7 Empirical Meaningfulness, Measurement-Dependent Constants, and Dimensional Analysis

Ehtibar N. Dzhafarov
University of Illinois at Urbana–Champaign

ABSTRACT

The "empirical meaningfulness" analysis in theory of measurement imposes a priori restrictions on statements involving a given set of quantities, by striking down as "empirically meaningless" those of logically possible statements whose truth value (true or false) is not invariant under mutual substitutions of "admissible" measurements of the quantities involved. However, any logically unambiguous statement that is "empirically meaningless" by this invariance criterion can be equivalently reformulated to become, by the same criterion, "empirically meaningful." This is achieved by explicating in the statement all its measurement-dependent constants, whose numerical values covary with choices of measurements within a specified class. Provided that the basic, nonderivable laws of a given area (such as mechanics or psychophysics) can be formulated in some specific measurements (such as mass in grams or in absolute threshold units), a simple algorithm described in this chapter determines the set of measurement-dependent constants that ensure the invariance of these basic laws under any specified class of transformations of these measurements: the choice of this class, and thereby of the measurement-dependent constants, is subject to no substantive constraints. The only context in which the invariance considerations may be restrictive is that of deciding whether a given statement is logically derivable from a given list of basic laws: if it is, then one should be able to make it invariant under the same class of transformations with the aid of the same set of measurement-dependent constants. Dimensional analysis in physics, for instance, can determine that a statement is not derivable from a given set of physical laws (such as the gravitation law and the second law of motion) by demonstrating that it cannot be made dimensionally homogeneous (invariant under scaling transformations) if one only utilizes the dimensional constants that have been explicated in these basic laws themselves, when presenting them in a dimensionally homogeneous form. Outside the context of derivability, however, the requirement of dimensional homogeneity does not restrict the class of possible laws of physics, as their dimensional homogeneity can always be achieved by an appropriate choice of dimensional constants.

INTRODUCTION

The main points of this chapter can be summarized as follows.

(1) Any grammatically correct sentence (statement, law) considered "empirically meaningless" by traditional criteria (based on the idea of *invariance under mutual substitutions of measurement functions from a given class of functions*) is logically *equivalent* to a sentence that, by the same criteria, is "empirically meaningful." The difference therefore is in *form* rather than *content,* having nothing to do with empirical truth or falsity. By a universal algorithm called *covariant substitution* any logically complete sentence can be (re)formulated in an "empirically meaningful" form: this is achieved by *explicating,* within the sentence, all its *measurement-dependent constants,* whose values covary with choices of measurement functions. Confusions and ambiguities in the content of a sentence may only arise when its formulation is *logically incomplete,* and they can always be resolved by standard logical analysis. Once a sentence is logically complete, it can be classified as *logically false* (i.e., stating something that can be shown to be false by mathematical means, like the sentence "this length is 5, in all possible units"), *logically true* (like the sentence "this length is 5, in some units"), or *empirical* ("this length is 5 m").

(2) The key concept in the analysis of sentences involving measurement functions is that of the measurement-dependent constants. In physics all or most of measurement-dependent constants are commonly known *dimensional constants*. By appropriate choice of those, *any* physical sentence can be written in a *dimensionally homogeneous* form, which is the physics version of "empirical meaningfulness." This is done by what I call Bridgman's algorithm, a particular case of covariant substitutuion. Dimensional analysis is an algebraic technique determining whether and how a sentence containing a given list of variables can be written in a dimensionally homogeneous form *using only those dimensional constants that have been explicated (by Bridgman's algorithm) in other, more basic sentences, from which the sentence in question is assumed to be logically derivable*. The sentence is determined to be *nonderivable* from these basic sentences when the list of their dimensional constants is not sufficient to write this sentence in a homogeneous form. Outside the context of derivability, dimensional analysis does not and cannot tell which sentences may and which may not be empirically true, which is why dimensional considerations cannot impose any restrictions upon possible *fundamental laws* of an area (such as mechanics, material science, or psychophysics), or upon its *situational sentences,* describing

specific circumstances to which the fundamental laws are to be applied to derive a given sentence.

(3) A decision on "empirical meaningfulness" or "meaninglessness" of a sentence containing measurement functions cannot be based on the *automorphisms of the empirical operations* used in constructing these measurement functions. Insofar as we are able to *empirically distinguish* between different measurement functions of one and the same quantity, at least one of these measurement functions must be defined *empirically*—by a set of *nonumerical* (or prenumerical) relations. A specific measurement function, such as length in meters, cannot be, however, defined within a relational system that contains automorphisms other than identity (e.g., the traditional system for length, with order and concatenation): in such a system no *ostensive* proposition can be formulated for its elements, which means that no specific length can be identified as "the length such that [. . . empirical relations involving this length. . .]." To *empirically* define a specific measure, one should be able to *empirically* ("qualitatively") refer to a specific length, which means that additional empirical relations should be appended to those already in the system, reducing the group of its automorphisms to identity. It is only an artefact of legitimate but arbitrary formalization (axiomatization) choices that some empirical relations (e.g., order and averaging in constructing an "interval scale" for temperature) are explicitly included in the *formal system*, whereas other relations (such as, "to be below the temperature of freezing water") are not, being instead used as elements of an *interpretation* of the formal system. One can always formalize a relational system so that its only automorphism is identity, forming thereby an *empirically complete relational system*. *Empirically incomplete relational systems*, those with nontrivial automorphism groups, can always be viewed as *groups* of complete systems.

The views presented here considerably overlap with those of three authors: P. W. Bridgman (1922) on dimensional analysis in physics, J. Michell (1986, 1990) on what can be loosely characterized as the emphasis on logical derivability, and W. W. Rozeboom (1962) on the critical importance of measurement-dependent constants in formulation of scientific propositions.

A BIT OF INFORMAL LOGIC: VARIABLES, CONSTANTS, AND SENTENCES

I begin by mentioning, without elaboration, some general logical concepts used throughout this paper. A *sentential function* (or *predicate*), such as "$x + f(y) = 6$, where $x, y \in Re$ and $f \in F$," is a (grammatically correct) formulation relating *constants* and *variables*. All variables considered here are either *variable functions* (in our example, f) or *variable quantities* (x and y). Variables of both kinds assume their *values* in certain sets ($x, y \in Re, f \in F$; note that the value of a variable function is a fixed function). All nonvariable terms in a sentential function are constants: fixed quantities (in our example, 6), fixed sets (*Re*, *F*), fixed relations and functions (. . . + . . . =), and fixed logical terms ("where," "and"). In this chapter, however, the term "constant" is used exclusively to designate fixed numbers.

A sentential function becomes a *sentence* (or *statement*) when all its variables are bound by *logical quantifiers,* such as "for all," "for some," "for precisely three," etc., referring to the variables' possible values. Sentences, but not sentential functions, can be evaluated in terms of their *truth values:* TRUE or FALSE. The following two sentences are obtained by quantifying, in two different ways, the sentential function above: (1) "for all x, $y \in Re$ and all $f \in F$, $x + f(y) = 6$" and (2) "for all x, $y \in Re$ there is an $f \in F$, such that $x + f(y) = 6$." If *Re* designates the class of reals, and F the class of positive linear functions, then the first sentence is false, the second is true.

Using variable functions and, especially, binding them by quantifiers "for any function," "there exists a function," etc., is not as common as doing the same with variable numbers. The main reason for this is that in many cases (though not always) a variable function can be *parametrized,* i.e., expressed through a *fixed function* of several *variable numbers*. The class of positive linear functions F, for example, can be parametrized by conventional coefficients $a \in Re^+$ (positive reals) and $b \in Re$, so that the sentential function above can be written as "$x + (ay + b) = 6$, where x, y, $b \in Re$ and $a \in Re^+$." The two sentences previously formed then become (1) "for all x, $y \in Re$, all $b \in Re$, and all $a \in Re^+$, $x + (ay + b) = 6$," and (2) "for all x, $y \in Re$ there are $b \in Re$ and $a \in Re^+$ such that $x + (ay + b) = 6$."

The order in which different quantifiers enter in a sentence is important in determining which variable numbers (or functions) depend on which. In the sentence "for all x, $y \in Re$ there is an $f \in F$, such that $x + f(y) = 6$" the choice of function f depends on the values of x, y, whereas in the sentence "there is an $f \in F$ (such that) for all x, $y \in Re : x + f(y) = 6$" it does not. In this chapter whenever it is not obvious or immaterial, interdependences between different variables and functions will be explicitly stated ("an $f \in F$ depending on x and y") or indicated by subscripts: "for all x, $y \in Re$ there is an $f_{xy} \in F$ such that $x + f_{xy}(y) = 6$," or "for all x, $y \in Re$ there are $b_{xy} \in Re$ and $a_{xy} \in Re^+$, such that $x + (a_{xy}y + b_{xy}) = 6$".

Since explicitly written quantifiers make formulations cumbersome, the usual convention is that if a variable number (or a variable function) is not explicitly bound by quantifiers, it is treated as bound by the generality quantifier "for all," referring to the class of the variable's possible values. I will only use this *generality convention* with respect to those variables that are not focal for the analysis. If, as it will usually be the case, the analysis focuses on variable functions (or their parameters), then our two example sentences may be written as (1) "for all $b \in Re$ and all $a \in Re^+$, $x + (ay + b) = 6$," and (2) "there are b_{xy}

$\in Re$ and $a_{xy} \in Re^+$, such that $x + (a_{xy}y + b_{xy}) = 6$" (the missing quantifiers are "for all x, $y \in Re$").

In this paper, sentences whose truth value (TRUE or FALSE) can be ascertained by logical (mathematical) means only, are called *logical sentences* (they are either *logically true* or *logically false*). Valid mathematical theorems are logical (and logically true) in this sense, and both example sentences above are logical, one being logically true, another logically false. Sentences that are not logical are called *empirical* (being *empirically true* or *empirically false*). If $\mathcal{A}$ denotes the class of all geometric points that can be placed on this page, then the following sentence is empirical (and probably empirically true): "for all $\mathscr{x}, \mathscr{y} \in \mathcal{A}$, distance between $\mathscr{x}$ and $\mathscr{y}$ in meters < 6." Note, however, that this sentence is empirical not because it refers to an empirical operation (measurement of distance in meters), but because its truth or falsity cannot be ascertained by purely logical or mathematical means. The sentence "there are $\mathscr{x}, \mathscr{y} \in \mathcal{A}$, such that distance between $\mathscr{x}$ and $\mathscr{y}$ in meters < 6" is logical, and logically true (because one can always find such pairs $\mathscr{x}, \mathscr{y} \in \mathcal{A}$, namely, $\mathscr{x} = \mathscr{y}$, for which the distance in meters is zero).

This following comment might seem superfluous, but is essential in the present context: all functions entering sentences under consideration are assumed to be *well defined*. This means that when one says $\log_e(x)$ there is an *effective mathematical procedure* to compute $\log_e(x)$ given x from its domain; when one says "numerical value t of temperature $\mathscr{t}$ in degrees Celsius," there is an *effective empirical procedure* that allows one, given temperature $\mathscr{t}$, to arrive at the number t. "Effective," here, means performable in a countable (generally infinite) number of steps defined by induction.

MEASUREMENT FUNCTIONS

The example with "numerical value t of temperature $\mathscr{t}$ in degrees Celsius," is that of a *measurement function* (MF). In general, a MF $x = x(\mathscr{x})$ is an effective empirical procedure by which a number x is assigned to any "instance" (or "magnitude") of an "empirical quantity" $\mathscr{x}$. Avoiding philosophical discussions, I will assume that the meanings of the terms "quantity" (like mass) and "magnitude" thereof (a value of mass) are understood. Different MFs measuring one and the same quantity are defined by their mathematical relations to each other, *conversion functions*, usually forming an N-parametric mathematical group of strictly increasing functions. The class of the "interval-scale" temperature MFs, for example (call this class TEMP), satisfies the following proposition: if MF $t(\mathscr{t})$ $\in$ TEMP, then for any real b and any positive real a, $t'(\mathscr{t}) = [at(\mathscr{t}) + b] \in$ TEMP. Here, the conversion functions form a two-parametric group of positive linear transformations.

Obviously, the conversion functions associated with a given class of MFs do not effectively define this class, unless at least one of the MFs is defined by some independent empirical procedure (an *anchoring* MF, e.g., temperature in degrees Celsius), allowing one to compute its values directly from an empirical quantity, rather than from other MFs for this quantity. Thus, an effective definition of the class TEMP of temperature MFs would be: $t(t) \in$ TEMP, if and only if for some real b and some positive real a, $at(t) + b = \bar{t}(t)$, where $\bar{t}(t)$ is the anchoring MF (the description of the empirical procedure follows).

Now we are ready to form various sentences involving temperature MFs. The most celebrated example of "empirical meaninglessness," given in virtually any treatise on the subject, is of the form

$$\text{if } \mathrm{B}(t_1, t_2) \text{ then } t(t_1)/t(t_2) = 2, \tag{7.1}$$

where $\mathrm{B}(t_1, t_2)$ can be replaced by some nonnumerical relation involving t_1 and t_2, such as "t_1 and t_2 are temperature magnitudes of objects $\mathbb{O}_1$ and $\mathbb{O}_2$, respectively [an identification of $\mathbb{O}_1$ and $\mathbb{O}_2$ in nontemperature terms follows]." Sentential function 7.1 is not, strictly speaking, a sentence, and it can only be considered a sentence under the convention of generality (free variables treated as bound by generality quantifiers). Then the explicit form of this sentence is

$$\begin{gathered}\text{for } \textit{all} \text{ MFs } t \in \text{TEMP and } \textit{all } t_1, t_2\text{:}\\ \text{if } \mathrm{B}(t_1, t_2) \text{ then } t(t_1)/t(t_2) = 2.\end{gathered} \tag{7.2}$$

This sentence can be trivially shown to be false by reductio ad absurdum, so in the terminology of this chapter the sentence is logical, and its truth value is FALSE. No empirical knowledge of temperature MFs or of the empirical relation $\mathrm{B}(t_1, t_2)$ is involved in this derivation; the latter is based exclusively on the specific combination of conversion functions with logical quantifiers (which is why the sentence is logical). The key words here are "specific combination of . . . with logical quantifiers," as can be seen from the fact that the following sentence is *logically true:*

$$\begin{gathered}\text{for } \textit{all } t_1, t_2 \textit{ there is} \text{ a MF } t \in \text{TEMP (depending on } t_1, t_2\text{) such that}\\ \text{if } \mathrm{B}(t_1, t_2) \text{ then } t(t_1)/t(t_2) = 2.\end{gathered} \tag{7.3}$$

Since 7.3 and 7.2 include one and the same sentential function, there can be nothing illegitimate in computing ratios of the temperature MFs per se; it is simply that some sentences about such ratios turn out to be logically false. The following sentence is yet another way of quantifying sentential function 7.1:

$$\begin{gathered}\textit{there is} \text{ a MF } t \in \text{TEMP such that for } \textit{all } t_1, t_2\text{:}\\ \text{if } \mathrm{B}(t_1, t_2) \text{ then } t(t_1)/t(t_2) = 2.\end{gathered} \tag{7.4}$$

This sentence is empirical, with its truth value depending on $\mathrm{B}(t_1, t_2)$. It would be (empirically) true, for instance, if $\mathrm{B}(t_1, t_2)$ describes such temperature pairs (t_1, t_2) that when water of temperature t_1 is mixed with equal amount of freezing water without heat loss, the eventual temperature of the mixture is t_2. For this $\mathrm{B}(t_1, t_2)$, temperature in degrees Celsius provides one possible solution.

With one important exception, considered in the next section, *all* examples of "empirically meaningless" propositions given in the literature (e.g., Falmagne & Narens, 1983; Pfanzagl, 1968; Roberts, 1979; Suppes, 1959; Suppes & Zinnes, 1963) are simply logically false sentences. In all such examples a sentential function, like 7.1, is *implicitly* interpreted under the generality convention, like 7.2, and the resulting sentence is shown to be false by reductio ad absurdum. Then, however, this logical falsity is attributed to the "inadmissibility" of the operations with MFs contained in the sentence, rather than to the logical structure of the sentence (and most importantly, its logical quantification). There is no big harm in calling things differently, and mental translation of "empirical meaninglessness" into logical falsity is not a demanding exercise. This terminology, however, is potentially misleading, because it might suggest that some empirical considerations, in addition to simple logical principles, are being involved—when in fact they are not.

MEASUREMENT-SPECIFIC SENTENCES AND INVARIANCE UNDER SUBSTITUTIONS OF MFS

I mentioned that there was an important exception to the rule that all "empirically meaningless" sentences are simply logically false when formulated unambiguously. This exception relates to *measurement-function-specific* (MF-specific) sentences, those referring to uniquely specified MFs (such as length in meters or Celsius temperature). Returning to the putative ratio of temperature MFs (from the class TEMP), an example might be (I begin using the generality convention here and omit the quantifiers "for all t_1, t_2")

$$\text{if } \mathrm{B}(t_1, t_2) \text{ then } \bar{\tau}(t_1)/\bar{\tau}(t_2) = 2, \tag{7.5}$$

where $\bar{\tau}(t)$ denotes a specific temperature MF belonging to TEMP, say Celsius temperature. To be well defined, this MF should either be anchoring itself (i.e., defined through an effective empirical procedure), or be reducible to an anchoring MF by a conversion $a\bar{\tau}(t) + b$. Assume for simplicity that $\bar{\tau}(t)$ (Celsius temperature) is an anchoring MF of the class TEMP.

According to the position considered (Falmagne, 1992; Falmagne & Narens, 1983; Narens & Mausfeld, 1992), sentence 7.5 is "empirically meaningless" because its truth value is not preserved under direct replacements of $\bar{\tau}(t)$ by other MFs from the class TEMP. Indeed, if one substitutes Fahrenheit temperature $\bar{\bar{\tau}}(t)$ for $\bar{\tau}(t)$, then the sentence

$$\text{if } \mathrm{B}(t_1, t_2) \text{ then } \bar{\bar{\tau}}(t_1)/\bar{\bar{\tau}}(t_2) = 2, \tag{7.6}$$

cannot be true if 7.5 is true (and vice versa). Now we have a serious discrepancy between "empirical meaninglessness" and logical falsity: both 7.5 and 7.6 are empirical sentences, and one of them may very well be empirically true. This approach has been criticized in the literature by pointing out its logical arbitrariness (Guttman, 1971; Michel, 1986, 1990). My analysis goes one step farther: I will show that insofar as the *content* (rather than a specific *form*) of a MF-specific sentence is concerned, the substitution criterion is not restrictive at all—*any such sentence can be equivalently reformulated so that its truth value will be preserved under substitutions of MFs within any class of MFs, however broad or arbitrary.* The dichotomy of form and content is not as vague or philosophical as it may appear. I call a characterization of a sentence *content related* if, and only if, whenever it holds for the sentence, it also holds for all sentences logically equivalent to it; if a characterization holds for a sentence but does not hold for at least one of its equivalents, then the characterization is *form related.* Logical truth value (TRUE or FALSE) is content related, by definition, and so are logical derivability and informal characterizations like "profound," or "interesting." "Empirical meaningfulness," by contrast, is only form related.

A systematic demonstration of this claim for sentence 7.5 involves two steps. First, we construct a sentence equivalent to 7.5 but referring to the *entire* class TEMP of MFs:

$$\begin{gathered}\text{for any MF } t \in \text{TEMP there is a real number } c_t \text{ such that}\\ \textit{if } \mathrm{B}(t_1, t_2) \text{ then } \frac{t(t_1) - c_t}{t(t_2) - c_t} = 2,\\ \text{where } c_t = 0 \text{ when } t \text{ is } \bar{t} \text{ (Celsius temperature).}\end{gathered} \tag{7.7}$$

It is easy to verify that this sentence has precisely the same truth value as 7.5, that is, they are indeed logically equivalent (interdeducible). For the moment I will leave open the question of what is the general algorithm by which this sentence is derived from 7.5. The second step consists in constructing direct logical *specializations* of this general sentence to specific MFs, such as Celsius $\bar{t}(t)$ and Fahrenheit $\bar{\bar{t}}(t)$:

$$\text{if } \mathrm{B}(t_1, t_2) \text{ then } \frac{\bar{t}(t_1) - 0}{\bar{t}(t_2) - 0} = 2 \quad \text{and} \quad \text{if } \mathrm{B}(t_1, t_2) \text{ then } \frac{\bar{\bar{t}}(t_1) - 32}{\bar{\bar{t}}(t_2) - 32} = 2. \tag{7.8}$$

(One could even attach subscripts $\bar{t}$ and $\bar{\bar{t}}$ to 0 and 32, respectively, to indicate that they are "measured in" °C and °F.) These two sentences are both equivalent to 7.5, they are both equivalent to 7.7 of which they are specializations, and they have a common "form" *up to a measurement-function-dependent* (MF-dependent) constant c_t. This constant is a mathematical function (or "reduction") of the conversion coefficients a, b that define the class TEMP, $c_t = -b/a$, and its sole purpose is to ensure the generalizability of 7.5 to the entire class TEMP.

One can clearly see now that the "empirical meaninglessness" of 7.5 is due to the fact that the constant 0, subtracted from the Celsius temperature values, has been overlooked, and the question of whether this constant is or is not MF dependent has not been raised. De facto the decision has been (unknowingly) made in favor of the MF independence of 0, resulting in the unjustifiable replacement of 7.5 with 7.6. To appreciate the peculiarity of the situation, note that this mistake would probably be avoided if the initial observations that led to 7.5 were made in °F rather than °C. Then the initial sentence would have the form of 7.8 (right), rather than 7.5, and it would be easy to realize that 32 is "measured in °F" and hence must change when we switch to other MFs.

The concept of a MF-dependent constant generalizes the familiar notion of a *dimensional constant*. It is a well-known fact that physical laws preserve their truth values (and their "forms") under changes of measurement units only if the numerical values of all dimensional constants in these laws are changed "correspondingly." In the abstract measurement literature, sentences containing MF-dependent constants were considered by Pfanzagl (1968) under the name of "meaningfully parametrized relations." The notion seems to have escaped the attention of later writers (e.g., Roberts, 1979, pp. 79–80, treats dimensional constants as essentially a nuisance for otherwise straightforward theory), but "meaningfully parametrized relations" have been reintroduced in Luce, Krantz, Suppes, and Tversky (1990, chap. 22). Using this language, the point of this section, systematically developed in the next one, is that any sentence can be "meaningfully parametrized," with respect to any class of MFs that includes those contained in the sentence. "Meaningful parametrization" of a sentence, therefore, is logically ensured and has nothing to do with its empirical content.

COVARIANT SUBSTITUTION

A MF-specific sentence, like 7.5, can be rewritten in an infinity of equivalent forms, containing different numerical constants. For example, every occurrence of $\bar{t}(t)$ in 7.5 can be replaced with $1 \cdot \bar{t}(t)^1 - 0$. How can one know which of these constants are and which are not MF dependent? Is the choice determined by a substantive theory of temperature? Is it guaranteed that MF-dependent constants can always be found, allowing one to rewrite sentences like 7.5 in a measurement-function-class (MF-class) form like 7.7? Precisely how do the MF-dependent constants, if found, covary with MFs within a given class, like TEMP? Are there any restrictions on the possible classes of MFs? The answers to all such questions are contained in the algorithm for the procedure I call *covariant substitution*. The essence of the algorithm is this. Let $S(\bar{x}(\mathscr{x}))$ be an arbitrary MF-specific sentence, referring to a well-defined MF (or a vector of MFs) $\bar{x}(\mathscr{x})$. Let all explicit numerical constants in this sentence be treated as "pure numbers." Let X be an arbitrary class of MFs that contains $\bar{x}(\mathscr{x})$, such that any MF $x(\mathscr{x}) \in X$ can be converted into $\bar{x}(\mathscr{x})$ by a one-to-one parametrizable conversion function $\bar{x} = f(x)$. The parameters $\mathbf{C}$ of the conversion functions are called *conversion* *coefficients*. The algorithm shows how one constructs from $S(\bar{x}(\mathscr{x}))$ a MF-class sentence (referring to *all* MFs from X) of the form

$$\text{for any } x(\mathscr{x}) \in X \text{ there is a vector } \mathbf{c} \text{ (depending on } x\text{) such that} \\ S^*(x(\mathscr{x}), \mathbf{c}), \\ \text{where } \mathbf{c} = \mathbf{c_0} \text{ when } x \text{ is } \bar{x}. \tag{7.9}$$

The components of vector $\mathbf{c}$ are MF-dependent constants: the algorithm shows that they are functions (or "reduced forms") of the conversion coefficients: $\mathbf{c} = \mathbf{c}(\mathbf{C})$. The specialization of 7.9 back to the original MF $\bar{x}(\mathscr{x})$ yields the sentence $S^*(\bar{x}(\mathscr{x}), \mathbf{c_0})$. This sentence is identical with the original $S(\bar{x}(\mathscr{x}))$, except that now all MF-dependent constants in it have been *explicated*. A specialization to another MF $\bar{\bar{x}}(\mathscr{x})$ from X will have the form $S^*(\bar{\bar{x}}(\mathscr{x}), \mathbf{c_1})$: not only $\bar{\bar{x}}(\mathscr{x})$ substitutes for $\bar{x}(\mathscr{x})$ in the original sentence, but also all MF-dependent constants change their numerical values "accordingly."

The algorithm is as follows.

Step 1. Formulate the MF-specific sentence, $S(\bar{x}(\mathscr{x}))$. Take $\bar{x}(\mathscr{x})$ to be the anchoring MF(s). Treat all explicit numerical constants in $S(\bar{x}(\mathscr{x}))$ as "pure numbers."

Illustration. Let mass $\mathscr{m}$, distance $\mathscr{l}$, and force $\mathscr{f}$ be measured in well-defined specific measures $\bar{m}(\mathscr{m})$, $\bar{l}(\mathscr{l})$, and $\bar{f}(\mathscr{f})$, say, kg-m-N. Then the following is a MF-specific form of Newton's law:

$$\mathrm{B}(\mathscr{m}_1, \mathscr{m}_2, \mathscr{l}, \mathscr{f}) \quad \text{iff} \quad \frac{\bar{m}_1\,\bar{m}_2}{\bar{f}\,\bar{l}^2} = \Gamma \qquad (\text{where } \Gamma^{-1} \approx 6.673 \cdot 10^{-11}). \tag{7.10}$$

The two explicit numerical constants (Γ and exponent 2) are treated as "pure numbers." [Hereafter, I omit arguments of MFs in all examples, writing m or $\bar{m}$ instead of the explicit $m(\mathscr{m})$ and $\bar{m}(\mathscr{m})$. Note that different subscripts, like in $\bar{m}_1$ and $\bar{m}_2$, refer to different arguments, rather than different MFs: $\bar{m}_1 = \bar{m}(\mathscr{m}_1)$ and $\bar{m}_2 = \bar{m}(\mathscr{m}_2)$.]

Step 2. Define the class(es) X of MFs related to $\bar{x}(\mathscr{x})$ through a certain (parametrizable) group of conversion functions: $x(\mathscr{x}) \in X$ if and only if for some vector of constants $\mathbf{C}$, $\bar{x}(\mathscr{x}) = g(x(\mathscr{x}), \mathbf{C})$. Constants $\mathbf{C}$ are conversion coefficients.

Illustration. Consider the following class MASS* of mass MFs: $m(\mathscr{x}) \in$ MASS* if and only if $\bar{m} = c_m m^{a_m}$, for some positive constants c_m and a_m. In traditional terminology, mass is measured "on a log-interval scale" (Stevens, 1974). Classes LENGTH* and FORCE* are defined analogously, with conversion functions $\bar{l} = c_l l^{a_l}$ and $\bar{f} = c_f f^{a_f}$, respectively.

Step 3. Substitute $g(x(\mathscr{x}), \mathbf{C})$ for $\bar{x}(\mathscr{x})$ in $S(\bar{x}(\mathscr{x}))$ and simplify the expression algebraically to reduce the number of constants to a minimum. Denote the resulting vector of constants by $\mathbf{c}$: these are MF-dependent constants. Express $\mathbf{c}$ as a function of conversion coefficients: $\mathbf{c} = \mathbf{c}(\mathbf{C})$.

Illustration. By substitution, algebra, and renaming,

$$\mathrm{B}(\mathscr{m}_1, \mathscr{m}_2, \mathscr{l}, \mathscr{f}) \quad \text{iff} \quad \frac{[c_m m_1^{a_m}][c_m m_2^{a_m}]}{[c_f f^{a_f}][c_l l^{a_l}]^2} = G_{lmf} \frac{(m_1 m_2)^{a_m}}{f^{a_f} l^{2a_l}} = \Gamma. \quad (7.11)$$

The MF-dependent constants **c** here are (G_{lmf}, a_l, a_m, a_f), expressed through the conversion coefficients $\mathbf{C} = (c_l, c_m, c_f, a_l, a_m, a_f)$ as $(G_{lmf}, a_l, a_m, a_f) = (c_l^{-2} c_m^2 c_f^{-1}, a_l, a_m, a_f)$.

Step 4. The proposition constructed at Step 3 is a sentential function, not a sentence. To make it a sentence, first, affix to it the quantifier(s) "for any $x \in X$ there are constants **c** (depending on x)." Next, determine conversion coefficients $\mathbf{C_0}$ in the equation $g(\bar{x}(\mathscr{x}), \mathbf{C_0}) = \bar{x}(\mathscr{x})$, and compute $\mathbf{c_0} = \mathbf{c}(\mathbf{C_0})$: these are the values of **c** when $x = \bar{x}$. Suffix the specifying proposition "where $\mathbf{c} = \mathbf{c_0}$ when $x = \bar{x}$" to the sentence. This is the resulting MF-class sentence, 7.9, containing MF-dependent constants **c** and their special values $\mathbf{c_0}$ for the original MF $\bar{x}(\mathscr{x})$. This sentence is logically equivalent to (interdeducible with) the original sentence $S(\bar{x}(\mathscr{x}))$.

Illustration. For LENGTH*-MASS*-FORCE* classes the resulting MF-class sentence, logically equivalent to 7.10, is

for any $l \in$ LENGTH*, $m \in$ MASS*, $f \in$ FORCE*
there are positive reals G_{lmf}, a_l, a_m, a_f, such that

$$\mathrm{B}(\mathscr{m}_1, \mathscr{m}_2, \mathscr{l}, \mathscr{f}) \quad \text{iff} \quad G_{lmf} \frac{(m_1 m_2)^{a_m}}{f^{a_f} l^{2a_l}} = \Gamma$$

where $G_{lmf} = a_l = a_m = a_f = 1$ when (l, m, f) is $(\bar{l}, \bar{m}, \bar{f})$. (7.12)

(If one omits the anchoring proposition "where $\mathbf{c} = \mathbf{c_0}$ when $x = \bar{x}$" from the sentence, the result is still a valid MF-class generalization, only it will now be logically weaker than, rather than equivalent to, the MF-specific sentence it is derived from.)

Step 5. To specialize the MF-class sentence 7.9 to any MF $\bar{\bar{x}}(\mathscr{x})$ from X, compute $\mathbf{c_1}$ as $\mathbf{c}(\mathbf{C_1})$, where $\mathbf{C_1}$ satisfies $g(\bar{\bar{x}}(\mathscr{x}), \mathbf{C_1}) = \bar{x}(\mathscr{x})$, and substitute $\bar{\bar{x}}(\mathscr{x})$ and $\mathbf{c_1}$ for $x(\mathscr{x})$ and **c**, respectively, in the sentential function $S^*(x(\mathscr{x}), \mathbf{c})$ of 7.9. (I omit an illustration since it is obvious.) This concludes the algorithm for covariant substitution.

It is clear now why the "direct substitution criterion" of meaningfulness does not work for MF-specific sentences: a correct substitution of one MF for another should be preceded by the explication of all MF-dependent constants and followed by changing their numerical values. When this is done, however, the substitution criterion becomes expressly nonrestrictive: *any* MF can be generalized to *any* class of MFs and thereby substituted for by *any* other MF from this class. To emphasize this fact, the classes of MFs for mass-length-force in our illustration have been chosen broader than the traditional "ratio scales" (obtained by putting $a_l = a_m = a_f = 1$). Perhaps the most important characteristic of MF-dependent constants is that they are merely mathematical reductions, $\mathbf{c} = \mathbf{c}(\mathbf{C})$, of conversion coefficients **C** that define the class X of MFs to which a given MF-specific sentence is being generalized. MF-dependent constants, therefore, have no substantive ("qualitative") meaning: they are not given by the theory of the quantities being measured, and their sole purpose is to ensure the generalizability of sentences from specific MFs to classes of MFs. The conversion coefficients **C**, obviously, vary from one class of MFs to another, and so do MF-dependent constants $\mathbf{c} = \mathbf{c}(\mathbf{C})$; in addition, for a given class of MFs, the MF-dependent constants will generally vary with the MF-specific sentence to be generalized.

DIMENSIONAL ANALYSIS IN PHYSICS: DERIVABILITY-FROM

Luce (1978) suggested that dimensional analysis in physics is a particular case of the "empirical meaningfulness" analysis. To the extent one subscribes to this point of view, my previous analysis should leave one wondering: if "empirical meaningfulness" either is a terminological replacement for logical falsity, or has nothing to do with the empirical content of a sentence altogether, why is then dimensional analysis clearly sound and useful?

The point of resemblance between dimensional analysis and the "empirical meaningfulness" analysis lies in the fact that usually physical sentences are written in a *dimensionally homogeneous* form. This means that a complete logical formulation of a physical sentence has the form of sentence 7.9, in which the classes X of MFs for all physical quantities $\mathscr{x}$ are defined as follows: $x(\mathscr{x}) \in X$ if and only if $Cx(\mathscr{x}) = \bar{x}(\mathscr{x})$ for some positive real C and some anchoring MF $\bar{x}(\mathscr{x})$ (defined by an effective empirical procedure). The conversion coefficients C are referred to as conversion *factors,* and MF-dependent constants c are called *dimensional* constants. The similarity conversions are rarely mentioned explicitly, due to their universal use, and the affix proposition (the first line) of sentence 7.9 is usually omitted. One simply says then that physical sentences hold "for all units of measurement" (provided one remembers that all dimensional constants are *unit covariant*). This is taken in the representational theory of measurement to constitute the essence of the restrictive power of dimensional analysis. According to this position, dimensional analysis operates by striking down formulations that are not dimensionally homogeneous (in Pfanzagl's terms, are not "meaningfully parametrized") and thereby cannot be "true laws of physics;" the main problem to be solved, therefore, is *why* physical sentences are dimensionally homogeneous (Krantz, Luce, Suppes, & Tversky, 1971, chap. 10). In some form or another this idea permeates the whole issue of "empirical meaningfulness" from its outset. This is certainly what Suppes (1959) meant by referring to "systematic language of physics" and quoting from Newton's *Principia,* or what Falmagne (1992) meant by saying that "only meaningful statements have reached posterity" and quoting from Galileo's *Dialogues.*

It must be clear from the algorithm for covariant substitution that *any sentence* *can be written in a dimensionally homogeneous form, by introducing appropriate dimensional constants:* homogeneity or inhomogeneity of the form of a sentence, therefore, has nothing to do with its content and cannot serve as a selection criterion for "possible laws." This is clearly explained in Bridgman's classical book (1922), and I will refer to covariant substitution restricted to similarity conversions as *Bridgman's algorithm.* As an example, consider the traditional classes MASS, LENGTH, TIME, FORCE of MFs m, l, t, f, related to the respective anchoring functions by positive similarities: $\bar{m} = c_m m$, $\bar{l} = c_l l$, $\bar{t} = c_t t$, $\bar{f} = c_f f$. The following two sentences are MF-specific versions of the gravitation law and the second law of motion:

$$\mathrm{B}^{\mathrm{GRV}}(m_1, m_2, l, f) \quad \text{iff} \quad \frac{\bar{m}_1 \bar{m}_2}{\bar{f}\,\bar{l}^2} = \Gamma;$$

$$\mathrm{B}^{\mathrm{MTN}}(m, l, t, f) \quad \text{iff} \quad \frac{\bar{m}}{\bar{f}} \cdot \frac{\mathrm{d}^2 \bar{l}}{\mathrm{d}\bar{t}^2} = 1. \tag{7.13}$$

By Bridgman's algorithm these sentences are generalized to the conventional MF classes as

$$\text{for any } l \in \text{LENGTH},\ t \in \text{TIME},\ m \in \text{MASS},\ f \in \text{FORCE}$$
$$\text{there are positive reals } G_{lmf},\ A_{ltmf}, \text{ such that}$$

$$\mathrm{B}^{\mathrm{GRV}}(m_1, m_2, l, f) \quad \text{iff} \quad G_{lmf}\frac{m_1 m_2}{f\,l^2} = \Gamma;$$

$$\mathrm{B}^{\mathrm{MTN}}(m, l, t, f) \quad \text{iff} \quad A_{ltmf}\frac{m}{f} \cdot \frac{\mathrm{d}^2 l}{\mathrm{d}t^2} = 1$$

$$\text{where } G_{lmf} = A_{ltmf} = 1 \text{ when } (l, t, m, f) \text{ is } (\bar{l}, \bar{t}, \bar{m}, \bar{f}). \tag{7.14}$$

To derive 7.14, every MF in 7.13 has been multiplied with its own conversion factor, and these factors have algebraically "coalesced" (using Bridgman's language) into two dimensional constants of a monomial structure: $G_{lmf} = c_l^{-2} c_t^0 c_m^2 c_f^{-1}$, $A_{ltmf} = c_l^1 c_t^{-2} c_m^1 c_f^{-1}$. In the technical language of dimensional analysis, called dimensional algebra, the same fact is expressed by introducing dimensional symbols L, T, M, F for the four basic quantities and presenting the dimensionality of G_{lmf} and A_{ltmf} as $\mathrm{L^2T^0M^{-2}F^1}$ and $\mathrm{L^{-1}T^2M^{-1}F^1}$, respectively (the exponents being those for the corresponding conversion factors multiplied by -1). According to Bridgman, this is the essential logic of where physics takes its dimensional constants from: all dimensional constants are "coalesced" conversion factors. Physical theory does not play any role here, in agreement with the general position of this chapter: all MF-dependent constants are merely mathematical reductions of conversion coefficients, and they can be computed for any MF-specific sentence generalized to any class of MFs. Thus, from the point of view of dimensional analysis, any sentence involving gravitation forces, masses, and distances could be the true gravitation law (even if it contained, say, masses added to distances), and any such sentence could be made dimensionally homogeneous by an appropriate choice of dimensional constants.

To understand the aspect of physical sentences that dimensional analysis does address, consider the following MF-specific sentences:

$$\begin{aligned} B_1(l, t) &\quad \text{iff} \quad \bar{t} + \bar{l} = 100, \\ B_2(l, t) &\quad \text{iff} \quad \bar{l} \cdot \bar{t}^{-1} = 100, \\ B_3(m, l, t) &\quad \text{iff} \quad \bar{m}^{-1} \cdot \bar{l}^{-2} \cdot \bar{t}^{1} = 100. \\ B_4(m, l, t) &\quad \text{iff} \quad \bar{m}^{-1} \cdot \bar{l}^{3} \cdot \bar{t}^{-2} = 100. \end{aligned} \tag{7.15}$$

Again, all these sentences can be put in a dimensionally homogeneous form by Bridgman's algorithm, each with its own dimensional constants; none of these sentences taken in isolation can be struck down as "impossible" or "meaningless." Suppose, however, that one asks whether these sentences can be *mathematically derived* from Newton's gravitation law and the second law of motion (sentences 7.14 or 7.13). In classical physics, this is what one would expect if, for example, $B_1(l, t)$ were describing period of revolution (t) of two celestial bodies about their gravitation center as a function of distance (l) between them (ignoring, for simplicity, sentences specifying initial conditions). Here is where dimensional analysis comes into operation, in the context of judging *derivability* of a given sentence from other sentences. It is easy to prove that since sentences 7.13 could be put in the dimensionally homogeneous form 7.14 by means of two dimensional constants, G_{lmf} and A_{ltmf}, any consequent of these sentences should be presentable in a dimensionally homogeneous form by means of dimensional constants that are functions of G_{lmf} and A_{ltmf}; moreover, these functions can only be monomials of the form $G^a_{lmf}A^b_{ltmf}$ (since all dimensional constants are monomials over conversion factors).

By simple algebra one can show now that the dimensional constants associated with $B_1(l, t)$ in 7.15, c_l and c_t (dimensional formulae $L^{-1}T^0M^0F^0$ and $L^0T^{-1}M^0F^0$), cannot be expressed as two monomials over G_{lmf} and A_{ltmf}: hence this sentence is not derivable from 7.13–7.14. This has nothing to do with the addition operation specifically, as can be seen from the fact that the same conclusion (nonderivability) applies also to $B_2(l, t)$, whose homogeneous reformulation requires one dimensional constant, c_{lt} ($L^{-1}T^1M^0F^0$). Both $B_1(l, t)$ and $B_2(l, t)$ might very well be empirically true (e.g., B_1 could describe spring length changing under an external force, B_2 could be stating the constancy of the speed of light, in some units)—dimensional analysis only tells us that they are *not derivable* from two *particular* sentences. The comparison of $B_3(m, l, t)$ with $B_4(m, l, t)$ is instructive, too. Both numerical expressions are monomial triples, their homogeneous formulations require one dimensional constant each, c_{ltm} and c'_{ltm} ($L^2T^{-1}M^1F^0$ and $L^{-3}T^2M^1F^0$, respectively). $B_3(m, l, t)$, however, is struck down as nonderivable, whereas $B_4(m, l, t)$ is not, because c_{ltm} cannot be presented as $G^a_{lmf}A^b_{ltmf}$ but $c'_{ltm} = G^{-1}_{lmf}A^1_{ltmf}$ (note that *not being nonderivable* in the considered sense is necessary but not sufficient for being de facto derivable).

This is the entire essence of dimensional analysis. (Algebraic techniques involved, however, based on the Vaschy-Buckingham Pi theorem, are quite a bit more powerful than in the illustrations given; see, e.g., Kurth, 1972; Langhaar, 1951.) It follows that any attempt to theoretically restrict (based on dimensional considerations only) a class of possible laws in a given area *without the context of derivability* is doomed to failure (see Palacios, 1964, chap. 7.4, "First Rule"). For instance, dimensional analysis cannot restrict the class of the *basic* laws of an area, because by definition they are not supposed to be derivable from other laws. In particular, if one wishes to present them in dimensionally homogenous forms, one is not restricted by any phsycial principle as to the number and character of the dimensional constants one has to introduce (explicate). The same clearly applies to psychophysical laws, such as Weber's law or near-miss to Weber's law (cf. Narens & Mausfeld, 1992).

It seems quite obvious that the differences between the dimensional analysis of the four sentences in 7.15 cannot be accounted for on the basis of the "empirical meaningfulness" analyses. If "100" is considered to be a "pure number" (as intended), all four sentences are "empirically meaningless" by the direct substitution criterion. If "100" is dimensioned, then $B_1(l, t)$ is "meaningless," and the remaining three sentences are "meaningful" (in Pfanzagl's terms, "meaningfully parametrized"). Without elaborating, if one sets up the "structure of physical quantities with basis," along the lines suggested by Krantz et al. (1971, chap. 10) and Luce et al. (1990, chap. 22), one will see that the truth value of B_2, B_3, B_4, but not of B_1, is preserved under "similarities of the structure." This is definitely not what dimensional analysis is about.

The restrictive power of dimensional analysis (its ability to detect nonderivable sentences) is due to the fact that the basic laws of (some areas of) physics, derivability from which is being tested, happen to be such that, when written in a dimensionally homogeneous form, the number of the resulting dimensional constants is less than maximal (the maximum number equals that of basic dimensions, e.g., it is 4 in the LENGTH-TIME-MASS-FORCE).[1] Specifically, the basic laws of some areas of physics can be decomposed into fewer sentences than there are basic dimensions, each sentence containing a single monomial over basic quantities, and thereby yielding a single dimensional constant by Bridgman's algorithm (Palacios, 1964). If the number of the MF-dependent constants in all derivations equaled or exceeded that of basic dimensions, then there would

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[1]A precise formulation should refer to the rank of the dimensional matrix associated with the constants, rather than their number. Those familiar with dimensional algebra might find useful the following theorem I state here without proof. Let Γ be a set of formulae containing variables $V_1, \ldots, V_m$ in specific units. Let $C_1, \ldots, C_k$ be the minimal set of dimensional constants that have to be introduced to write Γ in a dimensionally homogeneous form (this set is found by Bridgman's algorithm). Then the Pi theorem restricts the class of formulae $F(V_1, \ldots, V_m) = 0$ (in specific units) derivable from Γ if, and only if, the rank of the dimensional matrix for $C_1, \ldots, C_k$ is less than that for $C_1, \ldots, C_k, V_1, \ldots, V_m$.

be no advantage in using dimensionally homogeneous formulations over MF-specific ones: in the imaginary world with such a structure of physical laws, physicists could very well have adopted fixed "scientific" units of measurement for all physical sentences. Remaining in our own world, the real reason why only dimensionally homogeneous sentences "have reached posterity" (Falmagne, 1992) is not their "empirical meaningfulness," but their optimality with respect to derivability decisions involving relatively few MF-dependent constants. Note that sentences of physics are almost never derived directly from fundamental laws: in addition, one should also include "situational sentences" specifying boundary conditions and intervening external forces. These situational sentences bring in their own dimensional constants (again, by unrestricted application of Bridgman's algorithm), which in many cases are sufficient to annul the restrictive power of dimensional analysis.

It should also be clear why physical MFs are not embedded into classes of conversion functions broader than positive similarities. Physical theory itself does not limit MFs to particular classes. We have seen, for example, that the law of gravitation can be trivially presented in MFs defined up to power conversion functions (sentences 7.10, 7.12), as well as traditional similarity conversions (7.13, 7.14). However, in 7.14 the number of MF-dependent constants in the law reduces to just one, G_{lmf}, due to the algebraic "coalescing" of the conversion factors. By contrast, in 7.12 the three "dimensional exponents" remain separate, and their number equals that of the basic quantities. As a result, in deciding whether a given sentence is or is not derivable from the gravitation law, writing them in a "power-homogeneous" form 7.12 would provide no additional advantage over usual dimensionally homogeneous formulations 7.14. Luce et al. (1990), discussing power transformaton groups in the context of "real unit structures," point out that these transformations are "just how far the dimensional structure of physics can be generalized" (p. 124). It seems that the generalization could very well go much farther, but there is no useful purpose in its going even this far. This seems to explain why "at present there are no substantive examples of such a generalization" (ibid).

COMPLETE EMPIRICAL RELATIONAL SYSTEMS

To be well defined, any class of MFs should contain at least one anchoring MF, defined through an effective empirical procedure: it would do little good to know that different MFs for length are interrelated by positive similarities if none of them could be computed independently, "from empirical objects." H. Helmholtz (see Menger, 1959) has shown that empirical measurement procedures for quantities like length or mass can be described by a set of a few operations whose basic properties are formalized in a set of axiomatic sentences. Suppes and

Zinnes (1963) called such a theoretical construct (a set of magnitudes with relations defined through their axiomatic properties) an *empirical relational system* (ERS). For example, the "ratio-scale" representation of length is traditionally associated with the ERS $\mathcal{L}_1 = \{\mathscr{L}, l_1 \leq_l l_2, l_1 \oplus_l l_2 = l_3\}$, involving a linear ordering $\leq_l$ of length magnitudes $\mathscr{L}$ and a concatenation operation $\oplus_l$ with sum-like properties. A representation-uniqueness theorem tells us that there exists such a MF $\bar{l}(l)$ mapping $\mathscr{L}$ onto Re^+ that $l_1 \leq_l l_2$, iff $\bar{l}(l_1) \leq \bar{l}(l_2)$ and $l_1 \oplus_l l_2 = l_3$ iff $\bar{l}(l_1) + \bar{l}(l_2) = \bar{l}(l_3)$; the same holds for, and only for, any MF $l(l) = C\bar{l}(l)$, $C > 0$, a member of the class I have referred to as LENGTH. In the language of algebra, the ERS $\mathcal{L}_1$ is isomorphically mapped onto a numerical relational system (NRS) $L_1 = \{Re^+, l_1 \leq l_2, l_1 + l_2 = l_3\}$, the isomorphisms being defined up to positive scaling.

Consider now a "qualitative" sentential function (a predicate containing no MFs) $\mathrm{P}(l_1, l_2, \ldots)$ that can be expressed through the defining predicates of $\mathcal{L}_1$ exclusively, $\leq_l$ and $\oplus_l$ (interconnected by logical and mathematical terms). Let such a predicate be called "empirically definable in $\mathcal{L}_1$" (Luce et al., 1990, chap. 22). A necessary condition for this is that the following sentence be logically true:

$$\begin{aligned} &\text{for any } l_1, l_2, \ldots, l_1^*, l_2^*, \ldots, \text{ and for any } C > 0: \\ &\quad \text{if } l_1^* = \bar{l}^{-1}[C\bar{l}(l_1)],\ l_2^* = \bar{l}^{-1}[C\bar{l}(l_2)], \ldots, \\ &\quad\quad \text{then } [\mathrm{P}(l_1, l_2, \ldots) \text{ iff } \mathrm{P}(l_1^*, l_2^*, \ldots)]; \end{aligned} \tag{7.16}$$

$\bar{l}$ stands here for a specific MF $\in$ LENGTH. The transformation $\bar{l}^{-1}[C\bar{l}(l)]$ mapping $\mathscr{L}$ onto itself is an automorphism of $\mathcal{L}_1$; the automorphisms form a group, with the identity (or trivial automorphism) corresponding to $C = 1$. Consider now the numerical sentential function $\mathrm{P}^*(\bar{l}_1, \bar{l}_2, \ldots)$ obtained by a direct substitution of $\bar{l}(l_1)$ for l_1, $\bar{l}(l_2)$ for l_2, etc., accompanied by a direct substitution of $\leq$ for $\leq_l$ and $+$ for $\oplus_l$. The numerical predicate P^* can be referred to as *representing* an empirically definable (in $\mathcal{L}_1$) predicate. For the moment, I leave open the question of whether the predicate $\mathrm{P}^*(\bar{l}(l_1), \bar{l}(l_2), \ldots)$ is itself empirically definable in $\mathcal{L}_1$, *when viewed as a "qualitative" predicate over* $(l_1, l_2, \ldots)$. Obviously, the truth value of any sentence formed from $\mathrm{P}(l_1, l_2, \ldots)$ by quantification or specialization (on particular values of $l_1, l_2, \ldots$) should coincide with that of the sentence formed from $\mathrm{P}^*(\bar{l}_1, \bar{l}_2, \ldots)$ by the same quantification or specialization. Since this must also be true for any MF $C\bar{l}(l)$, one comes to the following conclusion: if a sentence $\mathrm{S}(\bar{l}(l))$ does not preserve its truth value under direct substitutions of $C\bar{l}(l)$ for $\bar{l}(l)$, then it must contain a predicate that does not represent an empirically definable predicate in $\mathcal{L}_1$. Such a sentence then can be labeled "empirically meaningless" with respect to $\mathcal{L}_1$. This is the essence of "empirical meaningfulness" understood on a "qualitative" level: invariance under mutual substitutions of MFs is justified as a *necessary condition* for "definability" in terms of a particular ERS.

As an example, consider the following two MF-specific sentences (to be read with the generality convention in mind):

$$\mathrm{B}_1(\ell_1, \ell_2, \ell_3) \quad \text{iff} \quad \bar{l}_1 + \bar{l}_2 = \bar{l}_3 \tag{7.17}$$

$$\mathrm{B}_2(\ell_1, \ell_2, \ell_3) \quad \text{iff} \quad \bar{l}_1 \cdot \bar{l}_2 = \bar{l}_3 \tag{7.18}$$

Sentence 7.18 does not pass the direct substitution test under similarity conversions, and one concludes that $\bar{l}_1 \cdot \bar{l}_2 = \bar{l}_3$ does not represent an empirically definable predicate in $\mathcal{L}_1$. Sentence 7.17 does pass the test, and one concludes that $\bar{l}_1 + \bar{l}_2 = \bar{l}_3$ *might* represent an empirically definable predicate in $\mathcal{L}_1$; in this case it obviously represents $\ell_1 \oplus_l \ell_2 = \ell_3$, which is empirically definable de facto. Clearly, the "empirical meaninglessness" of 7.18 thus understood does not refer to the sentence in isolation, but only in its relation to a particular ERS. In particular, it has nothing to do with truth or "scientific significance" of this sentence—the "meaninglessness" here is void of negative connotations, being a purely technical characterization (see, e.g., Narens, 1985, p. 155). Sentence 7.17, for example, is "empirically meaningless" with respect to the ERS $\mathcal{L}_0 = \{\mathscr{L}, \ell_1 \leq_l \ell_2\}$, that can be isomorphically mapped onto $\mathrm{L}_0 = \{Re^+, l_1 \leq l_2\}$ (the MFs $l(\ell)$ here are defined up to arbitrary strictly increasing conversion functions). Obviously, one can find an infinity of ERSs in which a given sentence is "empirically meaningless," and this should not concern a researcher any more than an abstract algebraic exercise. The "empirical meaninglessness" of 7.18 in $\mathcal{L}_1$ simply indicates that the factual empirical procedures that led to its formulation cannot be formalized by the axioms of $\mathcal{L}_1$. On this note the discussion might have ended, perhaps with pointing out, in addition, that the literature regrettably abounds with misleading statements suggesting that "empirical meaninglessness" indicates things like "concepts that have neither empirical nor qualitative interpretations in the substantive domain" (Narens & Mausfeld, 1992, p. 467).

The issue is somewhat deeper, however: when applied to numerical statements involving well-defined MFs, the notion of "empirical meaningfulness" cannot serve even the limited technical purpose just discussed. The reason for this is in that *no specific MF (such as length in meters) can be defined within an ERS that has nontrivial automorphisms (equivalently, an ERS that isomorphically maps onto a given NRS by more than one MF).* Thus, the MF $\bar{l}$ in sentences 7.17 and 7.18 is an empirical predicate, "$\bar{l}(\ell) = \bar{l}$," that is not empirically definable in $\mathcal{L}_1$ (or $\mathcal{L}_0$, or any other ERS whose isomorphisms onto a given NRS consist of more than one MF). Therefore, by the very fact of formulating sentences invoking this MF (whether these sentences are "meaningful" or "meaningless" in $\mathcal{L}_1$ or $\mathcal{L}_0$), one guarantees that ERSs like $\mathcal{L}_1$ and $\mathcal{L}_0$ cannot formalize the factual empirical procedures involved.

Indeed, in the language consistent with $\mathcal{L}_1$, the empirical predicate "$\bar{l}(\ell) = \bar{l}$" is defined as "$\ell \circledR_l \ell_0 = \bar{l}$," where ℓ_0 refers to some standard length ("yardstick") and $\circledR_l$ is the "empirical ratio" operation effectively defined through $\oplus_l$ and $\leq_l$ by the standard Archimedean algorithm (parallel concatenations of "the measured" and the "yardstick"; see, e.g., Narens, 1985). This semiformal definition of "$\bar{l}(l) = \bar{l}$" presumes that l_0 is "known and fixed"; that is, it can be uniquely identified by a designatory sentential function "such length that [a unique description follows]." It is obvious, however, that the description in brackets cannot be written in terms of operations $\oplus_l$ and $\leq_l$ only. Put formally, the predicate $\mathcal{I}_0(l)$ that says "l is the standard length l_0" is not empirically definable in $\mathcal{L}_1$. A formal proof consists in observing that the sentence

$$\text{for any } l_1, l_2 \text{ and any } C > 0, \text{ if } l_2 = \bar{l}^{-1}[C\bar{l}(l_1)] \text{ then } [\mathcal{I}_0(l_1) \text{ iff } \mathcal{I}_0(l_2)] \quad (7.19)$$

is logically false (compare this with 7.16). There is nothing surprising or contradictory in the fact that $\mathcal{L}_1$ does not allow one to effectively identify any member of the class LENGTH of its own isomorphisms. Indeed, the class of all MFs isomorphically mapping $\mathcal{L}_0 = \{\mathcal{L}, l_1 \leq_l l_2\}$ onto $L_0 = \{\mathcal{L}, l_1 \leq_l l_2\}$, too, contains any specific MF one can think of, say, the meter measure, but here it is quite obvious that the language of $\mathcal{L}_0$ is too limited to single out and identify this measure. One might say that "from the point of view" of $\mathcal{L}_1$ different MFs from the class LENGTH are not just intersubstitutable, they are indistinguishable. Insofar as one can effectively distinguish between different specific MFs within a class of "admissible transformations" for a given ERS, a relevant formalization of the empirical procedures requires a *completion* of this ERS by some additional predicates, that would reduce its automorphisms to identify.

For length MFs such a construction was proposed by D. Hilbert in his classical axiomatization of Euclidean geometry (Hilbert, 1902); in Krantz et al. (1971, chap. 2) this construction is considered under the name of Archimedean ordered rings. The ERS in question is $\mathcal{L}_2 = \{\mathcal{L}, l_1 \leq_l l_2, l_1 \oplus_l l_2 = l_3, l_1 \otimes_l l_2 = l_3\}$, where $\otimes_l$ is an operation with multiplication-like properties. This ERS can be isomorphically mapped onto $L_2 = \{Re^+, \bar{l}_1 \leq \bar{l}_2, \bar{l}_1 + \bar{l}_2 = \bar{l}_3, \bar{l}_1 \cdot \bar{l}_2 = \bar{l}_3\}$, the MF $\bar{l}(l)$ being defined *uniquely*. Equivalently put, the group of the automorphisms of this ERS is reduced to identity. I will call such an ERS *complete*. Observe that $\mathcal{L}_2$ is equivalent to the ERS $\mathcal{L}_2^* = \{\mathcal{L}, l_1 \leq_l l_2, l_1 \oplus_l l_2 = l_3, \mathcal{I}_0(l)\}$, where $\mathcal{I}_0(l)$ says "l is the standard length l_0 defined as [a unique identification of l_0 in nonlength terms follows]." One recognizes here the common practice of complementing descriptions of the empirical operations of "comparing the measured with a yardstick" (which is what $\leq_l$ and $\oplus_l$ provide) by a definition of the "yardstick" itself. A NRS uniquely isomorphic to $\mathcal{L}_2^*$ is, for example, $L_2^* = \{Re^+, \bar{l}_1 \leq \bar{l}_2, \bar{l}_1 + \bar{l}_2 = \bar{l}_3, \bar{l} = 1\}$.

It must be quite clear that *all* predicates involving length values are de facto empirically definable in $\mathcal{L}_2^*$ (equivalently, $\mathcal{L}_2$). Indeed, any predicate P(l_1, l_2, . . .) can be defined by the proposition "P(l_1, l_2, . . .) iff P*($\bar{l}_1$, $\bar{l}_2$, . . .)," where P* is as explained earlier. The predicate $\bar{l}(l) = \bar{l}$ is defined in $\mathcal{L}_2^*$ by the proposition "$\bar{l}(l) = \bar{l}$ iff $\mathcal{I}_0(l_0)$ and $l \circledR_l l_0 = \bar{l}$," where $\circledR_l$ is the "empirical ratio" referred to earlier. Once $\bar{l}(l) = \bar{l}$ is empirically definable, P*($\bar{l}_1$, $\bar{l}_2$, . . .) is empirically definable too, because it only involves numerical operations on $\bar{l}(l)$. Thus, both predicates $\bar{l}_1 \cdot \bar{l}_2 = \bar{l}_3$ and $B_2(l_1, l_2, l_3)$ in 7.18 are de facto and a priori empirically definable in $\mathcal{L}_2^*$. In an incomplete ERS, like $\mathcal{L}_1$, even if a numerical predicate, like $\bar{l}_1 + \bar{l}_2 = \bar{l}_3$, represents an empirically definable predicate, it would not be empirically definable itself.

Without elaborating, it trivially follows from the definitions introduced by Narens (1981) that an N-point-unique ERS can be made complete by appending to its defining predicates N arbitrary "yardstick-predicates." For example, the "interval-scale" representation of temperature is sometimes associated with the ERS $\mathcal{T}_1 = \{\mathfrak{T}, t_1 \leq_t t_2, t_1 \therefore_t t_2 = t_3\}$, involving a linear ordering $\leq_t$ of temperature magnitudes $\mathfrak{T}$ and an operation $\therefore_t$ with averaging-like properties. It is isomorphically mapped onto NRS $T_1 = \{Re, t_1 \leq t_2, t_1 + t_2 = 2t_3\}$, the isomorphisms being defined up to positive linear conversions. To make $\mathcal{T}_1$ complete, one can append to it such predicates as $\mathcal{W}_f(t)$, standing for "water eventually freezes at t," and $\mathcal{W}_e(t)$, standing for "water eventually evaporates at t." The unique isomorphic mapping $\bar{t}(t)$ of $\mathcal{T}_2 = \{\mathfrak{T}, t_1 \leq_t t_2, t_1 \therefore_t t_2 = t_3, \mathcal{W}_f(t), \mathcal{W}_e(t)\}$ onto $T_2 = \{Re, \bar{t}_1 \leq \bar{t}_2, \bar{t}_1 + \bar{t}_2 = 2\bar{t}_3, \bar{t} \leq 0, \bar{t} \geq 100)$ is the Celsius MF. Any predicate involving temperature magnitudes is empirically definable in $\mathcal{T}_2$, and any sentence involving Celsius MFs is "empirically meaningful." I will not discuss here whether the predicates $\mathcal{W}_f$ and $\mathcal{W}_e$ are as "fundamental" as, say, the operation $\therefore_t$, but they are clearly as well defined empirically and as rigorous theoretically (the description of $t_1 \therefore_t t_2 = t_3$ is as follows: "if one mixes two equal amounts of some substance, with initial temperatures t_1 and t_2, and if a heat loss is prevented, then the eventual temperature of the mixture is t_3").

Once a complete ERS is constructed and represented by a well-defined (unique) MF, this MF, *or some transformation thereof,* can, of course, be embedded as an anchoring MF in any class of MFs interrelated by conversion functions: the algorithm of covariant substitution guarantees, as I have shown, that both the truth value and the form (up to MF-dependent constants) of any sentence involving this MF will be preserved under all possible substitutions within any such class. The specific empirical operations formalized in an ERS are, in fact, quite irrelevant insofar as it is complete. I suggest that the only goal of measurement is to construct a unique numerical *identification* of the magnitudes of a quantity being measured. Once such an identification is empirically available, the class of conversion functions with which it will be associated will be determined by the objective structure of the empirical laws of an area. If the only empirically available numerical identification $\bar{l}$ of length, for example, were logarithmically related to the conventional meter measure, all sentences of mechanics would still be formulated as we know them, for the MFs within the class LENGTH. The only difference would be, of course, that the class will now be defined as "all MFs $l(l)$ such that $l = C \exp(\bar{l})$, for some positive real C." In the section on dimensional analysis I have discussed the reasons why using classes like LENGTH is convenient and desirable.

I suggest that construction of empirically complete ERSs and investigation of their mutual relations could be the central subject in a new revision of theory of measurement. Incomplete ERSs and their automorphisms can be treated as groups of complete ERSs (extensive, set-theoretic approach), or their parts (intensive, logical approach). In such a theory, which I tentatively call the *constructive* theory of measurement, *measurement is understood as an effective algorithm by which one constructs within a set of linearly ordered objects an everywhere dense subset of standard objects. To measure an object in the set is to indicate a unique chain of steps (generally, countably infinite) of the algorithm that leads to a standard object which, in some well-defined sense, is "infinitely close" to the object being measured.* Such an approach would bring measurement procedures back into the measurement theory, while preserving most of the mathematical results established within its framework. The notion of "empirical meaningfulness," however, would have no useful purpose.

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