediately follows from Theorem 42 on putting $\mu = 1$. □

Theorem 45. *If stimulus space* $\mathfrak{M}$ *is cross-unbalanced, then for any smooth arc* $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$, *the minimum level function* $\psi(\mathbf{x}(t), \mathbf{x}(t))$ *is continuously differentiable at every point* t, *with*

$$\frac{\mathrm{d}\psi(\mathbf{x}(t), \mathbf{x}(t))}{\mathrm{d}t} = F_1(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) - F_2(\mathbf{x}(t), -\mathring{\mathbf{x}}(t)) = F_2(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) - F_1(\mathbf{x}(t), -\mathring{\mathbf{x}}(t)).$$

This derivative is nonzero at some points of some smooth arcs.

Proof. Consider first an arc element $(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$ at which $F_1(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) \neq F_2(\mathbf{x}(t), -\mathring{\mathbf{x}}(t))$. From Theorems 43 and 44, and the previous corollary, as $\alpha \to 0+$,

$$\Omega_\alpha \mathbf{x}(t) \sim [F_1(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) - F_2(\mathbf{x}(t), -\mathring{\mathbf{x}}(t))]\alpha = [F_2(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) - F_1(\mathbf{x}(t), -\mathring{\mathbf{x}}(t))]\alpha,$$

whence

$$\lim_{\alpha\to 0+} \frac{\Omega_\alpha \mathbf{x}(t)}{\alpha} = F_1(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) - F_2(\mathbf{x}(t), -\mathring{\mathbf{x}}(t)) = F_2(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) - F_1(\mathbf{x}(t), -\mathring{\mathbf{x}}(t)).$$

But

$$\lim_{\alpha\to 0+} \frac{\Omega_\alpha \mathbf{x}(t)}{\alpha} = \lim_{\alpha\to 0+} \frac{\psi(\mathbf{x}(t+\alpha), \mathbf{x}(t+\alpha)) - \psi(\mathbf{x}(t), \mathbf{x}(t))}{\alpha} = \frac{\mathrm{d}\psi(\mathbf{x}(t), \mathbf{x}(t))}{\mathrm{d}t+}.$$

From Theorem 43,

$$\lim_{\alpha\to 0+} \frac{\Omega_{-\alpha} \mathbf{x}(t)}{\alpha} = -\lim_{\alpha\to 0+} \frac{\Omega_\alpha \mathbf{x}(t)}{\alpha}.$$

At the same time

$$\lim_{\alpha\to 0+} \frac{\Omega_{-\alpha} \mathbf{x}(t)}{\alpha} = \lim_{\alpha\to 0+} \frac{\psi(\mathbf{x}(t-\alpha), \mathbf{x}(t-\alpha)) - \psi(\mathbf{x}(t), \mathbf{x}(t))}{\alpha} = -\frac{\mathrm{d}\psi(\mathbf{x}(t), \mathbf{x}(t))}{\mathrm{d}t-}.$$

It follows that both unilateral derivatives exist as finite nonzero numbers, and

$$\frac{\mathrm{d}\psi(\mathbf{x}(t), \mathbf{x}(t))}{\mathrm{d}t+} = \frac{\mathrm{d}\psi(\mathbf{x}(t), \mathbf{x}(t))}{\mathrm{d}t-} = \frac{\mathrm{d}\psi(\mathbf{x}(t), \mathbf{x}(t))}{\mathrm{d}t}.$$

The last statement of the theorem holds because such arc elements should exist in a cross-unbalanced space.

Consider now an arc element $(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$ at which

$$F_1(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) - F_2(\mathbf{x}(t), -\mathring{\mathbf{x}}(t)) = F_2(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) - F_1(\mathbf{x}(t), -\mathring{\mathbf{x}}(t)) = 0.$$

The proof will be complete if we show that at such arc elements,

$$\frac{\mathrm{d}\psi(\mathbf{x}(t), \mathbf{x}(t))}{\mathrm{d}t} = 0.$$

But this immediately follows from combining Theorem 41 with $\mu = 1$ and $R(\alpha) = \alpha$ (Theorem 44):

$$\Omega_\alpha \mathbf{x}(t) = o\{\alpha\},$$
$$\Omega_{-\alpha} \mathbf{x}(t) = o\{\alpha\}. \quad \square$$

We see from this result that Nonconstant Self-Dissimilarity is a consequence of cross-unbalancedness. Since this property is global rather than local, however, we formulate it in the next subsection.

11.3. Cross-unbalanced case: global properties

Theorem 46. *In a cross-unbalanced stimulus space* $\mathfrak{M}$, *Nonconstant Self-Dissimilarity is manifest in* $\mathfrak{M}$, *that is,* $\psi(\mathbf{b}, \mathbf{b}) \neq \psi(\mathbf{a}, \mathbf{a})$ *for some pairs* $\mathbf{a}, \mathbf{b} \in \mathfrak{M}$.

Proof. Otherwise along any smooth arc $\mathbf{x}(t)$ we would have

$$\frac{\mathrm{d}\psi(\mathbf{x}(t), \mathbf{x}(t))}{\mathrm{d}t} = 0,$$

which would contradict Theorem 45. □

It is convenient at this point to introduce simplified notation. Let $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ be a piecewise smooth arc connecting $\mathbf{a}$ to $\mathbf{b}$. This arc can be presented as $\mathbf{a} \to \mathbf{x}_{[a,b]} \to \mathbf{b}$. In most cases the parametric domain $[a,b]$ does not matter, and we can write simply $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$. If an arc $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ is traversed "in the opposite direction", it will then be written as $\mathbf{b} \to \mathbf{x} \to \mathbf{a}$. Psychometric lengths of these arcs then will be denoted by $L^{(\iota)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}]$ and $L^{(\iota)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}]$, $\iota = 1,2$.

Theorem 47. *If stimulus space* $\mathfrak{M}$ *is cross-unbalanced, then for any piecewise smooth arc* $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$,

$$\begin{aligned} &L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ &\quad = L^{(2)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(1)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ &\quad = \psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a}). \end{aligned}$$

Proof. Let $\mathbf{x}$ be defined on some interval $[a, b]$. We have

$$L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] = \int_a^b F_1(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t,$$

$$L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] = \int_a^b F_2(\mathbf{x}(t), -\mathring{\mathbf{x}}(t))\,\mathrm{d}t,$$

$$L^{(2)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] = \int_a^b F_2(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t,$$

$$L^{(1)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] = \int_a^b F_1(\mathbf{x}(t), -\mathring{\mathbf{x}}(t))\,\mathrm{d}t.$$

Using Theorem 45,

$$\begin{aligned} &\int_a^b F_1(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t - \int_a^b F_2(\mathbf{x}(t), -\mathring{\mathbf{x}}(t))\,\mathrm{d}t \\ &\quad = \int_a^b F_2(\mathbf{x}(t), \mathring{\mathbf{x}}(t))\,\mathrm{d}t - \int_a^b F_1(\mathbf{x}(t), -\mathring{\mathbf{x}}(t))\,\mathrm{d}t \\ &\quad = \int_a^b \frac{\mathrm{d}\psi(\mathbf{x}(t), \mathbf{x}(t))}{\mathrm{d}t}\,\mathrm{d}t = \psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a}). \qquad \square \end{aligned}$$

Theorem 48. *If stimulus space* $\mathfrak{M}$ *is cross-unbalanced, then for any two piecewise smooth arcs* $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ *and* $\mathbf{a} \to \mathbf{y} \to \mathbf{b}$,

$$\begin{aligned} &L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] + L^{(1)}[\mathbf{b} \to \mathbf{y} \to \mathbf{a}] \\ &\quad = L^{(2)}[\mathbf{a} \to \mathbf{y} \to \mathbf{b}] + L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}]. \end{aligned}$$

In particular,

$$\begin{aligned} &L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] + L^{(1)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ &\quad = L^{(2)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] + L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}]. \end{aligned}$$

Proof. Follows from

$$L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] = \psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a}),$$
$$L^{(2)}[\mathbf{a} \to \mathbf{y} \to \mathbf{b}] - L^{(1)}[\mathbf{b} \to \mathbf{y} \to \mathbf{a}] = \psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a}),$$

which is true by Theorem 47. $\square$

This result deserves a commentary. Any pair of piecewise smooth arcs $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ and $\mathbf{b} \to \mathbf{y} \to \mathbf{a}$ can be called a *piecewise smooth loop* (formally introduced later, in Definition 20). The theorem says that the psychometric length of any such a loop in one observation area is the same as the psychometric length of this loop when traversed in the opposite direction in another observation area.

Theorem 49. *For any* $\mathbf{a}, \mathbf{b}$ *in a cross-unbalanced stimulus space* $\mathfrak{M}$,

$$\begin{aligned} &G_1(\mathbf{a}, \mathbf{b}) - G_2(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) - G_1(\mathbf{b}, \mathbf{a}) \\ &\quad = \psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a}) \end{aligned}$$

and

$$G_1(\mathbf{a}, \mathbf{b}) + G_1(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) + G_2(\mathbf{b}, \mathbf{a}).$$

Proof. By Theorem 47,

$$L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] - L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] = \psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a}).$$

Clearly then

$$\begin{aligned} G_1(\mathbf{a}, \mathbf{b}) &= \inf L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] \\ &= \inf L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] + [\psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a})] \\ &= G_2(\mathbf{b}, \mathbf{a}) + [\psi(\mathbf{b}, \mathbf{b}) - \psi(\mathbf{a}, \mathbf{a})]. \end{aligned}$$

The proof for $G_2(\mathbf{a}, \mathbf{b})$ and $G_1(\mathbf{b}, \mathbf{a})$ is identical. $\square$

11.4. Cross-balanced case

We now consider the case when

$$F_1(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) = F_2(\mathbf{x}(t), -\mathring{\mathbf{x}}(t))$$

for all arc elements $(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$.

Theorem 50. *If stimulus space* $\mathfrak{M}$ *is cross-balanced, then*

(i) *for any piecewise smooth arc* $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$,

$$L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] = L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}];$$

(ii) *for any* $\mathbf{a}, \mathbf{b} \in \mathfrak{M}$,

$$G_1(\mathbf{a}, \mathbf{b}) = G_2(\mathbf{b}, \mathbf{a}).$$

Proof. By simpler versions of the arguments in Theorem 47 and 49. $\square$

As immediate algebraic consequences of (i) and (ii) in this theorem, we obtain (i) and (ii) of the following corollary.

Corollary 14 (*to Theorem 50*). *If stimulus space* $\mathfrak{M}$ *is cross-balanced, then*

(i) *for any two piecewise smooth arcs* $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ *and* $\mathbf{a} \to \mathbf{y} \to \mathbf{b}$,

$$\begin{aligned} &L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] + L^{(1)}[\mathbf{b} \to \mathbf{y} \to \mathbf{a}] \\ &\quad = L^{(2)}[\mathbf{a} \to \mathbf{y} \to \mathbf{b}] + L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}]; \end{aligned}$$

(ii) *for any* $\mathbf{a}, \mathbf{b} \in \mathfrak{M}$,

$$G_1(\mathbf{a}, \mathbf{b}) + G_1(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) + G_2(\mathbf{b}, \mathbf{a}).$$

The significance of these two observations is that they coincide with the statements of Theorems 48 and 49.

These statements therefore hold for all stimulus spaces, cross-balanced as well as cross-unbalanced.

The relationship between cross-balancedness and Non Constant Self-Dissimilarity is straightforward only in one direction.

Theorem 51. *If* $\psi(\mathbf{x},\mathbf{x}) \equiv \text{const}$ (*Constant Self-Dissimilarity*), *then stimulus space* $\mathfrak{M}$ *is cross-balanced.*

Proof. This is merely the contraposition of Theorem 46. □

The reverse, however, does not hold: cross-balancedness does not imply Constant Self-Dissimilarity.

Lemma 9. *If Nonconstant Self-Dissimilarity is manifest in* $\mathfrak{M}$, *then one can find a smooth arc* $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ *and* $t \in [a, b)$ *such that* $\Omega_\alpha \mathbf{x}(t)$ *attains nonzero values within any right-hand vicinity of* $\alpha = 0$.

Proof. If for some $\mathbf{x}(t)$ connecting $\mathbf{a}$ to $\mathbf{b}$ and any $t \in [a, b)$ one could find a right-hand vicinity of $\alpha = 0$ on which $\Omega_\alpha \mathbf{x}(t) \equiv 0$, then $\psi(\mathbf{x}(t), \mathbf{x}(t))$ would be constant throughout the arc, and $\psi(\mathbf{a},\mathbf{a}) = \psi(\mathbf{b},\mathbf{b})$. Since there are $\mathbf{a},\mathbf{b}$ for which this is not the case, the statement of the lemma must hold. □

Theorem 52. *If stimulus space* $\mathfrak{M}$ *is cross-balanced, then Nonconstant Self-Dissimilarity may be manifest in* $\mathfrak{M}$ *only in two cases: if psychometric order* $\mu < 1$, *or else if* $\mu = 1$ *but characteristic function* $R(\alpha) = \alpha\ell(\alpha)$ *with*

$$\lim_{\alpha\to 0+} \sup \ell(\alpha) = \infty.$$

Proof. From Theorem 41, on putting $F_1(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) = F_2(\mathbf{x}(t), -\mathring{\mathbf{x}}(t))$ and $F_1(\mathbf{x}(t), -\mathring{\mathbf{x}}(t)) = F_2(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$, we get

$$\Omega_\alpha \mathbf{x}(t) = o\{R^\mu(\alpha)\}$$

for all points of all smooth arcs. If $\mu > 1$, then $R^\mu(\alpha) = o\{\alpha\}$, and

$$\Omega_\alpha \mathbf{x}(t) = o\{\alpha\}.$$

As this implies

$$\lim_{\alpha\to 0+} \frac{\Omega_\alpha \mathbf{x}(t)}{\alpha} = 0,$$

which would contradict Lemma 9, we conclude that $\mu \leqslant 1$.

Consider the possibility $\mu = 1$, and let $\mathbf{x}(t)$ and t be chosen as stated in Lemma 9. Then for some sequence $\alpha_n \to 0+$,

$$\lim_{n\to\infty} \frac{\Omega_{\alpha_n} \mathbf{x}(t)}{\alpha_n} \neq 0.$$

As $\Omega_\alpha \mathbf{x}(t) = o\{\alpha\ell(\alpha)\}$, where $\alpha\ell(\alpha) = R(\alpha)$, we must have, for the same sequence $\alpha_n \to 0+$,

$$0 = \lim_{n\to\infty} \frac{\Omega_{\alpha_n} \mathbf{x}(t)}{\alpha_n \ell(\alpha_n)} = \lim_{n\to\infty} \frac{\Omega_{\alpha_n} \mathbf{x}(t)/\alpha_n}{\ell(\alpha_n)},$$

whence it follows that $\ell(\alpha_n) \to \infty$. The latter implies $\lim_{\alpha\to 0+} \sup \ell(\alpha) = \infty$. □

11.5. Overall Fechnerian distance

The results obtained in the previous subsections can be summarized as follows.

Theorem 53 (*Second Main Theorem of Fechnerian Scaling*). *For any* $\mathbf{a},\mathbf{b} \in \mathfrak{M}$ *and any piecewise smooth arcs* $\mathbf{a} \to \mathbf{x} \to \mathbf{b}$ *and* $\mathbf{a} \to \mathbf{y} \to \mathbf{b}$,

(i) $L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] + L^{(1)}[\mathbf{b} \to \mathbf{y} \to \mathbf{a}] = L^{(2)}[\mathbf{a} \to \mathbf{y} \to \mathbf{b}] + L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}]$;

(ii) $G_1(\mathbf{a},\mathbf{b}) + G_1(\mathbf{b},\mathbf{a}) = G_2(\mathbf{a},\mathbf{b}) + G_2(\mathbf{b},\mathbf{a})$.

This is merely a restatement of the identical consequent parts of Theorems 48 and 49 and Corollary 14. The reason this is called a main theorem is that it justifies the introduction and interpretation of the following notion.

Definition 19. The quantity $G(\mathbf{a},\mathbf{b}) = G_1(\mathbf{a},\mathbf{b}) + G_1(\mathbf{b},\mathbf{a}) = G_2(\mathbf{a},\mathbf{b}) + G_2(\mathbf{b},\mathbf{a})$ is called the (overall) Fechnerian distance between $\mathbf{a}$ and $\mathbf{b}$.

Theorem 54. *Fechnerian distance* $G(\mathbf{a},\mathbf{b})$ *is a* (*symmetrical*) *metric on space* $\mathfrak{M}$, *invariant with respect to the observation area.*

Proof. Obvious. □

Definition 20. A pair of piecewise smooth arcs $\mathbf{a} \to \mathbf{x} \to \mathbf{b}, \mathbf{b} \to \mathbf{y} \to \mathbf{a}$ forms a piecewise smooth loop containing $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \begin{smallmatrix} \to \mathbf{x} \to \\ \leftarrow \mathbf{y} \leftarrow \end{smallmatrix} \mathbf{b}$. The quantity

$$\begin{aligned} L\left[\mathbf{a} \begin{matrix} \to \mathbf{x} \to \\ \leftarrow \mathbf{y} \leftarrow \end{matrix} \mathbf{b}\right] &= L^{(1)}[\mathbf{a} \to \mathbf{x} \to \mathbf{b}] + L^{(1)}[\mathbf{b} \to \mathbf{y} \to \mathbf{a}] \\ &= L^{(2)}[\mathbf{a} \to \mathbf{y} \to \mathbf{b}] + L^{(2)}[\mathbf{b} \to \mathbf{x} \to \mathbf{a}] \\ &= L\left[\mathbf{a} \begin{matrix} \to \mathbf{y} \to \\ \leftarrow \mathbf{x} \leftarrow \end{matrix} \mathbf{b}\right]. \end{aligned}$$

is called the psychometric length of this loop.

Theorem 55. $G(\mathbf{a},\mathbf{b}) = \inf L[\mathbf{a} \begin{smallmatrix} \to \mathbf{x} \to \\ \leftarrow \mathbf{y} \leftarrow \end{smallmatrix} \mathbf{b}]$, *where the infimum is taken over all piecewise smooth loops containing* $\mathbf{a}$ *and* $\mathbf{b}$.

Proof. Obvious. □

Fig. 27 illustrates these results.

A word of caution is due here. Refer to Section 4. Fechnerian distance $G(\mathbf{a},\mathbf{b})$ between $\mathbf{a}$ and $\mathbf{b}$ is the same in the first and in the second observation area only in space $\mathfrak{M}$, where mutual PSEs are labelled by the same stimulus labels (which only for convenience are referred to as stimuli). This means that $(\mathbf{a},\mathbf{b})$ in the first observation area may very well be physically different from $(\mathbf{a},\mathbf{b})$ in the second observation area.

11.6. Empirical evidence

Although we do not make use of this fact in the present paper, the results of the experiments described in Fig. 12 strongly favor the cross-unbalanced version of the theory over the cross-balanced one. The predictions of the two versions are shown in Figs. 28 and 29. We assume here that stimulus set $\mathfrak{M}$ is an interval of reals, that the Fechnerian topology coincides with the conventional one, and that for any two stimuli $a \leqslant b$ the smooth arcs connecting them are diffeomorphisms $[a',b'] \to [a,b]$ and $[a',b'] \to [b,a]$. With no loss of generality, we can choose these diffeomorphisms so that their derivatives identically equal 1 and -1 in these two cases, respectively. We deal here with the unidimensional case of MDFS (the previously developed specialization of the present theory to vectorial spaces with Euclidean topology, as described in Introduction), and details of the predictions shown in the left upper panels of Figs. 28 and 29 can be found in Dzhafarov (2002d).

The lower panels of these two figures illustrate the fact that our testing of these predictions based on the experiments described in Fig. 12 is rather crude, both due to the necessity of estimating tangent slopes by slopes of corresponding cords, and due to the only approximate determinability of the endpoints of these cords. There are three reasons for the latter: statistical deviation of frequencies from probabilities, crudeness in determining stimulus values at which discrimination probabilities reach their minima, and crudeness in canonically transforming these probabilities using the PSE curves shown in Figs. 13–15. What makes our empirical testing possible is that the number of computed points in Fig. 30 is very large: one "blue" point $(C_{1+} - C_{2-})$ vs. $(K_1 + K_2)/2$ and one "red" point $(C_{2+} - C_{1-})$ vs. $(K_1 + K_2)/2$ per every PSE pair shown in Figs. 13–15. As it turns out, the regression line drawn through these points is almost precisely the bisector, as predicted in Fig. 28, and very definitely is not the abscissa line predicted in 29.

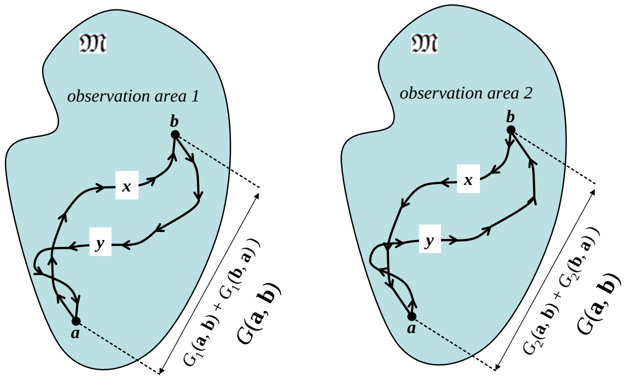

Fig. 27. Schematic illustration of the Second main theorem. Shown are two points connected by a piecewise smooth loop, in two observation areas. Psychometric length of the loop is the same in the two areas if traversed in the opposite directions, as shown. Fechnerian distance $G(\mathbf{a},\mathbf{b})$ is defined as $G_1(\mathbf{a},\mathbf{b}) + G_1(\mathbf{b},\mathbf{a})$ or $G_2(\mathbf{a},\mathbf{b}) + G_2(\mathbf{b},\mathbf{a})$: the two quantities coincide.

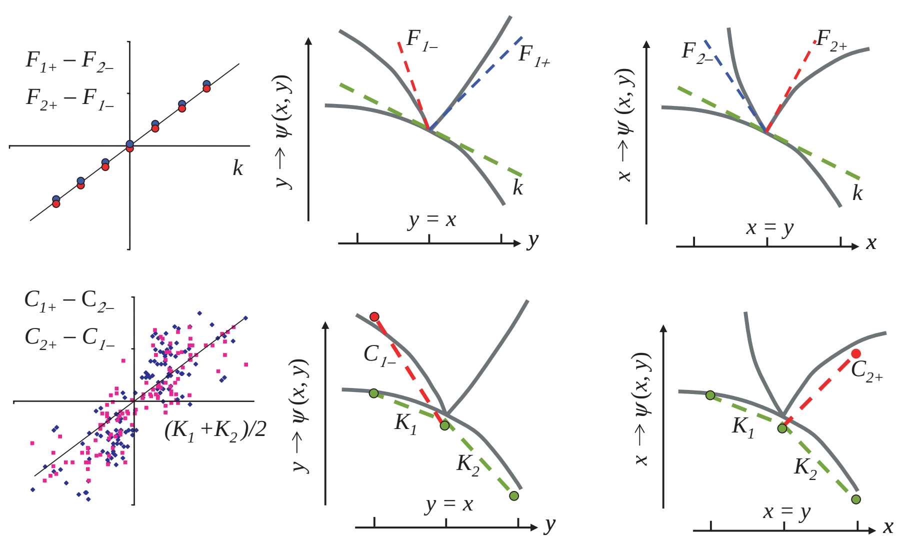

Fig. 28. Predictions derived from the cross-unbalanced version of the theory for stimulus space representable by an interval of reals (with topology and smooth arcs defined as explained in the text). Solid lines represent cross-sections of $\psi(x,y)$ made at $x = y$ (like the V-shaped cross-sections in the right-hand panels of Fig. 11) and a portion of the minimum level function (like the dashed curves in Fig. 11, right). The dashed lines are tangents drawn to these curves at the their intersection. k is the slope of the minimum level function, whereas F_{1+}, F_{1-}, F_{2+}, F_{2-} are absolute values of the slopes of the V-shaped cross-sections. The prediction is $F_{1+} - F_{2-} = F_{2+} - F_{1-} = k$ (left upper panel). In an experiment, however, the tangent slopes have to be estimated by the slopes of cords drawn between a few estimates of true probabilities (for simplicity, only one pair of cords is shown for the cross-sections). The expected pattern should therefore be more like the one in the left lower panel.

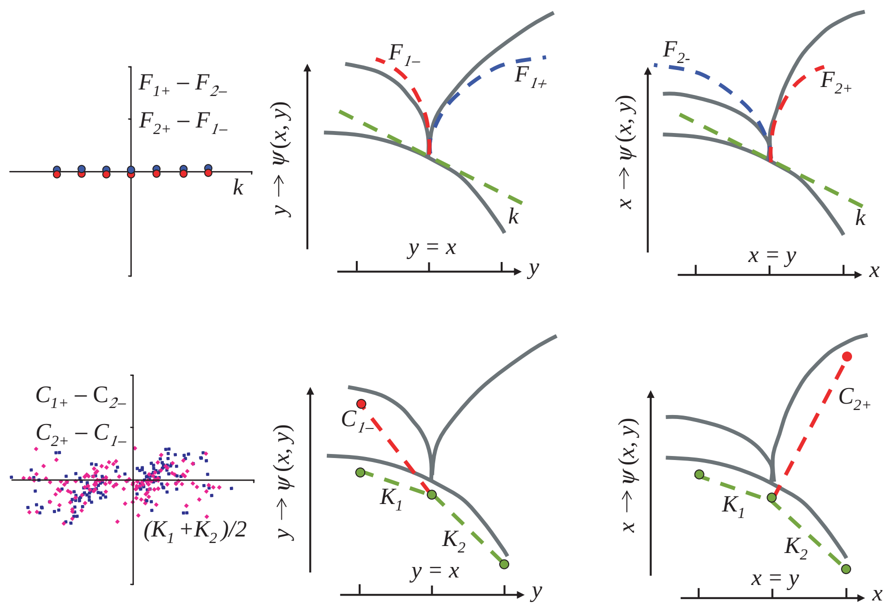

Fig. 29. Predictions derived from the cross-balanced version of the theory for stimulus space representable by an interval of reals. The logic is the same as in the previous figure, except that the tangent lines are replaced with curves $[F_{1+}R(\alpha)]^{\mu}$, $[F_{1-}R(\alpha)]^{\mu}$, $[F_{2+}R(\alpha)]^{\mu}$, $[F_{2-}R(\alpha)]^{\mu}$, the lowest-order terms in the asymptotic decompositions of, respectively, $\psi(x, x+\alpha) - \psi(x, x)$, $\psi(x, x-\alpha) - \psi(x, x)$, $\psi(y+\alpha, y) - \psi(y, y)$, $\psi(y-\alpha, y) - \psi(y, y)$. The prediction here is $F_{1+} - F_{2-} = F_{2+} - F_{1-} = 0$. The slopes of the cords C_{1+}, C_{1-}, C_{2+}, C_{2-} here do not estimate the values of F_{1+}, F_{1-}, F_{2+}, F_{2-}, but due to the symmetry of the identically colored dashed lines one should still expect $C_{1+} - C_{2-}$ and $C_{2+} - C_{1-}$ to be close to zero.

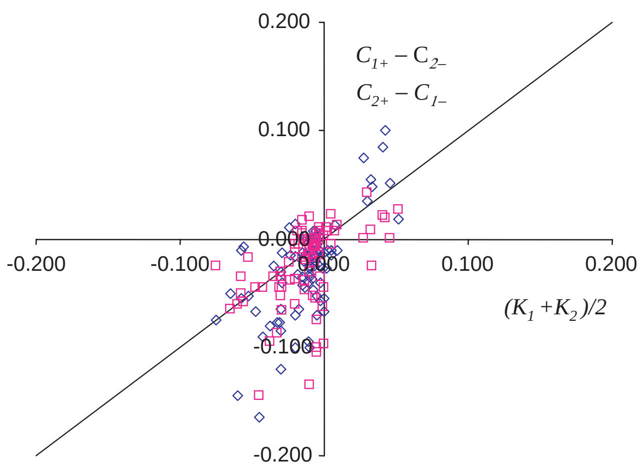

Fig. 30. Relationship between $C_{1+} - C_{2-}$ and $(K_1 + K_2)/2$ (blue symbols) and between $C_{2+} - C_{1-}$, and $(K_1 + K_2)/2$ (red symbols) computed from the experiments A, B, and C described in Fig. 12. The regression line is almost precisely the bisector.

Note that it could have happened that the regression line was clearly different from both the bisector and the abscissa, in which case the present theory would have been empirically falsified.

Fig. 31 provides an additional illustration. The results are taken from a pilot series for experiment B, conducted on a coarser spatial scale. The point of the demonstration is that the straight lines satisfying the prediction of the cross-unbalanced version of the theory, $|right\ slope| - |left\ slope| = 1$ (due to specially transformed abscissa), do have the visual appearance of tangent lines drawn to the three cross-sections of $\psi(x, y)$ at their minima.

11.7. Metrically equivalent transformations

Definition 21. Two discrimination probability functions $\psi(\mathbf{x}, \mathbf{y})$ and $\psi_{\dagger}(\mathbf{x}, \mathbf{y})$ are called weakly metrically equivalent if they induce identical Fechnerian metrics $G(\mathbf{a}, \mathbf{b})$ and $G_{\dagger}(\mathbf{a}, \mathbf{b})$, up to multiplication by a positive constant,

$$G_{\dagger}(\mathbf{a}, \mathbf{b}) = kG(\mathbf{a}, \mathbf{b}), \quad k > 0.$$

Two discrimination probability functions $\psi(\mathbf{x}, \mathbf{y})$ and $\psi_{\dagger}(\mathbf{x}, \mathbf{y})$ are called strongly metrically equivalent if any arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ which is smooth with respect to ψ is also smooth with respect to $\psi_{\dagger}$, and for any arc element $(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$,

$$F_{\dagger\iota}(\mathbf{x}(t), \mathring{\mathbf{x}}(t)) = kF_{\iota}(\mathbf{x}(t), \mathring{\mathbf{x}}(t)), \quad k > 0, \quad \iota = 1, 2.$$

where $F_{\iota}, F_{\dagger\iota}$ are submetric functions induced by, respectively, ψ and $\psi_{\dagger}$.

Two comments.

1. That strong metric equivalence implies weak metric equivalence is obvious.
2. Submetric functions, in accordance with Theorem 19, are determined by discrimination probability functions uniquely up to multiplication by a positive constant. The constant k in the definition therefore can always be eliminated (put equal to 1) by appropriate rescaling.

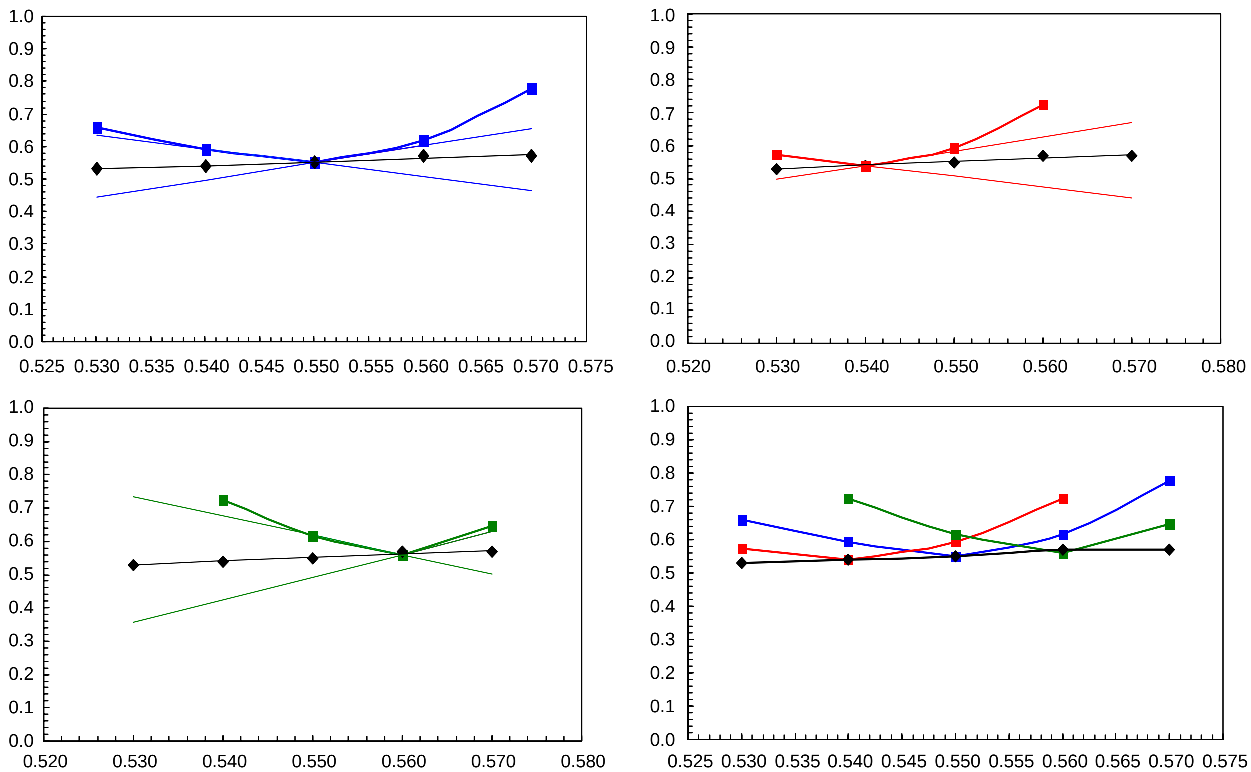

Fig. 31. Results of experiment B (synchronous apparent motions) for one participant, with $a = 25$, $\Delta = 5$ (pixels), and $n = 200$ (see Fig. 12). The data in this case conformed with the canonical form of Regular Minimality (i.e., PSEs were physically identical, $x = y$) and with symmetry, i.e., $\psi(x, y)$ and $\psi(y, x)$ values were close and could therefore be averaged. As a result, the right lower panel of the figure can be viewed both as $x \to \psi(x, y)$ and as $y \to \psi(x, y)$, an empirical analogue of either of the two right-hand panels in Fig. 11. The abscissa axis was transformed so that the slope of the minimum level function $\psi(x, x)$ be precisely 1. The prediction of the cross-unbalanced version of the Fechnerian theory in this case is that for the tangents drawn to each of the V-shaped cross-sections shown, |right slope| − |left slope| = 1. The straight lines shown in the left and upper panels of the figure were drawn under this constraint.

On the level of generality of the present theory the notion of weak metric equivalence seems too complex to analyze. A priori, one can think of the possibility that $\psi(\mathbf{x}, \mathbf{y})$ and $\psi_{\dagger}(\mathbf{x}, \mathbf{y})$ induce different sets of smooth curves, different submetric functions, perhaps even different oriented Fechnerian distances, but $G_{\dagger}(\mathbf{a}, \mathbf{b})$ at the end happens to be equal to $kG(\mathbf{a}, \mathbf{b})$. Having mentioned this for completeness sake, we will not pursue this topic. Instead we will focus here on the strong metric equivalence, and pose the following question:

> *what are necessary and sufficient conditions for an increasing transformation $B[\psi(\mathbf{x}, \mathbf{y})]$ of $\psi(\mathbf{x}, \mathbf{y})$ to be strongly metrically equivalent to $\psi(\mathbf{x}, \mathbf{y})$?*

The reason this is of interest is that the transition from $\psi(\mathbf{x}, \mathbf{y})$ to $B[\psi(\mathbf{x}, \mathbf{y})]$ in this case can be viewed as a result of changed *response bias* (propensity to responding "different") that leaves intact all perceptual dissimilarities among the stimuli.

We need a definition and a lemma.

Definition 22. Set

$$\mathfrak{s}[\psi] = \{x : x = \psi(\mathbf{x}(a), \mathbf{x}(t)) \text{ for some smooth arc } \mathbf{x}(t) : [a, b] \to \mathfrak{M} \text{ and some } t \in [a, b)\}$$

is called the smooth codomain of discrimination probability function $\psi(\mathbf{x}, \mathbf{y})$.

Lemma 10. *Smooth codomain $\mathfrak{s}[\psi]$ of $\psi(\mathbf{x}, \mathbf{y})$ includes interval $(\inf_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{x}), \sup_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{x}))$ and points $\min_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{x})$, $\max_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{x})$ if they exist.*

Proof. By Corollary 10 to Axiom 7, for any $\mathbf{a} \in \mathfrak{M}$ one can find a sufficiently close $\mathbf{b} \in \mathfrak{M}$ such that $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ is a smooth arc, $\mathbf{x}(a) = \mathbf{a}, \mathbf{x}(b) = \mathbf{b}$, and $a \neq b$. This means that $\psi(\mathbf{a}, \mathbf{a}) \in \mathfrak{s}[\psi]$. Since $\psi(\mathbf{x}, \mathbf{x})$ is continuous, its values form an interval. $\square$

Theorem 56. *Discrimination probability function $\psi(\mathbf{x}, \mathbf{y})$ is strongly metrically equivalent to its increasing transformation $B[\psi(\mathbf{x}, \mathbf{y})]$ if and only if*

$$B(x) = kx + l + f(\max\{x, M\}),$$

where

1. $M = \sup_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{x})$;
2. $k > 0$ *and* l *are some constants*;
3. $f(x)$ *is a positive, increasing, and continuous function, with* $f(M) = f'(M) = 0$;
4. $f(x)$ *is continuously differentiable at all* $x \in \mathfrak{s}[\psi]$, $x \geqslant M$.

Proof. We prove the necessity first. By Comment 5 to Definition 5, both $\psi(\mathbf{x}(a), \mathbf{x}(t))$ and $B[\psi(\mathbf{x}(a), \mathbf{x}(t))]$ are continuously differentiable on $(a, b]$, for any smooth arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$. It follows that $B(x)$ is continuously differentiable at all $x \in \mathfrak{s}[\psi]$. In particular, it is differentiable on $(\inf_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{x}), \sup_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{x})) \cup \{\min_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{x}), \max_{\mathbf{x}} \psi(\mathbf{x}, \mathbf{x})\}$, where the second set may be empty, a singleton, or a pair (Lemma 10). For every smooth arc $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$ then, as $\alpha \to 0+$,

$$\begin{aligned} &B[\psi(\mathbf{x}(a), \mathbf{x}(a+\alpha))] - B[\psi(\mathbf{x}(a), \mathbf{x}(a))] \\ &= B[\{\psi(\mathbf{x}(a), \mathbf{x}(a+\alpha)) - \psi(\mathbf{x}(a), \mathbf{x}(a))\} + \psi(\mathbf{x}(a), \mathbf{x}(a))] \\ &\quad - B[\psi(\mathbf{x}(a), \mathbf{x}(a))] \\ &\quad \sim B'[\psi(\mathbf{x}(a), \mathbf{x}(a))][\psi(\mathbf{x}(a), \mathbf{x}(a+\alpha)) - \psi(\mathbf{x}(a), \mathbf{x}(a))] \\ &\quad \sim B'[\psi(\mathbf{x}(a), \mathbf{x}(a))][F_1(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) R(\alpha)]^{\mu}. \end{aligned}$$

Due to strong metric equivalence, we also should have

$$\begin{aligned} &B[\psi(\mathbf{x}(a), \mathbf{x}(a+\alpha))] - B[\psi(\mathbf{x}(a), \mathbf{x}(a))] \\ &\quad \sim [h F_1(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) R_0(\alpha)]^{\mu_0} \end{aligned}$$

for some $h > 0, \mu_0 > 0$ and some characteristic function $R_0(\alpha)$. Hence

$$\begin{aligned} &B'[\psi(\mathbf{x}(a), \mathbf{x}(a))][F_1(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) R(\alpha)]^{\mu} \\ &\quad \sim [h F_1(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) R_0(\alpha)]^{\mu_0} \end{aligned}$$

and

$$B'[\psi(\mathbf{x}(a), \mathbf{x}(a))] F_1^{\mu}(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) \sim F_1^{\mu_0}(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) \frac{[h\alpha\ell_0(\alpha)]^{\mu_0}}{[\alpha\ell(\alpha)]^{\mu}},$$

where $\ell(\alpha), \ell_0(\alpha)$ are the slowly varying components of $R(\alpha) = \alpha\ell(\alpha)$ and $R_0(\alpha) = \alpha\ell_0(\alpha)$. Clearly, this is possible if and only if

$$\frac{[h\alpha\ell_0(\alpha)]^{\mu_0}}{[\alpha\ell(\alpha)]^{\mu}} \to k > 0,$$

whence $\mu_0 = \mu$ (otherwise the ratio will tend to 0 or ∞) and $\ell_0(\alpha) \sim \frac{k}{h}\ell(\alpha)$. From

$$B'[\psi(\mathbf{x}(a), \mathbf{x}(a))] F_1^{\mu}(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) \sim F_1^{\mu}(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) k$$

we conclude

$$B'[\psi(\mathbf{x}(a), \mathbf{x}(a))] = k.$$

The function $B(x)$ described in the statement of the theorem is obviously the only function satisfying this condition, and increasing, continuous, and continuously differentiable on $\mathfrak{s}[\psi]$.

The sufficiency is verified directly:

$$\begin{aligned} &B[\psi(\mathbf{x}(a), \mathbf{x}(a+\alpha))] - B[\psi(\mathbf{x}(a), \mathbf{x}(a))] \\ &\quad \sim k[\psi(\mathbf{x}(a), \mathbf{x}(a+\alpha)) - \psi(\mathbf{x}(a), \mathbf{x}(a))] \\ &\quad \sim k F_1^{\mu}(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) R(\alpha), \\ &B[\psi(\mathbf{x}(a+\alpha), \mathbf{x}(a))] - B[\psi(\mathbf{x}(a), \mathbf{x}(a))] \\ &\quad \sim k[\psi(\mathbf{x}(a+\alpha), \mathbf{x}(a)) - \psi(\mathbf{x}(a), \mathbf{x}(a))] \\ &\quad \sim k F_2^{\mu}(\mathbf{x}(a), \mathring{\mathbf{x}}(a)) R(\alpha). \qquad \square \end{aligned}$$

Corollary 15 (*to Theorem 56*). *An increasing and strongly metrically equivalent transformation* $B[\psi(\mathbf{x}, \mathbf{y})]$ *of* $\psi(\mathbf{x}, \mathbf{y})$ *induces the same psychometric order* μ *and asymptotically the same characteristic function* $R(\alpha)$ *as* $\psi(\mathbf{x}, \mathbf{y})$.

Proof. As follows from the proof of the theorem, $R_0(\alpha) \sim R(\alpha) \cdot \text{const}$, where the constant can be set to 1 by Theorem 19. $\square$

In Theorem 56, for reasons to become apparent in Section 13, we ignored the fact that the codomains of $\psi(\mathbf{x}, \mathbf{x})$ and $B[\psi(\mathbf{x}, \mathbf{y})]$ are confined to interval $[0, 1]$. Clearly, nothing changes in the proof if we stipulate that k, l, and $f(x)$ should be chosen to ensure

$$0 \leqslant k\psi(\mathbf{x}, \mathbf{y}) + l + f(\max\{\psi(\mathbf{x}, \mathbf{y}), M\}) \leqslant 1$$

for all $\mathbf{x}, \mathbf{y}$.

It is interesting to note that, as a special case, strong metric equivalence of $B[\psi(\mathbf{x}, \mathbf{y})]$ and $\psi(\mathbf{x}, \mathbf{x})$ is satisfied by linear transformations

$$B[\psi(\mathbf{x}, \mathbf{y})] = k\psi(\mathbf{x}, \mathbf{y}), \quad 0 < k \leqslant 1$$

and

$$B[\psi(\mathbf{x}, \mathbf{y})] = k\psi(\mathbf{x}, \mathbf{y}) + (1 - k), \quad 0 < k \leqslant 1$$

corresponding to the two branches of Luce's (1963) two-state model for response bias (of which the second branch accords with Blackwell's (1953) "guessing" model).

12. Brief overview

This completes the construction of Fechnerian metric in stimulus space $\mathfrak{M}$. It was effected by purely psychological means, without referring to any physical properties of stimuli involved. The highlights of this construction are as follows.

We begin with stimulus spaces $\mathfrak{M}_1^*, \mathfrak{M}_2^*$ corresponding to two observation areas, with discrimination probabilities $\psi^* : \mathfrak{M}_1^* \times \mathfrak{M}_2^* \to [0, 1]$. We lump together psychologically equal stimuli, those that cannot be distinguished in terms of the values of ψ^* (Definition 1). We thus form two "reduced" spaces (of equivalence classes of stimuli, also referred to as stimuli) $\tilde{\mathfrak{M}}_1, \tilde{\mathfrak{M}}_2$, and the corresponding discrimination probability function $\tilde{\psi}$: $\tilde{\mathfrak{M}}_1 \times \tilde{\mathfrak{M}}_2 \to [0, 1]$.

In terms of this function we formulate the defining property of discrimination, Regular Minimality (Axiom 1): every stimulus $\mathbf{x} \in \tilde{\mathfrak{M}}_1$ has its PSE (point of subjective equality) $\mathbf{h}(\mathbf{x}) \in \tilde{\mathfrak{M}}_2$, every stimulus $\mathbf{y} \in \tilde{\mathfrak{M}}_2$ has its PSE $\mathbf{g}(\mathbf{y}) \in \tilde{\mathfrak{M}}_2$, and $\mathbf{h} \equiv \mathbf{g}^{-1}$. PSE for a given stimulus $\mathbf{a}$ is the stimulus least discriminable from $\mathbf{a}$. Regular Minimality is the cornerstone of the present development.

Based on Regular Minimality we relabel our stimuli (i.e., bijectively map $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ onto $\mathfrak{M}$, a space of stimulus labels) so that stimuli in $\mathfrak{M}_1^*$ and $\mathfrak{M}_2^*$ that are mutually PSEs map into identical labels. We obtain thus a new space $\mathfrak{M}$ (of stimulus labels, for brevity also referred to as stimuli) and the corresponding discrimination probability function $\psi : \mathfrak{M} \times \mathfrak{M} \to [0, 1]$. In this "canonical representation", Regular Minimality has its simplest possible form, (5) or (26). We stipulate that in general $\psi(\mathbf{x}, \mathbf{x})$ can vary with $\mathbf{x}$ (Nonconstant Self-Dissimilarity) and $\psi(\mathbf{y}, \mathbf{x})$ need not equal $\psi(\mathbf{x}, \mathbf{y})$ (Asymmetry).

Regular Minimality also ensures that psychometric increments $\Psi^{(1)}(\mathbf{x}, \mathbf{y})$, $\Psi^{(2)}(\mathbf{x}, \mathbf{y})$ of the first and second kind (Definition 3), our basic building blocks, are nonnegative quantities vanishing only at $\mathbf{x} = \mathbf{y}$. Psychometric increments allow us to define convergence in stimulus space $\mathfrak{M}$ (Axiom 2 and Definition 4), and to postulate that with respect to this convergence $\psi(\mathbf{x}, \mathbf{y})$ is a continuous function. We thus impose on stimulus space $\mathfrak{M}$ a topology, which turns out to be Urysohn (hence also Hausdorff). We can now define homeomorphic images of real intervals in $\mathfrak{M}$ (arcs $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$), and we postulate that $\mathfrak{M}$ is an arcwise connected space (Axiom 4).

Next we define smooth arcs $\mathbf{x}(t) : [a, b] \to \mathfrak{M}$, those along which $\Psi^{(1)}(\mathbf{x}(t), \mathbf{x}(t'))$ and $\Psi^{(2)}(\mathbf{x}(t), \mathbf{x}(t'))$ have certain differentiability properties (Definition 5). We assume further that any two psychometric increments

$$\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t \pm \alpha)) \text{ and } \Psi^{(\kappa)}(\mathbf{y}(\tau), \mathbf{y}(\tau \pm \alpha))$$

(where $\iota, \kappa = 1$ or 2), taken along any two smooth arcs are asymptotically comeasurable, that is, their ratio tends to a finite positive number as $\alpha \to 0+$ (Axiom 5). Using the very conservatively defined notion of an arc element $(\mathbf{x}(t), \mathring{\mathbf{x}}(t))$ (Definition 8) and a quasi-multiplication operation $(\mathbf{x}(t), k\,\mathring{\mathbf{x}}(t))$ on its directional component (Definition 9), we can denote the asymptotic ratio of $\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t \pm \alpha))$ (varying across all possible points on all possible smooth arcs) to a fixed $\Psi^{(\kappa)}(\mathbf{y}(\tau), \mathbf{y}(\tau + \alpha))$ by $V_\iota(\mathbf{x}(t), \pm \mathring{\mathbf{x}}(t)), \iota = 1, 2$. We endow this quantity with certain continuity properties (Axiom 6), and we derive the asymptotic decomposition of psychometric increments (along smooth arcs),

$$\Psi^{(\iota)}(\mathbf{x}(t), \mathbf{x}(t \pm \alpha)) = V_\iota(\mathbf{x}(t), \pm \mathring{\mathbf{x}}(t)) R^\mu(\alpha), \quad \iota = 1, 2,$$

where $\mu > 0$ is the psychometric order of stimulus space $\mathfrak{M}$ and $R(\alpha)$ is its characteristic function, unit-exponent regularly varying (Definitions 10 and 11).

The quantity

$$F_\iota(\mathbf{x}(t), \pm \mathring{\mathbf{x}}(t)) = V_\iota^{1/\mu}(\mathbf{x}(t), \pm \mathring{\mathbf{x}}(t)), \quad \iota = 1, 2$$

is called the submetric function (Definition 12), and its properties (derived from Axioms 6 and 7) allow us to integrate it along any piecewise smooth arc to obtain the psychometric length of this arc (of the first or second kind, according as $\iota = 1$ or 2; Definition 16).

The infimum of psychometric lengths of the ιth kind taken across all piecewise smooth arcs connecting $\mathbf{a}$ to $\mathbf{b}$ is taken as oriented Fechnerian distance $G_\iota(\mathbf{a}, \mathbf{b})$ from $\mathbf{a}$ to $\mathbf{b}$ (in the ιth observation area; Definition 17). This metric is shown to metrize the previously constructed (in Section 5) topology of stimulus space $\mathfrak{M}$ (i.e., metrics G_1 and G_2 induce this topology). In Section 10 we establish a variety of properties of G_1 and G_2. In particular, $G_\iota(\mathbf{x}(t), \mathbf{y}(\tau))$ ($\iota = 1, 2$) is differentiable in t and in τ almost everywhere. The internal consistency of Fechnerian metric is established by showing that $G_\iota(\mathbf{a}, \mathbf{b})$ equals the infimum of Burkill integrals of function G_ι over all piecewise smooth arcs connecting $\mathbf{a}$ to $\mathbf{b}$.

In Section 11 we prove the "second main theorem of Fechnerian scaling", according to which

$$G_1(\mathbf{a}, \mathbf{b}) + G_1(\mathbf{b}, \mathbf{a}) = G_2(\mathbf{a}, \mathbf{b}) + G_2(\mathbf{b}, \mathbf{a}) = G(\mathbf{a}, \mathbf{b})$$

and we propose to take this symmetric and observation-area-invariant quantity $G(\mathbf{a}, \mathbf{b})$ as the Fechnerian distance between $\mathbf{a}$ and $\mathbf{b}$. $G(\mathbf{a}, \mathbf{b})$ is the infimum of the psychometric lengths of all piecewise smooth loops containing $\mathbf{a}$ and $\mathbf{b}$.

Finally, at the end of Section 11 we consider the question of characterizing the monotone transformations of $\psi(\mathbf{x}, \mathbf{y})$ that preserve the set of smooth arcs and the numerical values of the submetric functions (then they obviously preserve all Fechnerian distances, both oriented and overall, and we prove that they also preserve the psychometric order and characteristic function of the stimulus space).

13. Some open questions

13.1. *Transformations of ψ and response bias*

The reader may have noticed that the theory presented in this paper makes no critical use of the fact that the values of $\psi(\mathbf{x}, \mathbf{y})$ are probabilities, or even that they are confined to the interval [0, 1]. Axioms 1–7 and all the definitions may very well apply to a function $\phi(\mathbf{x}, \mathbf{y})$ with another codomain. In particular, if the theory applies to $\psi(\mathbf{x}, \mathbf{y})$, then it will also apply to any continuously differentiable transformation thereof,

$$\phi(\mathbf{x},\mathbf{y}) = \varphi[\psi(\mathbf{x},\mathbf{y})].$$

Any arc $\mathbf{x}(t) : [a,b] \to \mathfrak{M}$ which is smooth (straight) with respect to ψ will also be smooth (straight) with respect to ϕ, and submetric functions (hence also the oriented and overall Fechnerian distances) will be as well-defined for ϕ as they are for ψ, with all theorems of the present development applying without modifications. The numerical values of the submetric functions, however (hence also those of the oriented and overall Fechnerian distances) will be different for ϕ and for ψ. The question arises therefore: is there a principled way of choosing the "right" transformation $\varphi[\psi(\mathbf{x},\mathbf{y})]$ of $\psi(\mathbf{x},\mathbf{y})$? In particular, is there a principled way of justifying the use of "raw" discrimination probabilities?

One possible approach to this issue is to relate it to another issue: to that of the possibility of experimental manipulations or spontaneous changes of context that change discrimination probabilities but leave intact subjective dissimilarities among the stimuli. In other words, we relate the issue of possible transformations of discrimination probabilities to that of response bias.

Suppose that according to some theory of response bias, discrimination probability functions can be presented as $\psi_{\mathscr{B}}(\mathbf{x},\mathbf{y})$, where $\mathscr{B}$ is value of response bias, varying within some abstract set (of reals, real-valued vectors, functions, etc.). Intuitively, this means that although $\psi_{\mathscr{B}_1}(\mathbf{x},\mathbf{y})$ and $\psi_{\mathscr{B}_2}(\mathbf{x},\mathbf{y})$ for two distinct response bias values may be different, the difference is not in "true" subjective dissimilarities but merely in the "overall readiness" of the perceiver to respond "different" rather than "same". If Fechnerian distances are to be interpreted as "true" subjective dissimilarities, one should expect then that Fechnerian metrics corresponding to $\psi_{\mathscr{B}_1}(\mathbf{x},\mathbf{y})$ and $\psi_{\mathscr{B}_2}(\mathbf{x},\mathbf{y})$ are identical (up to multiplication by positive constants). This may or may not be true for Fechnerian metrics computed directly from $\psi_{\mathscr{B}}(\mathbf{x},\mathbf{y})$, and if it is not, it may be true for Fechnerian metrics computed from some transformation $\varphi[\psi_{\mathscr{B}}(\mathbf{x},\mathbf{y})]$ thereof. The solution for the problem of what transformations of discrimination probabilities one should make use of can now be formulated as follows:

choose $\phi_{\mathscr{B}}(\mathbf{x},\mathbf{y}) = \varphi[\psi_{\mathscr{B}}(\mathbf{x},\mathbf{y})]$ *so that* $G(\mathbf{a},\mathbf{b})$ *computed from* $\phi_{\mathscr{B}}(\mathbf{x},\mathbf{y})$ *is invariant (up to positive scaling) with respect to* $\mathscr{B}$.

Based on Theorem 56, this idea can be developed further if we agree to understand response bias in the sense of *strong* metric equivalence (Definition 21); that is, if we require that not only $G(\mathbf{a},\mathbf{b})$ but also the sets of smooth arcs and the submetric functions $F_1(\mathbf{x}(t),\mathring{\mathbf{x}}(t)), F_2(\mathbf{x}(t),\mathring{\mathbf{x}}(t))$ remain invariant as we change $\mathscr{B}$. The solution then acquires the following form:

choose $\phi_{\mathscr{B}}(\mathbf{x},\mathbf{y}) = \varphi[\psi_{\mathscr{B}}(\mathbf{x},\mathbf{y})]$ *so that*

$$\phi_{\mathscr{B}}(\mathbf{x},\mathbf{y}) = k_{\mathscr{B}}\phi(\mathbf{x},\mathbf{y}) + l_{\mathscr{B}} + f_{\mathscr{B}}(\max\{\phi(\mathbf{x},\mathbf{y}), M\}),$$

where $\phi(\mathbf{x},\mathbf{y})$ *is* $\phi_{\mathscr{B}}(\mathbf{x},\mathbf{y})$ *at some specific value of* $\mathscr{B}$, *coefficients* $k_{\mathscr{B}}, l_{\mathscr{B}}$, *and function* $f_{\mathscr{B}}$ *are as in Theorem* 56, *and* $M = \sup_{\mathbf{x}} \phi(\mathbf{x},\mathbf{x})$.

It is easy to verify that any two transformations $\varphi_1[\psi_{\mathscr{B}}(\mathbf{x},\mathbf{y})], \varphi_2[\psi_{\mathscr{B}}(\mathbf{x},\mathbf{y})]$ with this property will be strongly metrically equivalent, which means that φ, if it exists, is determined essentially uniquely.

The approach proposed is, of course, open-ended, as the solution now depends on one's theory of response bias, independent of Fechnerian scaling. Thus, if one adopts Luce's (1963) or Blackwell's (1953) linear model of bias, φ is essentially the identity function and one should deal with "raw" discrimination probabilities. If one adopts the conventional d' measure of sensitivity, φ can be chosen as the inverse of the standard normal integral,

$$\psi(\mathbf{x},\mathbf{y}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\varphi[\psi(\mathbf{x},\mathbf{y})]} e^{-z^2/2}\,\mathrm{d}z.$$

As we do not know which model of response bias should be preferred, the question of whether and how one should transform ψ before computing Fechnerian distances should be viewed as open.

13.2. Overall psychometric transformation in the large

As explained in Section 8.3, the overall psychometric transformation Φ of psychometric increments is determined by discrimination probabilities ψ only asymptotically uniquely. One is free to choose its variant in the large, and the question arises whether there is a principled way for doing this. We mention this issue for completeness sake only, as it is not important in the present context, for arcwise connected stimulus spaces: in most of our considerations $\Phi(h)$ only interests us in the arbitrarily small vicinity of $h = 0$ where it vanishes; in the remaining cases the choice of Φ is shown to be irrelevant. When developing more general theory, however, in which stimulus space $\mathfrak{M}$ need not be arcwise connected, the issue becomes critical. Its analysis is given elsewhere (Dzhafarov & Colonius, in press).

13.3. "Adaptation" or "representative design"?

It is easier to introduce this problem using the special case of the theory when stimulus space $\mathfrak{M}^{(k)}$ is an open connected region of Euclidean space Re^k, with the ordinary continuously differentiable arcs playing the role of smooth arcs. Suppose that we have computed Fechnerian distances $G^{(k)}(\mathbf{a},\mathbf{b})$ in this space, using a discrimination probability function $\psi^{(k)}(\mathbf{x}, \mathbf{y})$. Suppose now that we fix some of the coordinates of this space at certain values, so that the remaining, variable dimensions form a subspace $\mathfrak{M}^{(l)}$ of $\mathfrak{M}^{(k)}$, an open connected region of Re^l, $l<k$. Formally, we can now use the projection $\psi^{(l)}(\mathbf{x}, \mathbf{y})$ of $\psi^{(k)}(\mathbf{x}, \mathbf{y})$ on these variable dimensions to compute Fechnerian distances $G^{(l)}(\mathbf{a}, \mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathfrak{M}^{(l)}$. The theory applies to this reduced space as well as it does to the original, "complete" one. (Of course, this "complete" space itself is obtained by fixing all but some k of potential dimensions along which stimuli could vary.) For any two points $\mathbf{a}, \mathbf{b}$ lying in $\mathfrak{M}^{(l)}$, however, the new Fechnerian distance $G^{(l)}(\mathbf{a}, \mathbf{b})$ in space $\mathfrak{M}^{(l)}$ will not be the same as the original Fechnerian distance $G^{(k)}(\mathbf{a}, \mathbf{b})$ computed in space $\mathfrak{M}^{(k)}$. It is easy to see that $G^{(l)}(\mathbf{a}, \mathbf{b}) \geqslant G^{(k)}(\mathbf{a}, \mathbf{b})$, because the set of piecewise smooth closed loops in $\mathfrak{M}^{(l)}$ that contain $\mathbf{a}$ and $\mathbf{b}$ is only a subset of those in $\mathfrak{M}^{(k)}$. Does this mean that by excluding some of the physical dimensions of stimuli from consideration we get Fechnerian distances "wrong" ? There seem to be three different ways of approaching this problem.

1. *Pragmatic* (*or formalist*) *approach*. One can simply posit that Fechnerian distances are always defined with respect to some stimulus space, however arbitrarily chosen, and insofar as Fechnerian distances can be computed (i.e., the space satisfies all our axioms) these distances cannot be "incorrect".
2. "*Adaptation-to-space*" *approach*. A similar but less formalistic position is to assume that perceptual dissimilarities among stimuli may change depending on which parameters of stimuli are held constant in an experiment and which vary. As pointed out to one of us by Jun Zhang (personal communication, April 2002), it is likely that in a real experiment $\psi^{(l)}(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathfrak{M}^{(l)}$ will not be equal to $\psi^{(k)}(\mathbf{x}, \mathbf{y})$. In other words, $\psi^{(l)}(\mathbf{x}, \mathbf{y})$ will not be merely the projection of $\psi^{(k)}(\mathbf{x}, \mathbf{y})$ on l variable dimensions, it may very well be an entirely different function, " adapted" specifically to space $\mathfrak{M}^{(l)}$. $G^{(l)}(\mathbf{a}, \mathbf{b})$ then may very well be smaller than $G^{(k)}(\mathbf{a}, \mathbf{b})$, for some or all stimulus pairs. Even if this is not the case, however, and $\psi^{(l)}(\mathbf{x}, \mathbf{y})$ happens to equal $\psi^{(k)}(\mathbf{x}, \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathfrak{M}^{(l)}$, one can still view Fechnerian distances $G^{(l)}(\mathbf{a}, \mathbf{b})$ as reflecting "true" perceptual dissimilarities among stimuli in $\mathfrak{M}^{(l)}$, with these dissimilarities being generally different from those when the same stimuli are viewed in the context of a larger stimulus space.
3. "*Representative design*" *approach*. Another approach is to consider the possibility that any given type of stimuli is associated with a "representative design" (borrowing the term from Brunswick, 1956), a space $\mathfrak{M}^{(l)}$ such that for any $\mathbf{a}, \mathbf{b} \in \mathfrak{M}^{(l)}$ and any $\mathfrak{M}^{(k)}$ that contains $\mathfrak{M}^{(l)}$ as its subspace, if the Fechnerian distance between $\mathbf{a}$ and $\mathbf{b}$ computed in $\mathfrak{M}^{(k)}$ is $G(\mathbf{a}, \mathbf{b})$, then there are piecewise smooth closed loops lying entirely in $\mathfrak{M}^{(l)}$ whose psychometric lengths converge to $G(\mathbf{a}, \mathbf{b})$. In other words, there might be a "correctly chosen" stimulus space $\mathfrak{M}^{(l)}$, in the sense that if immersed in a larger space, the Fechnerian distance between any two of its points does not change (because $\mathfrak{M}^{(l)}$ contains all the loops whose psychometric lengths converge to the Fechnerian distance between any two of its points). This happens, for example, when subspace $\mathfrak{M}^{(l)}$ is *perceptually separable* from the complementary subspace $\mathfrak{M}^{(k-l)}$ of a larger space $\mathfrak{M}^{(k)}$, in the sense of Dzhafarov (2002c, 2003c). In the case when for every pair of points $\mathbf{a}, \mathbf{b} \in \mathfrak{M}^{(k)}$ one can find a geodesic loop, the piecewise smooth loop containing $\mathbf{a}$ and $\mathbf{b}$ whose psychometric length equals $G(\mathbf{a}, \mathbf{b})$, the representative design can be defined even simpler: a stimulus space that with every pair of stimuli it contains also contains the geodesic loop between them. It is noteworthy that the language of Fechnerian scaling allows one to define Brunswick's notion of "representative design" in rigorous mathematical terms.

This discussion is trivially generalizable to arbitrary arcwise connected stimulus spaces $\mathfrak{M}$ and their subspaces (i.e., subsets of $\mathfrak{M}$ satisfying the same axioms), as well as to still more general spaces. The choice among the three approaches just mentioned remains an open issue.

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